Advanced Computational Techniques

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Lecture 09

Linear System of Equations

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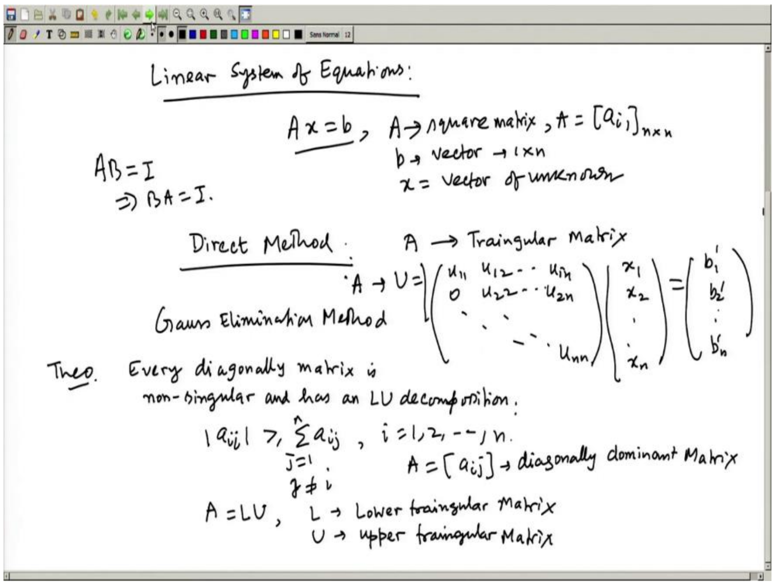


Now, we will talk about solving a linear system of equations. So, we have a system

Ax=b, A→ square matrix, A = $[a_{ij}]$ n x n

b → vector → 1 x n

x = vector of unknown

we have shown before that a system is equivalent under all the elementary operations, row and column operations. Instead of solving Ax = b, we can solve an equivalent system which is more convenient to get a solution. And if it is a square matrix, it will have a unique right inverse. So, that means AB = I, then this matrix a square and this is unique. So, this implies BA=I. Now, direct methods, there are few direct methods for solving. One of these is that if we can reduce this

A → a triangular matrix, triangular matrix is a matrix which is lower triangular or upper triangular.

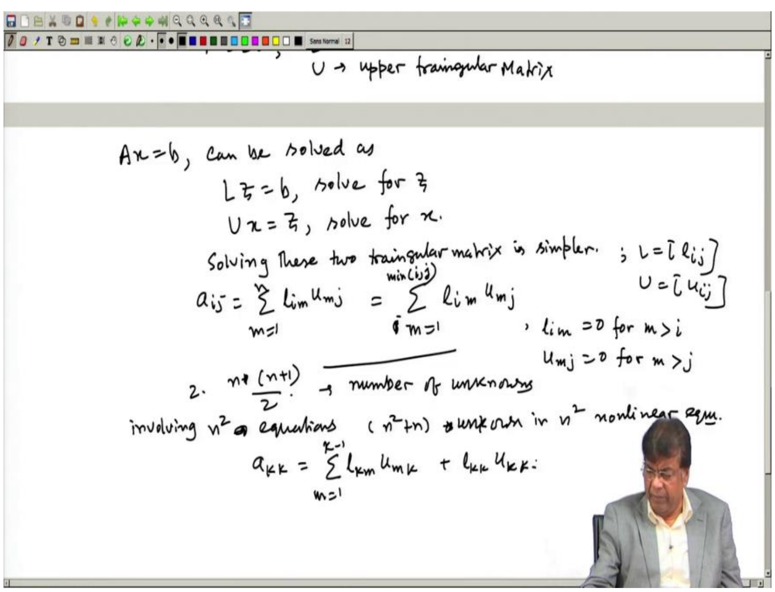
$$A \rightarrow U = \begin{pmatrix} u_{11} & u_{12} \ldots u_{1n} \\ 0 & u_{22} \ldots u_{2n} \\ . & . \quad . u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} = \begin{pmatrix} \bar{b_1} \\ \bar{b_2} \\ \bar{b_n} \end{pmatrix}$$

