Advanced Computational Techniques
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Lecture 07
Numerical Quadrature (Contd.)

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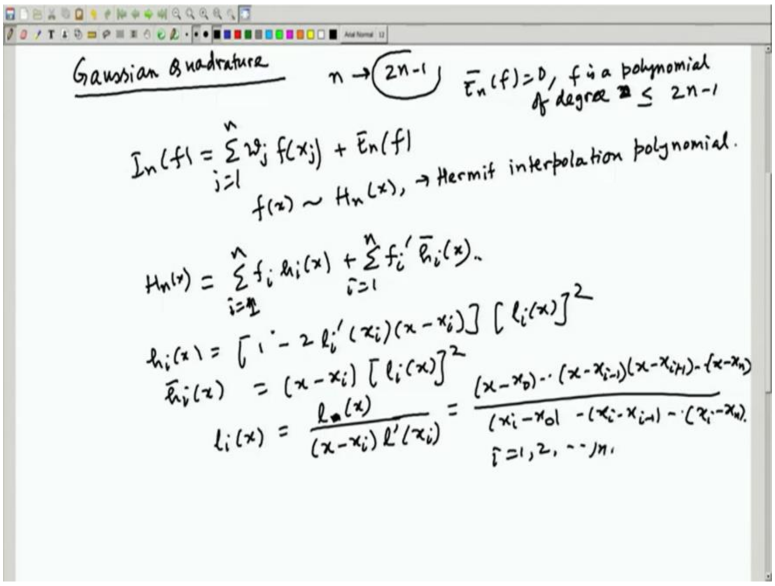

Okay, so, the continuation, we will talk about Gaussian quadrature formula. So, this is a higher order formula of large degree of precision. So, what we said in the previous class that you have n grid points or n node points and from there, we want to find out a quadrature formula of degree of precision 2n-1. So, recall that the degree of precision means the error E_n(f) is 0 when f is a polynomial of degree less than equal to 2n-1.

So, basically what we are looking for a polynomial integration formula like this

$$I_n(f) = \sum_{j=1}^{n} w_j f(x_j) + E_n(f)$$

using n node points so for that what we do is we approximate these function f(x) by a polynomial what is called the Hermite interpolation polynomial.

So, because this Hermite interpolation polynomial it uses n points and becomes a degree of 2n -1 we are defining as

$$H_n(x) = \sum_{i=1}^{n} f_i h_i(x) + \sum_{i=1}^{n} f_i' \, \overline{h}_i(x)$$

This is the formula for the harmonic interpolation polynomial where

$$h_i(x) = [1 - 2l_i'(x_i)(x - x_i)][l_i(x)]^2$$

$\overline{h_\iota(x)} = (x - x_i)[l_i(x)]^2$

$l_i(x) = \frac{l(x)}{(x-x_i)\, l'(x_i)} = \frac{(x-x_0)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)\ldots(x_i-x_{i-1})\ldots(x_i-x_n)}$ i=1,2,…,n

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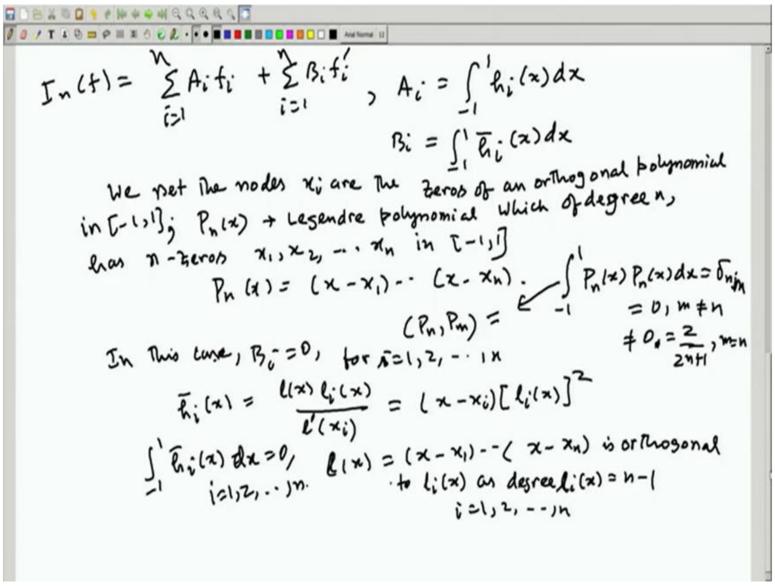

