Advanced Engineering Mathematics
Lecture 1

1 Derivative at a point

If a function f is defined on a neighbourhood of a point c and $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ or $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists, then the function is said to be derivable at point $x=c$ and the derivative is denoted by $f'(c)$. $f'(c)$ is called the derivative of the function f at point $x=c$.
A function f is said to be derivable from the right or left at a point c if $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ or $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ exists finitely.

We define $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ as $Rf'(c)$ and $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$ as $Lf'(c)$.

Example 1.1. Let $f(x)=|x| \quad \forall x\in\mathbb{R}$. Verify whether f is differentiable at $x=0$.

Sol. Given function

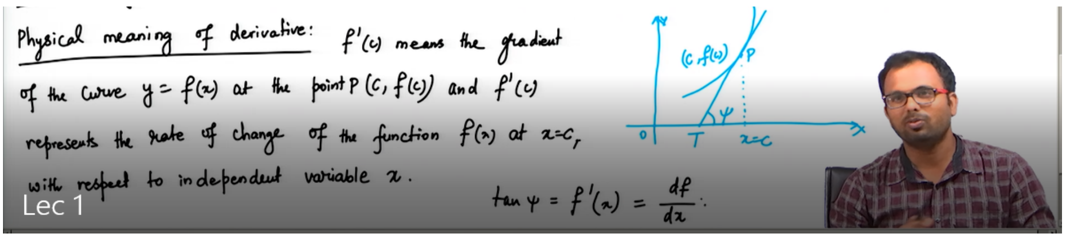
$$\begin{aligned} f(x) &= |x| \\ &= x \qquad x>0 \\ &= -x \quad\; x<0 \\ &= 0 \qquad x=0 \end{aligned}$$

By definition,

$$\begin{aligned} \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} &= \lim_{h\to 0^+}\frac{h-0}{h} = 1 = Rf'(0) \\ \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} &= \lim_{h\to 0^-}\frac{-h-0}{h} = -1 = Lf'(0) \end{aligned}$$

i.e, $Rf'(0)\neq Lf'(0) \Rightarrow f$ is not differentiable at $x=0$. But, $|f(x)-f(0)| = ||x|-0| = |x| < \varepsilon$ whenever $0<|x|<\delta \Rightarrow f$ is continuous at 0.
Remark. Every differentiable function is continuous but converse may not be true.
Physical meaning of derivative: Derivative of a function f at a point $x=c$ i.e. $f'(c)$ means the gradient of the curve $y=f(x)$ at point $P(c,f(c)$ and $f'(c)$represents the rate of change of the function $f(x)$ at $x=c$ with respect to the independent variable x.

$$\tan\psi = f'(x) = \frac{df}{dx}$$



Example 1.2. Let us consider the function

$$\begin{aligned} f(x) &= x\sin\frac{1}{x} \quad \text{if} \quad x\neq 0 \\ &= 0 \qquad\qquad \text{if} \quad x=0 \end{aligned}$$

Then, verify that $f(x)$ is differentiable at $x=0$.

Sol. Here $|f(x) - f(0)| = |x \sin \frac{1}{x} - 0| = |x \sin \frac{1}{x}| = |x|| \sin \frac{1}{x}|$

$$
\begin{aligned}
&|f(x) - f(0)| = |x \sin \frac{1}{x} - 0| = |x \sin \frac{1}{x}| = |x|| \sin \frac{1}{x}| \\
\Rightarrow &|f(x) - f(0)| \leq |x| < \varepsilon \quad \text{whenever} \quad 0 < |x| < \delta \\
\Rightarrow &f \ \text{ is continuous at } \ x = 0.
\end{aligned}
$$

By definition,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \to 0} \sin \frac{1}{x}$$

But $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist. $f'(0) = \lim_{x \to 0} \sin \frac{1}{x}$ does not exist. This implies that the function f is not differentiable at $x = 0$.

Example 1.3. Let us define

$$
\begin{aligned}
f(x) &= x^2 \sin \frac{1}{x} \quad \text{if} \quad x \neq 0 \\
&= 0 \qquad\qquad \text{if} \quad x = 0
\end{aligned}
$$

Sol. By definition we can show that f is differentiable. However, when $x \neq 0$, then $f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} - \frac{1}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.