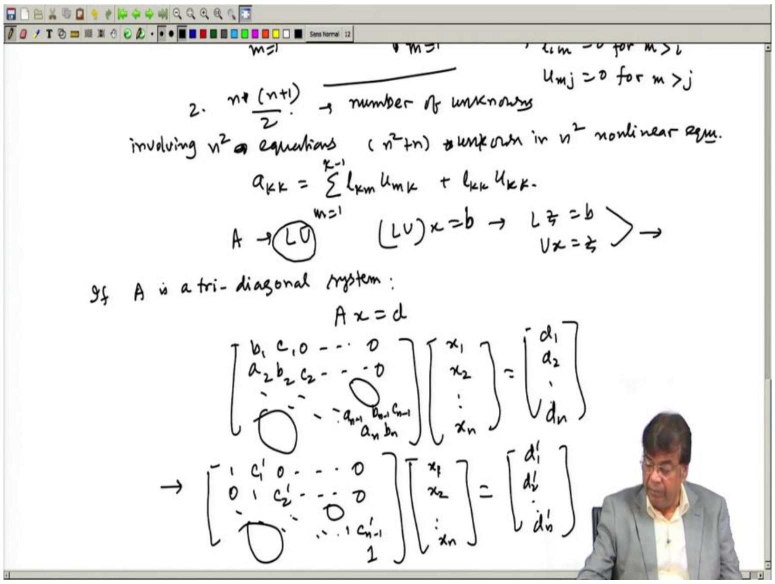
So, if we can reduce A to the upper triangular matrix, the solution is easy to perform. Similarly, if we can reduce A to the upper or lower triangular matrix, by the Gauss Elimination Method. What we do is we reduce this A to a triangular system, an upper triangular system, and we obtain the solution from there.

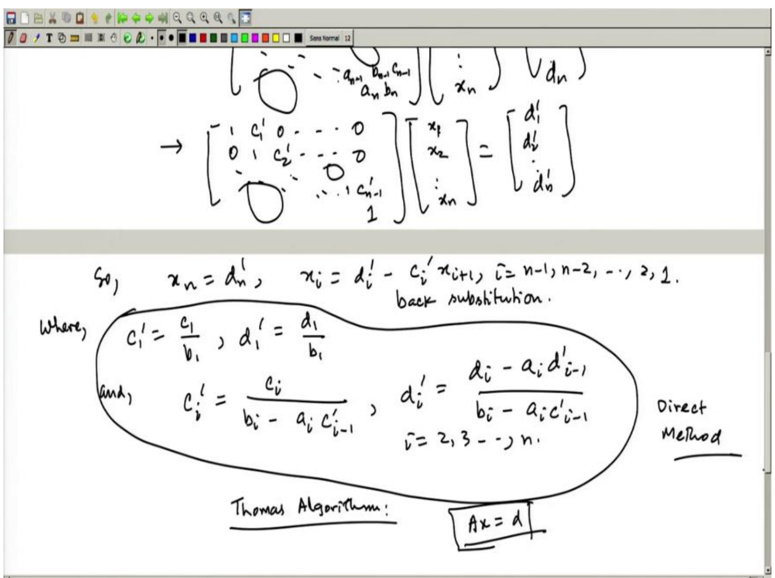
Now, we can write every diagonally dominant matrix so that every diagonally dominant matrix is non-singular and has an LU decomposition. Now, what is diagonally dominant is that if we have a situation

$| a_{ii} | \geq \sum_{j=1, j \neq i}^{n} a_{ij}$, i=1, 2…n. $A = (a_{ij}) \rightarrow$ diagonally dominant matrix.

if the matrix is diagonally dominant then it is the non-singular and can have a LU decomposition. So, that means we can write the matrix A=LU, and A can be factorized by this manner, where L→ lower triangular matrix and U → upper triangular matrix. So, if we can have a decomposition of A into a lower triangular and an upper triangular matrix.







Ax=b, can be solved as

Lz=b, solve for z

Ux=z, solve for x

with these two steps we can find out the solution of the system two triangular matrix are much system is simple. Solving these two triangular matrices is simpler than solving directly a matrix given by this way.

L= l_{ij} , U= u_{ij}

$a_{ij} = \sum_{m=1}^{n} l_{mn}\ \ u_{mj} = \sum_{m=1}^{min(ij)} l_{mn}\ u_{mj},\qquad l_{im} = 0$ for m>i,

$u_{mj} = 0$ for m>j

Now, finding this l_{im} and u_{mj} from here it is not a very simple procedure because we have number of equations and number of unknowns are not telling. So, we have the number of unknowns is 2. $\frac{n\,(n+)}{2} \rightarrow$ number of unknowns and for that we have n^2 involved in n^2 equations and which are not linear by the way, so that means $(n^2 + n)$ unknown in n^2 nonlinear equations. it is not a very simple procedure to find out these to convert it to A. There are several techniques for converting to lower triangular and upper triangular form. So, which we are not going to talk about. So, one of these is very simple way is to construct say if we find the diagonal elements.

$$a_{kk} = \sum_{m=1}^{k-1} l_{km}\ u_{mk} + l_{kk}\ u_{kk}$$

and this once we get ij = k if we put, so once we get that and from there one can find out the other elements. I am not going to talk about much detail. There are several ways to decompose A into a LU decomposition and once we are lucky enough to get a LU decomposition, so (LU)x=0 can be solved in two step, one is l_x = b, l_z = b and then $u_x = $ z these two solutions can be made in that some manner. Now, this Gauss elimination and there are several other variations of that like Gauss Jordan methods and there are several faster ways of solving, there are quite a few direct methods there and one of them is the LU decompositions. And we will also talk about of course, now, we will talk about iterative methods, we have the iterative methods.

