Advanced Computational Techniques
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Lecture 06
Numerical Quadrature

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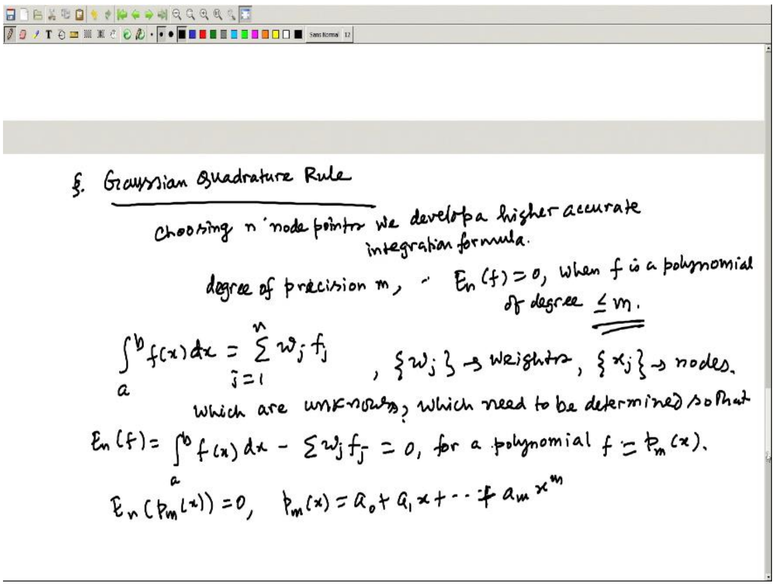

Now, we have the previous one we have discussed about some formula Simpson one third rule and trapezoid formula and now our intention is that the number of node points will choose the same, but the accuracy we would like to have more as high as possible. So, one of this formula is the Gauss quadrature formula. So, there is class of formula.

So, this is called the quadrature rule. Now, our intention is that choosing in node points we develop a higher accurate integration formula. The process whatever we have discussed in the previous one that is effects, we are representing by polynomial. So, if we have endpoints, we can construct a polynomial of degree $n-1$ and in that process what is referred as the degree of precision.

Degree of precision means the numerical integration is exact for a polynomial, if I call the degree of precision as m, so that means all the $E_n(f)$ equal to exact or rather I would say the

$$E_n(f) = 0$$

when f is a polynomial of degree ≤ m. So, degree of precision we define in this way that I am representing or writing a formula and such that these the error are vanished when f is a polynomial of degree ≤ m.

For example, the trapezoidal formula if I consider a polynomial function as a first-degree polynomial, a linear polynomial so the trapezoidal formula is exact. Similarly, the Simpson's one third rule is exact for a polynomial of degree 1 or degree 2 at most. So, similarly, there are other formula like Newton Cotes formula which you can refer to any book. So, those can be increased the degree of precision as we increase the number of node points.

Now, increasing the number of node points means more number of calculations. So, that is another bottleneck creates. So, there is another advantage of Gaussian quadrature is if you have a similarity. So, what similarity means if the function has a similarity at a certain point within the interval [a,b], then those representation is not possible or by the formula whatever we have derived.

So, because of these two bottlenecks, we now look for a formula say

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} w_j\, f_j$$

Now, w_j are called the weights and x_j as we define is here the nodes, which are unknown.

So, suppose I want a formula with $n = 1$ which need to be determined so that

$$E_n(f) = \int_a^b f(x)\ dx - \sum w_i f_j = 0$$

for a polynomial f=$p_m(x)$

$$E_n(p_m(x))=0,\ p_m(x) = a_0 + a_1 x + \cdots + a_m x^m$$

$$E_n(a_0 + a_1 x + \cdots + a_m x^m) = 0$$

So, in that case we call that formula is degree of precision m.