So, if I apply this formula then what will happen is this that $I_n(f)$ can be written as

$$I_n(f) = \sum_{i=1}^{n} A_i f_i + \sum_{i=1}^{n} B_i f_i'$$

where

$A_i = \int_{-1}^{1} h_i(x)dx$

and

$B_i = \int_{-1}^{1} \overline{h_\iota(x)}\, dx$

So, with a degree of precision 2n-1. Now, we set the nodes x_i are the zeroes of an orthogonal polynomial in [-1, 1].

So, one of the orthogonal polynomial is the Legendre polynomial. So, $P_n(x)$ is the Legendre polynomial which is of degree n and has as n zeros $x_1, x_2, \dots, x_n$ in [-1, 1]. So, that means this $P_n(x)$ can be expressed as

$P_n(x) = (x - x_1) \dots (x - x_n)$

So, to make it a little correction that this correction to be noted that the node points are from 1 to n instead of zero. So, orthogonality means

$$(P_n, P_m) = \int_{-1}^{1} P_m(x) P_n(x) dx = 0, \mathrm{m} \neq n \neq 0$$

$$= \frac{2}{2\mathrm{n}+1}, m = n$$

So, in that case this kind is also referred to as the inner product. Now, we choose the zeros as the node points as zeros and in that process B_i becomes 0 for i = 1,…,n.

Because see this

$$\overline{h_i(x)} = \frac{l(x) l_i(x)}{l'(x_i)} = (x - x_i)[l_i(x)]^2$$

We have defined before. So, now, if we take the integration

$\int_{-1}^{1} \overline{h_i(x)}\, dx = 0$

$$l(x) = (x - x_i) \dots . (x - x_n)$$

$$l_i(x) \; as \; degree \; l_i(x) = n - 1$$

So, what we find that if I choose the node points as x_1 , x_2,…, x_n the zeros of the Legendre polynomial which are orthogonal, so B_i becomes zero.

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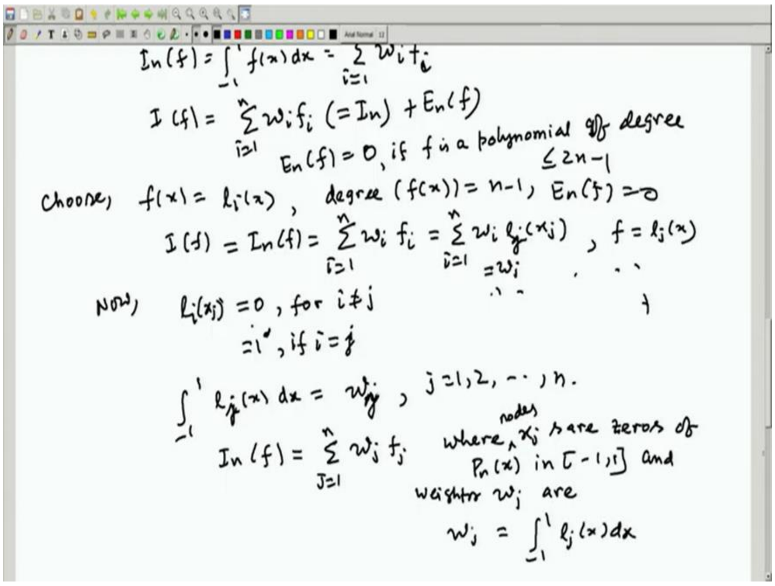

So, in that case what we find the formula becomes. So, we get the formula as

$$I_n(f) = \int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f_i$$

Another thing is error

$$I(f) = \sum_{i=1}^{n} w_i f_i (= I_n) + E_n(f)$$

f is a polynomial of degree less than equal to 2n-1. If this happens, then this error is zero altogether.