Now, another simpler situation if A is a tri-diagonal matrix, if A is a tri-diagonal system. in that case also we can have a very simple direct method to solve the system of equation. if you have

Ax= d

$$A = \begin{pmatrix} b_1\ c_1, 0 \ldots.0 \\ a_1\ \ b_2\ c_2 \ldots 0 \\ .\quad .\quad . \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \cdots \\ d_n \end{pmatrix}$$

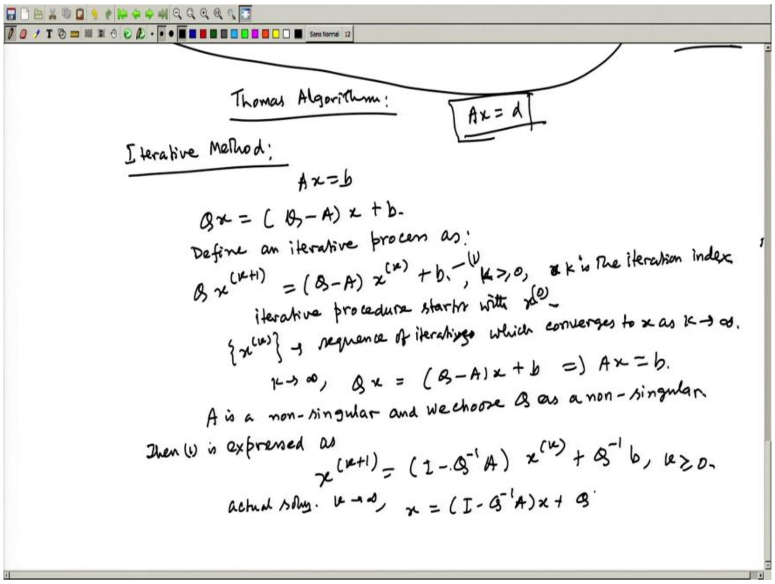
If a system which needs to be solved is this type so then one simple procedure is that we do the elementary row operations and all and reduce this A to a triangular form. then we reduce it to first bring in the diagonal position or the entries one and prior to the diagonal so we go to steps one we bring the diagonal elements as 1 and then the next step what we do is we eliminate all these elements before the diagonal. after a two-step what I get is

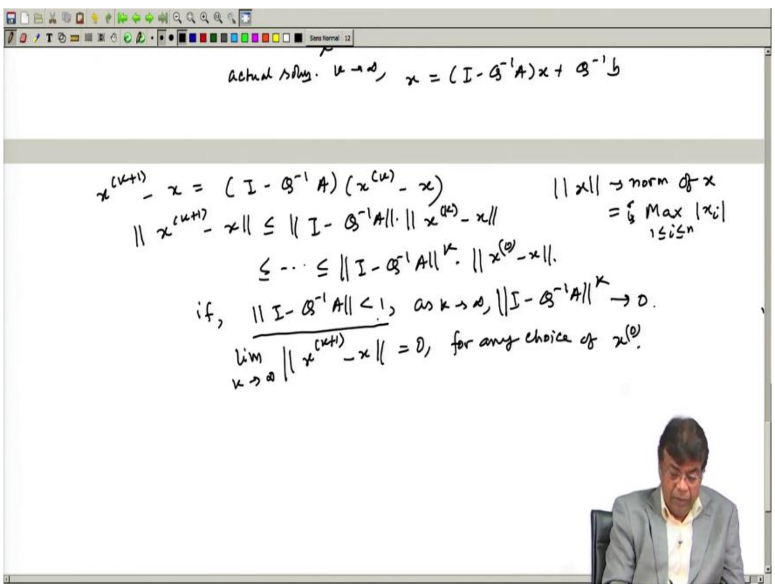
$$\begin{pmatrix} 1 & c_1^- \; 0 \ldots .0 \\ 0 & 1 \; c_2^- \ldots 0 \\ . & . \quad . \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} = \begin{pmatrix} d_1^- \\ d_2^- \\ d_n^- \end{pmatrix}$$ once we have this system then it is very easy to solve.