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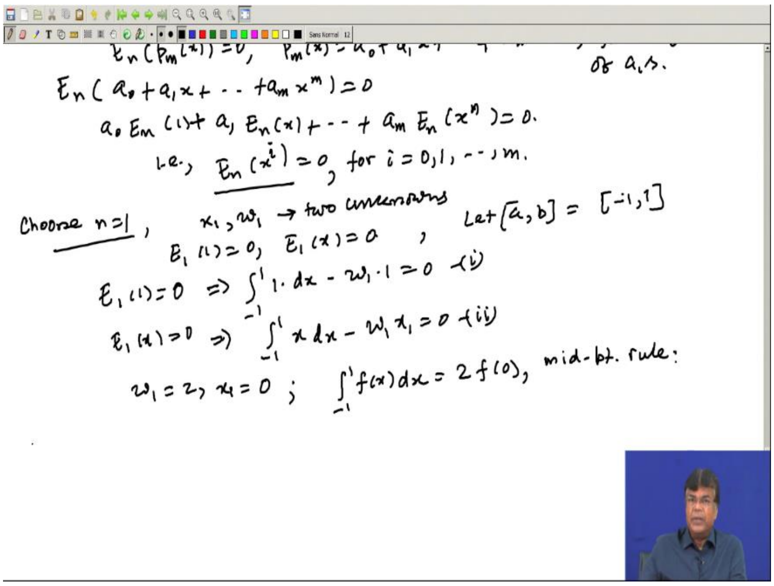

So, that means, now this is a by definition this is the linear form. So, what we can write is

$$a_0 E_n(1) + a_1 E_n(x) + \cdots + a_m E_n(x^n) = 0$$

$$E_n(x^i) = 0, for\ i = 0,1, \ldots., m$$

So, this is our requirement.

So, we need to find out to an unknown because w_j and f_x and x_j to an unknown to be obtained and in such a manner that this happens. So now we choose, suppose I would like to derive a formula like this. So, this is our job in hand that we are writing the formula as

$$I_n(f) \equiv \int_a^b f(x) = \sum_{j=1}^{n} w_j f_j$$

Now, let us choose $n = 1$. So, that means you have one node, one weight function. So, the relations we have so these are two unknowns. So, these relations what we have is

$E_1(1)$=0, $E_1(x)$=0

So, this is the two conditions we can impose a $E_1(1)$=0 and $E_1(x)$=0

Now what is $E_1(1)$?

$E_1(1)$=0 $\int_{-1}^{1} 1\, dx - w_1 1 = 0$ -(i)

$E_1(x)$=0$\int_{-1}^{1} x\, dx - w_1 x_i = 0$ -(ii)

So, if I now solve this two, so what I get is $w_1 = z$ and $x_1 = 0$. x1 equal to 0. So, in that case, we get the formula

$$\int_{-1}^{1} f(x)\,dx = 2f(0)$$

What I find that degree of precision is in this case is 1. So, any polynomial of degree 0 and 1 are exact. So, this is the way one formula we have derived for a simple case.

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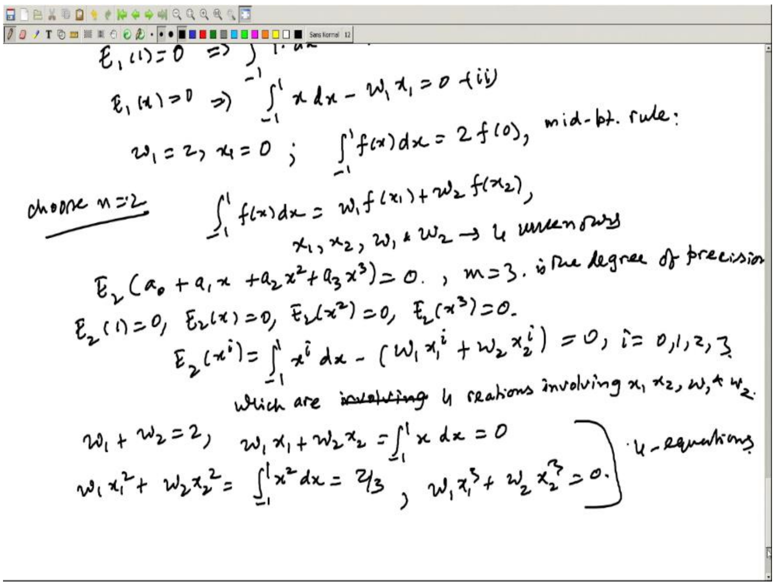

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2),$$

Similarly, choose another say let us go for $n = 2$ we are into the stick now, finding the simplicity that $n = 1$ this is the way we have derived.

