Advanced Engineering Mathematics
Lecture 10

1 Integral Calculus

$$I = \int_a^b f(x)\, dx$$

If the integrand of the integral becomes infinite in the interval $[a, b]$ or f has points of infinite discontinuity or when both or either one of the limit points/ end points $[a, b]$, is infinite then the integral I is called improper integral.

Example 1.1. $\int_0^1 \frac{1}{x^2} dx, \int_1^2 \frac{1}{(2-x)(1-x)} dx, \int_0^1 \frac{1}{(x-\frac{1}{2})^2} dx, \int_0^\infty e^x dx, \int_{-\infty}^0 \frac{1}{e^x} dx$

Convergence at the left end point.

$$I = \int_a^b f(x) dx$$

Let, at $x = a$, $f(x)$ has infinite discontinuity. In this case, we first find the value of $\int_{a+\varepsilon}^b f(x)\, dx$ then pass the limit $\varepsilon \to 0^+$ i.e,

$$I = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\, dx$$

Convergence at the right end point.

$$I = \int_a^b f(x) dx$$

Let, at $x = b$, $f(x)$ has infinite discontinuity. In this case, we first find the value of $\int_a^{b+\varepsilon} f(x)\, dx$ then pass the limit $\varepsilon \to 0^-$ i.e,

$$\begin{aligned} I &= \lim_{\varepsilon \to 0^-} \int_a^{b+\varepsilon} f(x) dx \\ &= \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x) dx \end{aligned}$$

Convergence inside the interval. Let at $x = c \in (a, b)$, $f(x)$ has an infinite discontinuity. In this case, we first separate $I = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$, then we do the same as before

$$\begin{aligned} I &= \int_a^b f(x)\, dx \\ &= \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \\ &= \lim_{\varepsilon \to 0^-} \int_a^{c+\varepsilon} f(x)\, dx + \lim_{\varepsilon \to 0^+} \int_{c+\varepsilon}^b f(x)\, dx \end{aligned}$$

Convergence at $+\infty, -\infty$.

$$\begin{aligned}
I &= \int_a^{\infty} f(x)\,dx \\
&= \lim_{t\to\infty} \int_a^t f(x)\,dx \\
I_1 &= \int_{-\infty}^{b} f(x)\,dx \\
&= \lim_{t\to-\infty} \int_t^b f(x)\,dx \\
I_2 &= \int_{-\infty}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx
\end{aligned}$$

Example 1.2. $I = \displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Sol. At $x = 1$, there is an infinite discontinuity.

$$\begin{aligned}
I &= \lim_{\varepsilon\to 0^+} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx \\
&= \lim_{\varepsilon\to 0^+} \left[\sin^{-1} x\right]_0^{1-\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \sin^{-1}(1-\varepsilon) \\
&= \sin^{-1} 1 \\
&= \frac{\pi}{2}
\end{aligned}$$

Example 1.3. $I = \displaystyle\int_0^{\pi} \frac{1}{\sin x}\,dx$

Hint.

$$\begin{aligned}
I &= \int_0^{\pi} \frac{1}{\sin x}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{\sin x}\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{\sin x}\,dx
\end{aligned}$$

Example 1.4. $I = \displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx$

Sol.

$$\begin{aligned}
I = \int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx &= \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t \\
&= \lim_{t\to\infty} \tan^{-1} t \\
&= \frac{\pi}{2}
\end{aligned}$$

Example 1.5. $I = \displaystyle\int_a^b \frac{1}{(x-a)^n}\,dx$

Sol.

$$\begin{aligned}
I &= \int_a^b \frac{1}{(x-a)^n}\,dx \\
&= \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b \frac{1}{(x-a)^n}\,dx \\
&= \lim_{\varepsilon\to 0^+} \left[\frac{1}{1-n}(x-a)^{(1-n)}\right]_{a+\varepsilon}^b \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{1-n}\left(\frac{1}{(b-a)^{(n-1)}} - \frac{1}{\varepsilon^{n-1}}\right)
\end{aligned}$$

It is convergent if $n < 1$ and divergent if $n \geq 1$.