What is need to be done is just a back substitution and we get the solution. So, the last one is given by, solution is $x_n = d_1^-$, and in general that is by back substitution so once first we find out the last one and then from there we find out the $x_i = d_1^- - c_1^- x_{i+1}$, i = n-1, n-2 …2 , 1 by back substitution. now the question is how to get the algorithm is required to get these reduced elements a d_i^- and c_i^-. these reduced elements can be obtained by this manner where $c_i^- = \frac{c_1}{d_1}$, $d_1^- = \frac{d_1}{b_1}$ we are making a 1 to 1 correspondence between the elements which was given that is $a_i b_i c_i$ and now we have c_i^- d_1^- this is the first one and then what we have is $c_i^- = \frac{c_i}{b_i - a_i c_{i-1}^-}$, $d_i^- = \frac{d_i - a_i d_{i-1}^-}{b_i - a_i c_{i-1}^-}$, i=2,3,….n. first in the algorithm what we have to find out first these coefficients c_i^- and d_i^-, once we obtain these coefficients then we go to the back substitution process and obtain the solution x_i^-. So, this is the procedure is a direct method, this method is a special kind of form that is a tridiagonal matrix. If A is a tridiagonal matrix, then we get the special form like this, and this is called the Thomas algorithm. in many cases later on in many applications. what we will do, we will try to get a system which is in a tridiagonal form Ax= d once you have that by n number of steps we can get the solution.

So, this is the direct method. Now, all the time we may not be so lucky, so we talk about the two procedure one is the Gauss elimination method by what step by step we reduce A to a triangular or in particular a upper triangular matrix and that is the usual Gaussian elimination procedure. So, once you have an upper triangular matrix you solve that upper triangular matrix to get the solution.

Another LU decomposition that means, you reduce these A to a product of two triangular matrix L and U and solve the two coupled system one by one in a sequential manner that is first LU =l z=b and then Ux= z and you get the solution x. So, this is the few direct methods.





Now, in most of the situation in practical purpose we may not be lucky enough to go to a reduced form either in a triangular matrix or the given matrix may not be a tri diagonal, so in those cases we have to go by iterative method. So, we talk about few iterative methods to solve this. what we do is Ax= b we split Q; we write this as this form

Qx=(Q-A) x + b

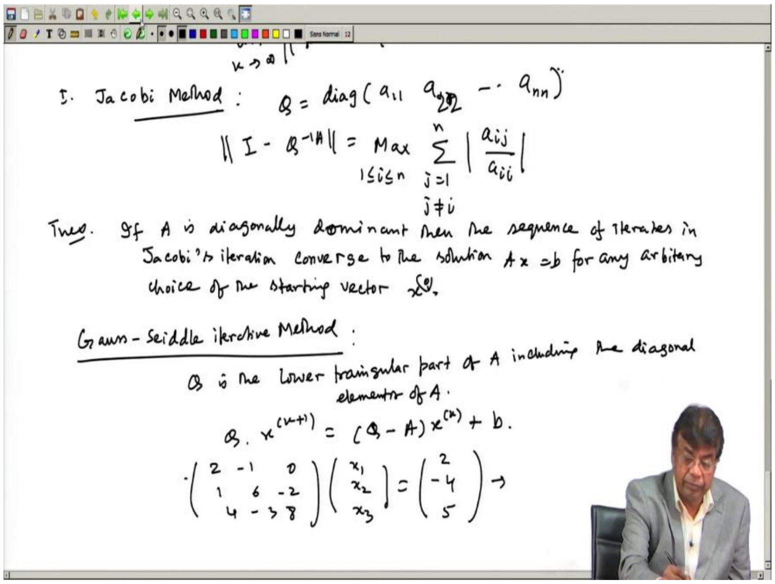
and then we apply an iterative procedure. define an iterative procedure as; $Qx^{(k+1)}$= (Q-A) $x^{(k)}$+b, k$\geq$ 0, k is the iteration index. Now, the iterative procedure starts with initial approximation with an $x^{(0)}$ we develop a sequence of iterates. So, this is the sequence of iterates. And what we need that these converge which converges to x as k $\rightarrow \infty$.