So, we have now 4 unknown w we have the unknowns now as x1, x2, w1 and w2 they are the four unknowns. Now, what are the equations we have?

$$x_1, x_2, w_1, w_2$$

$$E_2(a_0 + a_1 x + a_2 x^2\ a_3 x^3) = 0,\ m=3$$

because we have to have four conditions to be prescribed. So, that means we are putting m =3 this is the degree of precision because this is in our hands.

$$E_2(1) = 0, E_2(x) = 0, E_2(x^2) = 0,, E_2(x^3) = 0$$

So, depending on the number of unknowns we need to find out we put the degree of precision as this manner. A formula to obtain like this way

$$E_2(x^i) = \int_{-1}^{1} x^i \, dx - w_1 x_1 i + w_2 x_2 i \,) = 0 \,, i = 0,1,2,3$$

So, this gives you four relations involving but they are non-linear. These need not be a linear relation which are involving four relations four unknowns,

$$x_1, x_2, w_1, w_2$$

So, if I go in the same manner, I get the set of equations as

$$w_1 + w_2 = 2, w_1 x_1 + w_2 x_2 = \int_{-1}^{1} x \, dx = 0$$

then one can find out in the same way

$$w_1 {x_1}^2 + w_2 {x_2}^2 = \int_{-1}^{1} x^2 \, dx = 2/3 \,,$$

$$w_1 {x_1}^3 + w_2 {x_2}^3 = 0$$

So, all these four equations need to be solved, so this is four equations but obviously they are nonlinear and this needs to be solved to determine this, w_1, w_2 and all these things.

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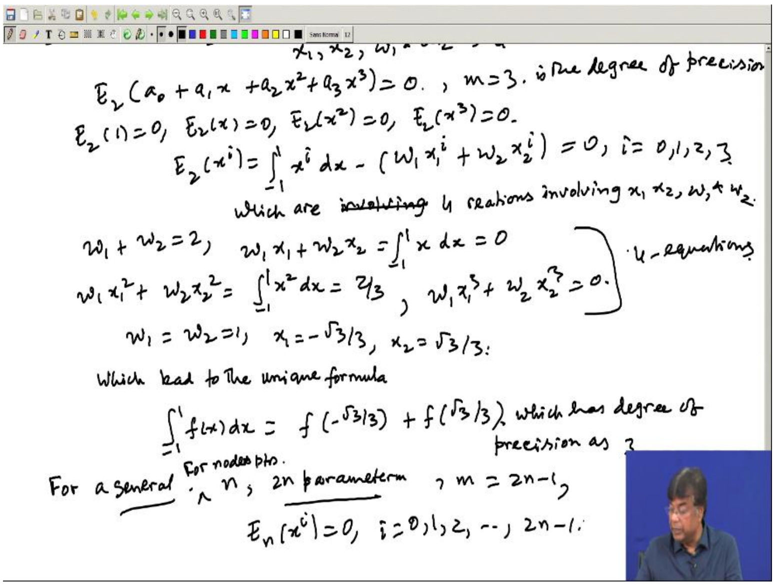

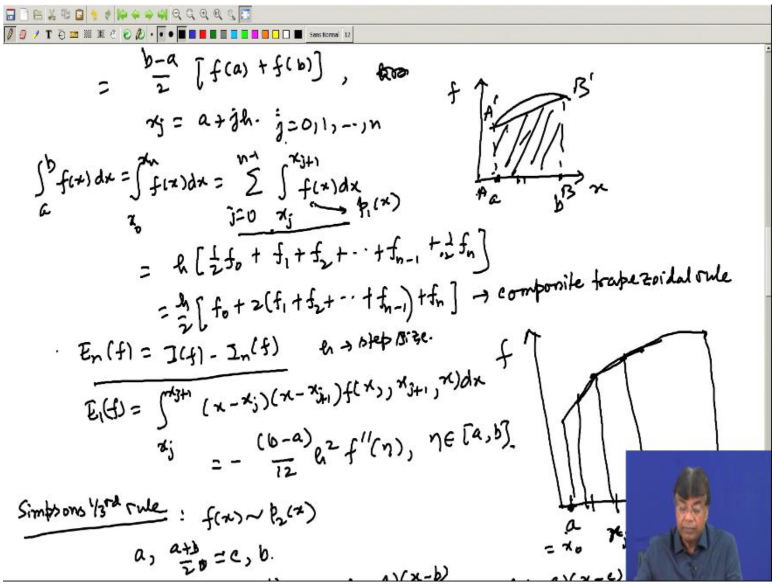

Now, if we solve this what we find that

$$w_1 = w_2 = 1\,, x_1 = -\frac{\sqrt{3}}{3}, x_2 = \frac{\sqrt{3}}{3}$$

So that process which lead to the unique formula

$$\int_{-1}^{1} f(x)\,dx = f(-\sqrt{3}/3) + f(\sqrt{3}/3)$$

So, this degree of precision is 3 which has **degree of precision** is 3. So, this is how one can proceed. So, that means, for the general case if we now go on like this way so, what we find in general if we choose the node points as n and there are 2n parameters.

