Advanced Computational Techniques
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Lecture 05
Spline Interpolation (Contd.)

So, we started with the cubic spline last class, we have discussed over the cubic spline, how to determine the cubic spline in a form of a piece wise polynomial set. Now, the derivations we have discussed in previous class, so what it consists of a set of tridiagonal system, which provides the second order derivatives that we define as m, so those Mk's are need to be determined by solving a set of linear algebraic equation which forms a tridiagonal system. And once we did that, then in each sub intervals one can determine the cubic polynomial which represents the cubic spline at in totality.

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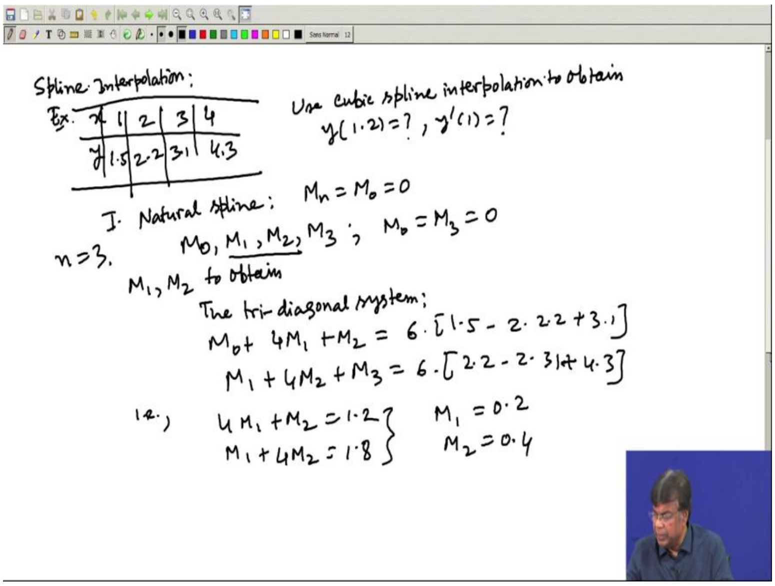

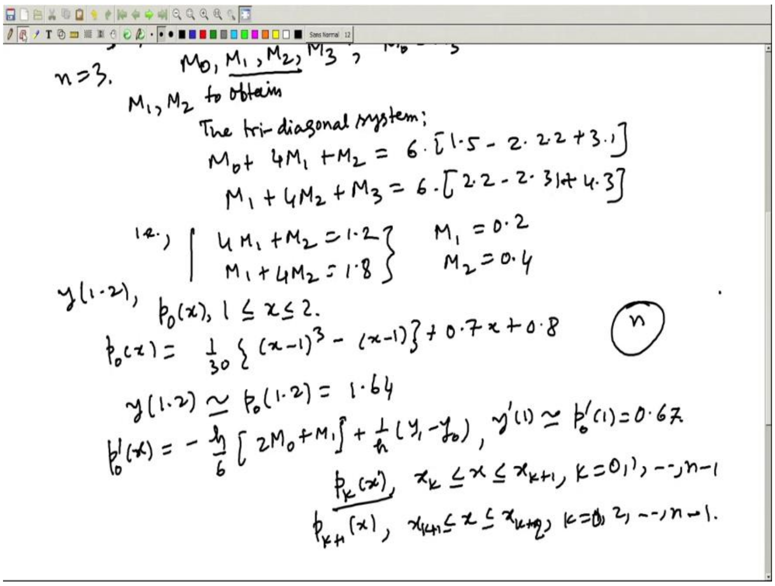

Show let us consider this spline interpolation example, an example of this, so let we have a set of data points as given by x, y say,

x	1	2	3	4
y	1.5	2.2	3.1	4.3

so we have a set of data points given by this way. So, what we have to do is determine use cubic spline interpolation to obtain

$$y(1.2) = ?.\, y'(1) = ?$$

this is the two values need to be considered.

So, first we consider a natural spline. So, natural spline as we have defined in this case

$$M_n = M_0 = 0$$

so, that means that the two end the curvature is 0, so here is n = 3, so we get a tridiagonal system of order 2, 2/2, so that means, we need to find out M0 is given, so $M_0, M_1, M_2, M_3; M_0 = M_3 = 0$ at the two end is given to be 0, so you need to find out

$$M_1, M_2$$

to obtain. Now, this M_1, M_2 are satisfying this following equation the tridiagonal system as we have derived in last class is given by.

It becomes

$$M_0 + 4M_1 + M_2 = 6[1.5 - 2.2.2 + 3.1],$$

if we do the calculation and the other one is

$$M_1 + 4M_2 + M_3 = 6[2.2. -2.3 + 4.3]$$

So, from there we get that is we have two system as

$$4M_1 + M_2 = 1.2 \quad M_1 = 0.2$$

And $M_1 + 4M_2 = 1.8 \quad M_2 = 0.4$

So, the cubic spline is determined by this manner and M1= 2, so that means, now we need to find out y(1.2) so, that means we need to find out the $p_0(x)$.