So, that means this becomes an exact integration formula. If f is a polynomial of degree 2n-1 that is the degree of precision. Now, if weights and nodes are chosen in such a way that the formula is exact when it is a polynomial of degree 2n-1. Now, if we choose

$$f(x) = l_i(x)$$

So, then what happened to that case? In that case $degree\ (f(x)) = n - 1$. So, that means

$$E_n(f) = 0$$

So,

$$I(f) = I_n(f) = \sum_{i=1}^{n} w_i f_i = \sum_{i=1}^{n} w_i l_{j\,(x_j)}, f = l_j(x)$$

Now, we know that

$$l_j(x_j) = 0, for\ i \neq j$$

=1, if i=j

So, with that definition what we find that

$$\int_{-1}^{1} l_j(x)\, dx = w_j, j = 1,2, \dots, n$$

if I choose f equal to $l_j(x)$. So, that means we get a formula for the weight.

So, in that way we obtain a formula given by this way. Now, so, that means, we have this

$I_n(f) = \sum_{j=1}^{n} w_j f_j$

These are called the nodes and weights are w_j

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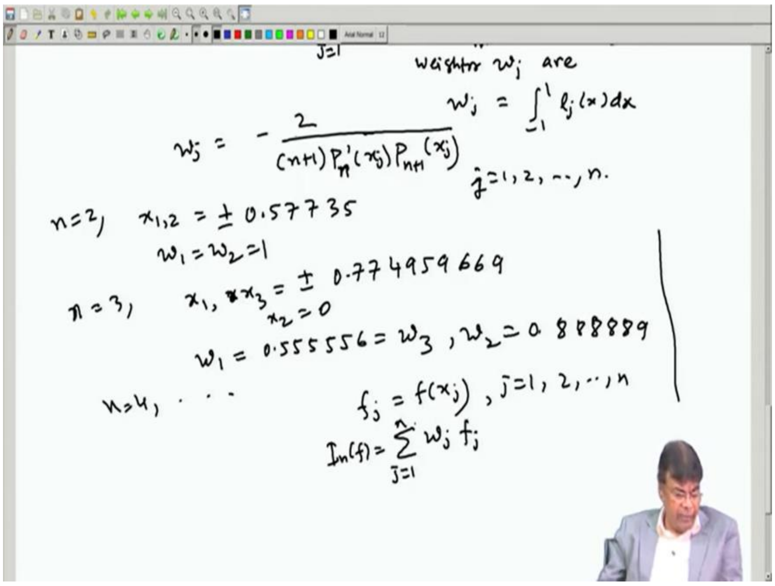

Now, if we consider a node that zeros of $p_n(x)$ and w_j is given by this way so this w_j can be written as

$$w_j = -\frac{2}{(n+1)P_n'(x_j)P_{n+1}\,(x_j)},$$

J=1,2,…,n

a little bit of algebra and all and orthogonality property of Legendre polynomial. So, this is the weight function. So, this is the way one can derive the Gauss Legendre quadrature formula and here the orthogonality property of the function has to be taken into account that is within -1 to 1 the function is considered to be orthogonal.

So, zeros of those polynomial which are orthogonal in -1 to 1. So, now, if we do little calculation, what we can do is, if we choose n =2 which we have derived already, so, in that case the weights are

$$x_{1,2} = \pm 0.57735$$

$$w_1 = w_2 = 1$$

These are the weights. So similarly, if we choose n =3, so

$$x_1, x_3 = \pm 0.774959669$$

$$x_2 = 0$$

These are the zeros of the Legendre polynomial.

$$w_1 = 0.555556 = w_3\ , w_2 = 0.888889$$

Similarly, n=4 one can find out. So, once we have that then what we have to do is, we have to find out $f_j = f(x_j), j = 1,2,..,n$

$$I_n(f) = \sum_{j=1}^{n} w_j f_j \cong \int_{-1}^{1} f(x)dx$$

So, this is the Gauss Legendre integration formula. Now degree of precision as we said is 2n-1 much better than polynomial interpolation formula like Simpson’s one third or Gauss integration formula and all these things.