So, if we have 2n parameters so we need to have a degree of precision, so n need to be 2n-1. So, that case node points n and 2n parameters involved x and w weight function and the weight function at the node points and the degree of precision has to be 2n-1 so that we will have 2n equations.

So, that means 2n number of equations need to be solved. So, that means

m=2n-1

$$E_n(x^i) = 0\,, i = 0,1,2,\ldots,2n-1$$

so, this need to be solved which is not a very simple job which will lead to a set of nonlinear equations for the nodes $\{x_i\}$ and weights $\{w_i\}$.

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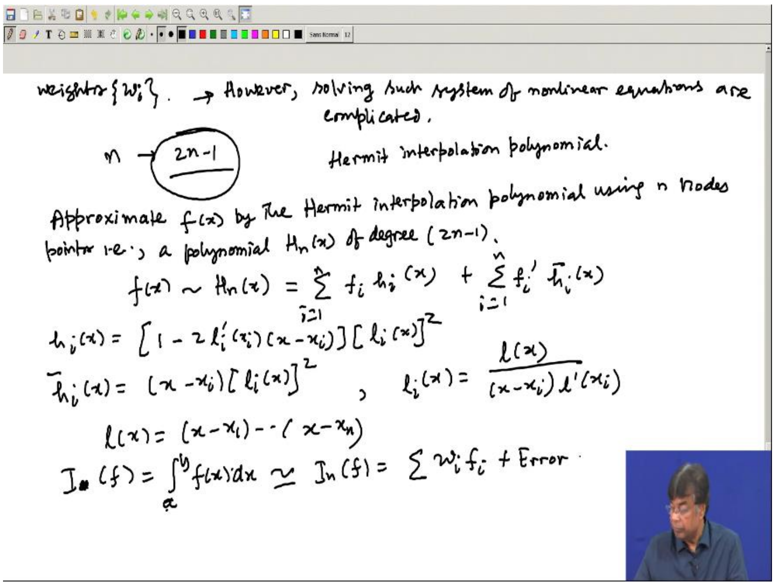

Now, one can try but solving such system of nonlinear equations are complicated. So, this is not a proper way to go. So, what we do is now, this gauss quadrature can be derived by what we will do is we make use of the Hermite's interpolation polynomials. So, Hermite interpolation polynomials what it was doing that if you have n node points you can construct a polynomial of degree

$$n \rightarrow 2n-1$$

So, that means a degree of precision of $n \rightarrow 2n-1$ can be obtained, if we have a Hermite interpolation polynomial if we use that means instead of representing function by a simple Lagrange or Newton's formula. So, if we use the Hermite interpolation polynomial so that is the idea we will add up here.

So, approximate $f(x)$ by the Hermite interpolation polynomial using n node points that is a polynomial of $H_n(x)$ of degree $2n-1$. So, that means

$$f(x) \sim H_n(x) = \sum_{i=1}^{n} f_i h_j(x) + \sum_{i=1}^{n} f_i{}' \overline{h_j}(x)$$

So, how it looks $H_n(x)$? So now, as you remember that Hermite interpolation polynomial it has not only become equal to the function and the function derivative at those node points.

$$f(x) \sim H_n(x) = \sum_{i=1}^{n} f_i h_j(x) + \sum_{i=1}^{n} f_i{}' \overline{h_j}(x)$$

$$h_i(x) = [1 - 2l'_i(x_i)(x - x_i)][l_i(x)]^2$$

$$\overline{h_i(x) =} (x - x_i)[l_i(x)]^2, \qquad l_i(x) = \frac{l(x)}{(x - x_i)l'(x_i)}$$

$$l(x) = (x - x_i) \dots (x - x_n)$$

So, in other words this is nothing but the factor $(x - x_i)$ is missing. So, what we find now