So, which is defined between

$$p_0(x) = 1 \le x \le 2$$

$$p(x) = \frac{1}{30}\{((x-1)^3) - (x-1)\} + 0.7x + 0.8$$

$$y(1.2) \cong p_0(1.2) = 1.64$$

$$p_0^{\prime(x)} = \frac{-h}{6}[2M_0 + M_1] + \frac{1}{h}(y_1 - y_0)$$

$$y'(1) \cong p_0'(1) = 0.67$$

So, this is how the spline interpolation formula goes so, one thing to note that when you have a large system then these tri-diagonal system or large number of n, so these data points if we are using a huge number of data points in that case so this, what we find that will have the number of n value of n is high in that case the tri-diagonal system become a large number of systems. So, in that case, we cannot just solve by gauss elimination or it is called the elimination technique.

So, there are some algorithm, that we will discuss later on. So, those algorithms need to be imposed to so, to obtain that and once we have done then we find out the

$$p_k(x), \; x_k \le x \le x_{k+1}, k = 0,1,\dots,n-1$$

So, whichever way that notation so, it can be also we can say this

$$p_{k+1}(x), \; x_k \le x \le x_{k+1}, k = 1,2,\dots,n$$

So in that case k will be varying from 0 to n. So, that means, this is the way we have already defined, so if we say this then it will be

$$p_{k+1}(x)$$

So, this is about the spline interpolation polynomial.

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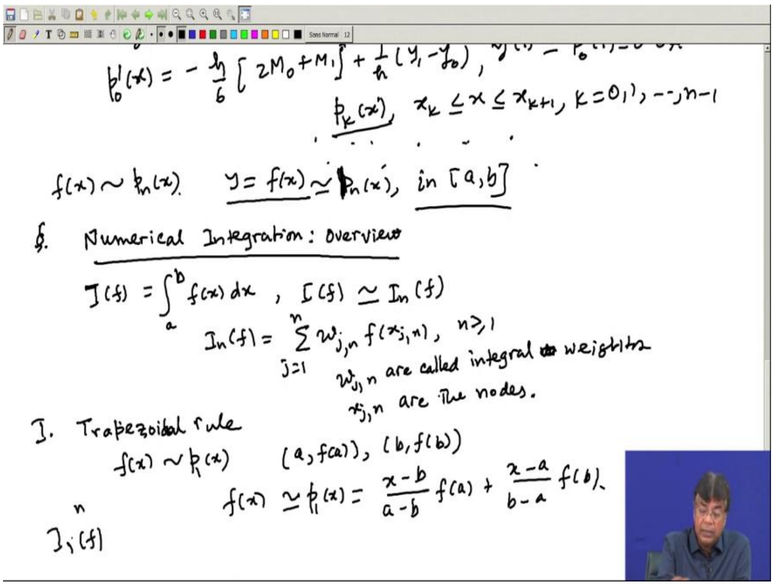

Now, one of the importance or one of the application for interpolation representation of a function by interpolation polynomial, or in a form of a smooth function f(x) when you represent by a polynomial is one of the reason now, there can be the function y equal to f(x) is either as we said either not known the function form or the function is so complicated to handle that one or maybe we may have a set of data points instead of the function form. So, in either of these cases, what we are doing is we are representing this function by

$$y = f(x) \sim p_k(x), \ \ in \ [a, b]$$

One of the application of this interpolation polynomial is the numerical integration.

So, this is our next topic is numerical integration. So, again, we will go for a little bit of overview, so not in all details, because in many elementary calculus and all numerical methods, these numerical integrations are described in details, so we will basically talk about the foundation of the numerical integration technique and talk about the higher order approximation for the numerical integration.

So, suppose you need to find out

$$I(f) = \int_a^b f(x)dx, \ I(f) \cong I_n(f)$$

$$I_n(f) = \sum_{j=1}^{n} w_{j,n} f(x_{j,n}), \ \ n \geq 1$$

So, $w_{j,n}$ are called the weights, integral weights and these $x_{j,n}$ are the node points. So, the formula whatever we want to establish are based on the consideration of the nodes $x_{j,n}$ number of nodes and weights. So, n can be the number of that gives the j= 1 to n, the number of nodes should be greater than equal to 1. So, that means n = 0 is of no meaning so because this summation and we are starting from j=1

Now, this several formulas are derived one the most-simple formula trapezoidal rule. These we represent f(x)~ $p_1(x)$ that is a linear form.