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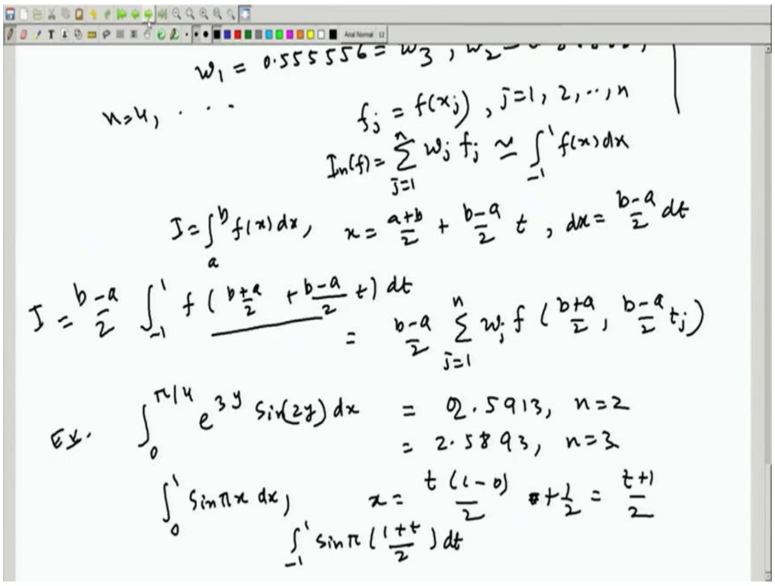

Now, one thing is , for example, in many cases we may not be having the limits of integration from -1 to 1. In that case, if we have a case

$\mathrm{I} = \int_a^b f(x)dx$

in that case we can substitute $x = \frac{a+b}{2} + \frac{b-a}{2}\ t, dx = \frac{b-a}{2} dt$

then

I=$\frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b+a}{2}+\frac{b-a}{2}\ t\right) dt$

if we are using the Gauss Legendre polynomial,

I=$\frac{b-a}{2}\sum_{j=1}^{n} w_i f(\frac{b+a}{2},\frac{b-a}{2}\ t_j)$ where t_j's are the zeros of Legendre polynomials in [-1, 1].

So, for example,

$\int_0^{\frac{\pi}{4}} e^{3y}\ sin\ sin\ (2y)dx = 2.5913, n = 2; = 2.5893$, n = 3

that is using two node points and two weights. So, that is how the gauss Legendre polynomial goes.

Similarly, say another example,

$$\int_0^1 sin\pi x\ dx$$

Substitute

$x = t\frac{(1-0)}{2}+\frac{1}{2}=\frac{t+1}{2},$

we get

$\int_{-1}^{1} sin\pi(\frac{1+t}{2})\ dt$

So, we have to reduce the integration limit to -1 to 1and then we have to use the Gauss Legendre polynomials in the function.

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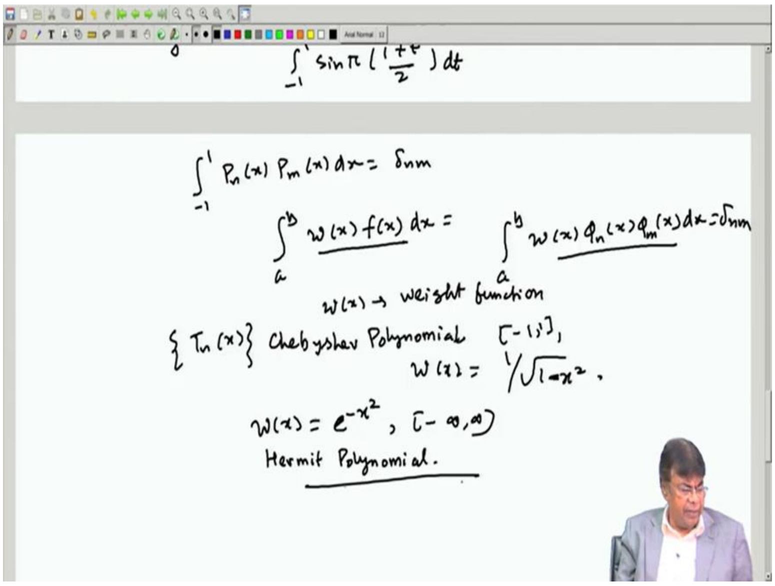