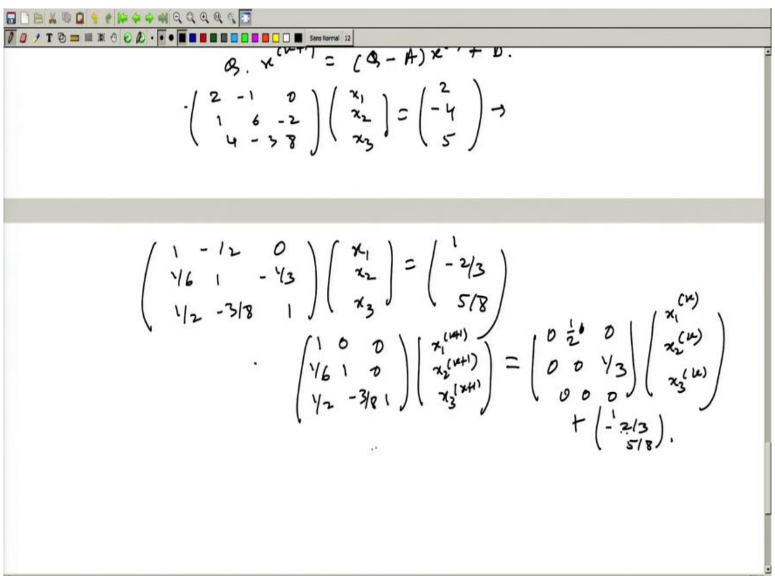
So, this is the procedure for, this is the basic philosophy for an iterative procedure. So, once we have a convergence so then as k tends to infinity what I get is Qx= (Q -A) x+b. So, that means this implies Ax = b. So, that means the system of the solution x is satisfying the system. So, now in this procedure of course, A is a non-singular matrix, so it as a unique solution and we choose Q as non-singular matrix, that means, determinant of A is nonzero then we can write this (1) expressed as $x^{(k+1)}$= (I-Q^{-1}A) $x^{(k)}$+ Q^{-1}b, k ≥0, that means, this is the iterates. And what we need actual solution is actual solution that means as k tends to infinity. So, we get x= (I-$Q^{-1}A)x + Q^{-1}$b. greater than, that get cancelled.

So, if we take the norm, this we define as the $||x|| \rightarrow$ norm of x, this can be also defined as x= maximum of 1 ≤ i ≤ n $|x_i|$, maximum value of this. these are x is a vector in this case, $|| x^{(x+1)}$-x|| ≤ || I-Q^{-1}A ||.|| $x^{(K)}$-x|| if we repeat the steps during this process what we get is ≤…. ≤ || I $-$ Q^{-1}A|| ^{k}. || $x^{(0)}$-x || .

So, in that process if we repeat several times, so this is into x^0-x. Now x^0-x is finite

If $||I - Q^{-1}A||$ <1, then as k $\rightarrow \infty$ || I $-$ Q^{-1}A|| $^{k}\rightarrow$0 in that case limit k $\rightarrow \infty||x^{(k+1)}$-x ||=0, for any choice of x^0 whatever your initial guess is you get a converged solution. So, if we can have a situation where Q is a non-singular matrix is choose in such a way that these s k tends to infinity so this is given by this norm is less than 1.





Now, one of the iterative method is Jacobi's method, Jacobi method where Q is taken to a the diagonal matrix, Q = diag $(a_{11}, a_{22,} a_{nn})$ Q is chooses the diagonal elements of A. In that case

$||I$-Q^{-1} A$||$= Max 1≤i≤n $\sum_{j=1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right|$, j $\neq$ i this is of course, less than 1 if it is diagonally dominant. So, we can put a theory that if, theorem is if A is diagonally dominant then the sequence of iterates in Jacobi's method converge to the solution $Ax = b$ for any arbitrary choice of the starting vector.

Now, this is the Jacobi iterative procedure. So, one of the is a linear iterative procedure. So, another modification of this is the Gauss-Seidel iterative procedure, so iterative method. So, in that case what we do in the that Q is the lower triangular part of A including the diagonal elements of A. if we choose the matrix A in this manner in this case Q $x^{(k+1)}$= (Q-A) $x^{(k)}$+b. For example, Q is lower triangular part including the diagonal elements and you have a matrix system of equation say

$$\begin{pmatrix} 2 & -1 & 4 \\ 1 & 6 & -2 \\ 4 & -3 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} \to \begin{pmatrix} 1 & -1/2 & 0 \\ 1/6 & 1 & -1/3 \\ 1/2 & -3/8 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1 \\ -2/3 \\ 5/8 \end{pmatrix}$$

$$Q=\begin{pmatrix} 1 & 0 & 0 \\ 1/6 & 1 & 0 \\ 1/2 & -3/8 & 1 \end{pmatrix}\begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{pmatrix}=\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix}+\begin{pmatrix} 1 \\ -2/3 \\ 5/8 \end{pmatrix}$$

this is the way iterative goes for the Gauss-Seidel method. That is, it. We will go to the next class now.