$$I(f) = \int_a^b f(x)dx \cong I_n(f) = \sum w_i f_i + Error$$

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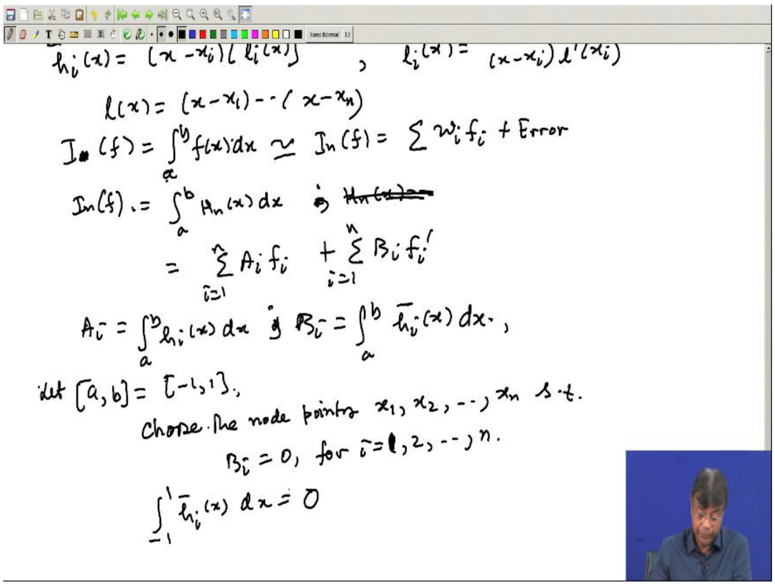

So, now if we substitute

$$I_n(f) = \int_a^b H_n(x)dx$$

$$= \sum_{i=1}^n A_i f_i + \sum_{i=1}^n B_i f'_i$$

$$A_i = \int_a^b h_i(x)dx, \quad \boldsymbol{B}_i = \int_a^b \overline{h_i}(x)dx$$

This is how and degree of precision is $2n - 1$. So, this is the formula $I_n(f)$. Now thing is that how to obtain that again. So, to obtain the node points, so now first we considered

Let $$[a, b] = [-1,1]$$

So, and in this case what we find that a, b is given by this way and choose the node points

$$x_1, x_2, \dots, x_n$$

$$\boldsymbol{B}_i = 0, for\ i = 1,2, \dots, n$$

$$\int_{-1}^{1} \overline{h}_i(x) dx = 0\ , i = 1,2, \dots, n$$

So, this is the choice we have to make.

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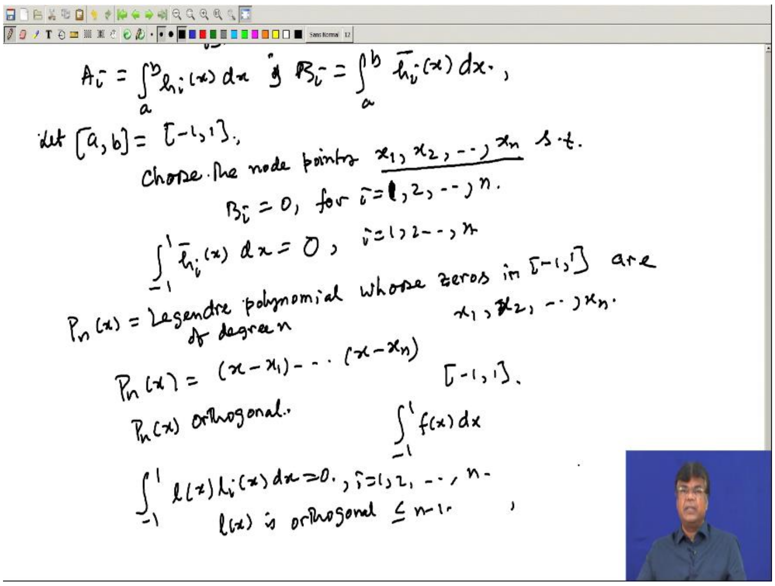

$$P_n(x) = (x - x_i) \dots (x - x_n)$$

$P_n(x)$ is called the Legendre polynomial whose zeroes in [-1,1].

Legendre polynomial of degree n with zeros and what we know that this $P_n(x)$ are orthogonal.

So, this is the thing we will exploit and derive the Gauss Legendre polynomial when we consider integration between [-1,1] so, that means, here we are not finding the solving a set of equations

$$\int_{-1}^{1} f(x)\, dx$$

Instead, what we are finding these node points are prescribed from the given orthogonal polynomial, zeros of the orthogonal polynomial.

So, if we consider

$$\int_{-1}^{1} l(x)\, l_i(x) dx = 0\ , i = 1,2, \dots , n$$

because this $l(x)$ is orthogonal, degree $l_i(x) \leq n - 1$. I think we have to continue, this so maybe in the next class we will finish this one. Okay, thank you.