So, that means, what we do is say

$$f(x) \sim p_1(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$$

So, once we have represented this function given by this way, then what we do is we integrate this polynomial, So, that means I 1f in this case n equal to 1.

$$I_1(f) = \int_a^b p_1(x)\, dx = \frac{b-a}{2} [f(a) + f(b)]$$

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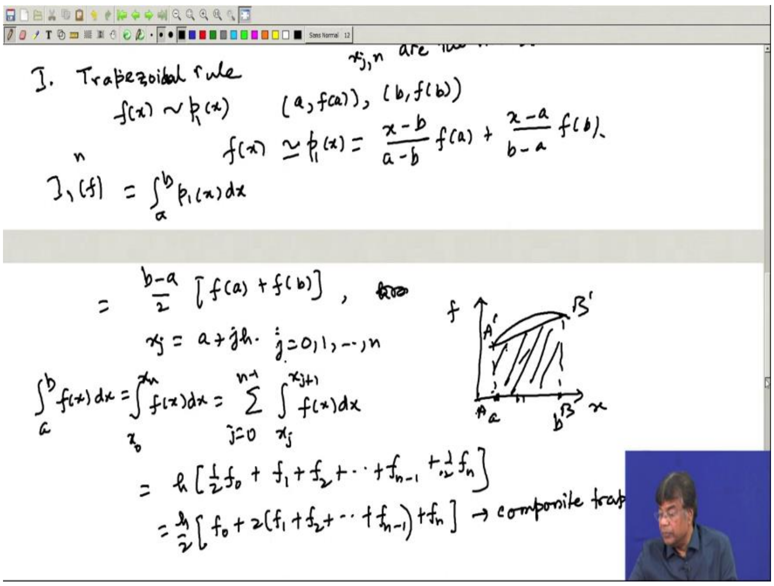

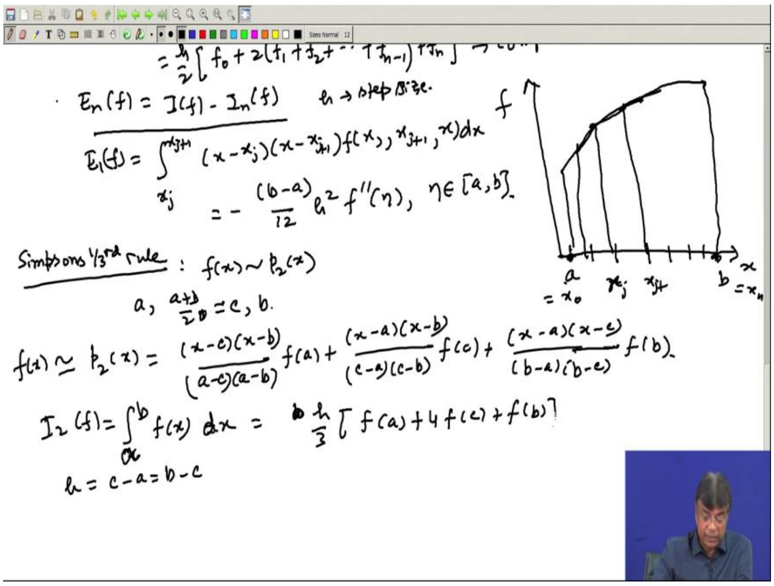

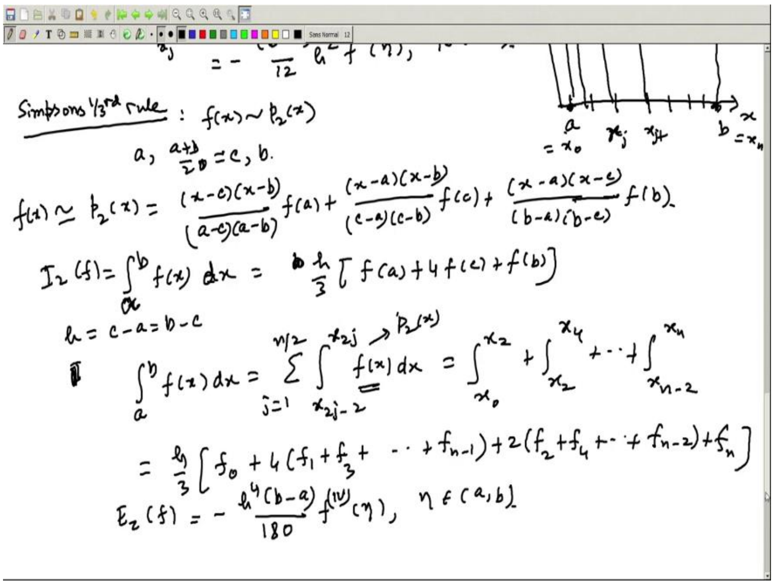

So, what we get is any I 1f, we can write as.

$$I_1(\mathrm{f}) = \int_a^b p_1\,(\mathrm{x})\,dx = \frac{b-a}{2}\,[f(a) + f(b)]$$

So, we call this h or the differences b-a. So, this is a trapezoid representation. So, that means, this integration means finding this area enclosed by this, zones instead what we did is we are approximating this integration by this trapezoidal rule.