So, now, one thing is that here we have taken the orthogonality

$\int_{-1}^{1} P_n(x) P_m(x) dx = \delta_{nm}$

Now, if we have

$$\int_a^b w(x) f(x) dx = \int_a^b w(x) \emptyset_n(x) \emptyset_m(x) dx = \delta_{mn}$$

As always we cannot have a choice that the integration is between -1 to 1 so then only we can apply the Legendre polynomial orthogonality, maybe we can have in some cases that it is some other weight function w(x).

So, in that case we have to look for a polynomial. So, a polynomial in such a way that this is becoming orthogonal with respect to the weight function w(x). So, for example, say Chebyshev polynomials, so, this is $\{T_n(x)\}$is called the Chebyshev polynomial which is orthogonal in -1 to 1 with respect to the weight function w(x)= $\frac{1}{\sqrt{1-x^2}}$.

So, we have to convert in this way with respect to this weight function . Similarly, the Hermite polynomials. So, Hermite polynomials weight function is W(x)=e^{-x^2} where integration is $[-\infty, \infty]$.

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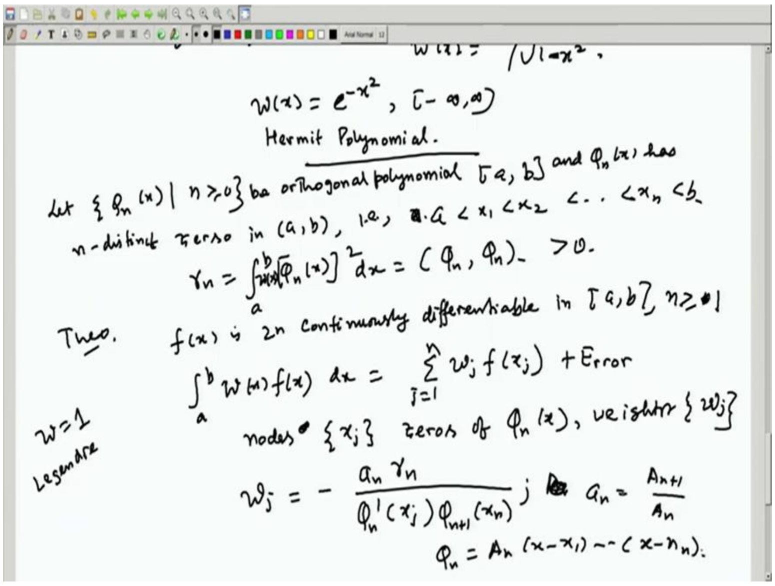

So, in general we can state a theorem for the general Gauss quadrature. So, let $\{n \geq 0\}$ be orthogonal polynomial, polynomial in [a,b] and $\emptyset_n(x)$ has n distinct zeroes in [a, b]

i.e. a $< x_1 < x_2 < \cdots < x_n < b$

$$Y_n = \int_a^b [\emptyset_n(x)]^2 dx = (\emptyset_n, \emptyset_n) > 0$$

that is the inner product between these two which is greater than zero then we can put a theorem that f(x) is 2n continuously differentiable in [a,b]

Differentiable in [a, b] say some interval for n ≥ 1 then we can write

$$\int_a^b w(x) f(x) dx = \sum_{j=1}^{n} w_j f(x_j) + error$$

in the Gauss Legendre polynomial we have w =1 in Legendre formula.

Weights w_j are given by

$w_j = -\frac{a_n y_n}{\emptyset_n'(x_j)\emptyset_{n+1}(x_n)}$

where

$a_n = \frac{A_{n+1}}{A_n}$

$$\emptyset_n = A_n(x - x_1) \ldots (x - x_n)$$

So, this is the theorem one can express in general that means in this case so obviously when w =1, this is the Gauss Legendre formula which we have derived. So accordingly and of course, a , b is transformed to a=-1, b =1. So, in general case we can have a formula like this. Okay, so, this is about the Gaussian quadrature formula which are much accurate compared to other formulas. Thank you.