So, by this trapezoid a b and say let us call this a' b' and this is a b so, that means, the trapezoid a' b' sure this area of this trapezoid, is given by this way. Now, as we could see that this approximation will be better if we take more number of points. So, if I define the same way if

We define,

$$x_j = a + jh \ , j = 0,1, \dots, n$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx = \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} f(x)dx$$
$$= h\left[\frac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2}f_n\right]$$

So, the rather way you should write it this way is,

$$= \frac{h}{2}[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n]$$

So, this is called the composite trapezoidal.

Now, composite trapezoidal rule so, if you choose more number of points so, n becomes larger. So, that means your approximations are becoming much better show that means, in other words.

So, what they did is this is x and this is f, so instead of taking a single polynomial $p_1(x)$ so, this is a b or rather this is equal to x_0 and this is equal to x_n so you are choosing discrete number of points say x_j and x_{j+1} so, that means, these two points and then we are approximating by a straight line, similarly, this point we are approximating by a straight line.

So, that means, this gives a much better approximation for the area which is enclosed by the region or which the required region are much better approximation if we do by this manner. Should, now this error function. So, error is we define from this

$$E_n(f) = I(f) - I_n(f)$$

so, obviously this what is $I(f)$ we have no idea so, only thing is that, we need to know the $E_n(f)$, order of h what should be the order of h in order which order of h that is the maybe we can call this as h the steps size, so the order of the steps size is very important to know to to have the precision of the polynomial for the numerical integration.

So, in this case $E_n(f)$ here it is $E_1(f)$, n equal to 1, in general case, so every sub-interval otherwise we are integrating, see, this division is exact. So, when you are replacing this

$$E_1(f) = \int_{x_j}^{x_{j+1}} (x - x_j)(x - x_{j+1}) f(x_j, x_{j+1}, \mathrm{x}) \mathrm{dx}$$

that is the where the numerical approximation comes that means, the function is represented by a polynomial in each sub interval.

So, these

$$E_1(f) = \int_{x_j}^{x_{j+1}} (x - x_j)(x - x_{j+1}) f(x_j, x_{j+1}, \mathrm{x}) \mathrm{dx}$$

$$= \frac{(b-a)}{12} h^2 f''(n), n \in [a, b]$$

Now, what is important from here is, important conclusion is that the order of approximation is of h square in this case, so that means, what we find that, if we need a better approximation for the numerical integration formula, so we increase the number of n or the division in between the two points a and b, where we are performing the integration. So, this was about the numerical integration by considering the function a polynomial f(x) is represented by a linear polynomial in the same manner now, we have that is called the Simpson’s one third rule Simpson’s one third rule.

So. in this case what we do is we represent

$$\mathrm{f(x)} \sim p_2(x)$$

that is a parabolic form. So, in this case, so using three points, let us call

$$a, \frac{a+b}{2} = c, b$$

these are three points, so if we represent by second degree polynomial, so f(x) now, by Lagrange formula.

$$f(x) \sim p_2(x) = \frac{(x-c)(x-b)}{(a-c)(a-b)} f(a) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c) + \frac{(x-c)(x-c)}{(b-a)(b-c)} f(b)$$

So, this cubic polynomial representation though if we find out the

$$I_2(f) = \int_a^b f(x)dx = \frac{h}{3}[f(a) + 4f(c) + f(b)]$$

and if we do by parts formula so, what I get a form as instead of h

$$h = c-a = b-c$$

$$= \frac{h}{3}[f(a) + 4f(c) + f(b)]$$

So, this is the formula for Simpson one third rule, that is there is the one third. So, that is why it is called Simpson. The same way we can represent in a form that means, if we

$$\int_a^b f(x)dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx = \int_{x_0}^{x_2} \quad + \int_{x_2}^{x_4} \quad + \cdots + \int_{x_{n-2}}^{x_n}$$

So, where n is an even number, so in this case and in each sub-interval we are representing this f(x) by $p_2(x)$. So, if I go in the same manner, so we find the final form is given by

$$= \frac{h}{3}[f_0 + 4(f_1 + f_3 + \cdots + f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2}) + f_n]$$

So, this is the composite formula and what we can show is that the error in this case can be expressed in a form as

$$E_2(f) = \frac{h^4(b-a)}{180} f^{(iv)}(n), \; n \epsilon (a,b)$$

Now, obviously, this we do not have the derivative can not be obtained, so simply so this exactly finding this error value is of no use only thing is that what we have to see that, if I reduce the h, I get the approximation much better. So, this is very evident that trapezoidal formula compared to that the Simpson's one third rule is very easy to implement.

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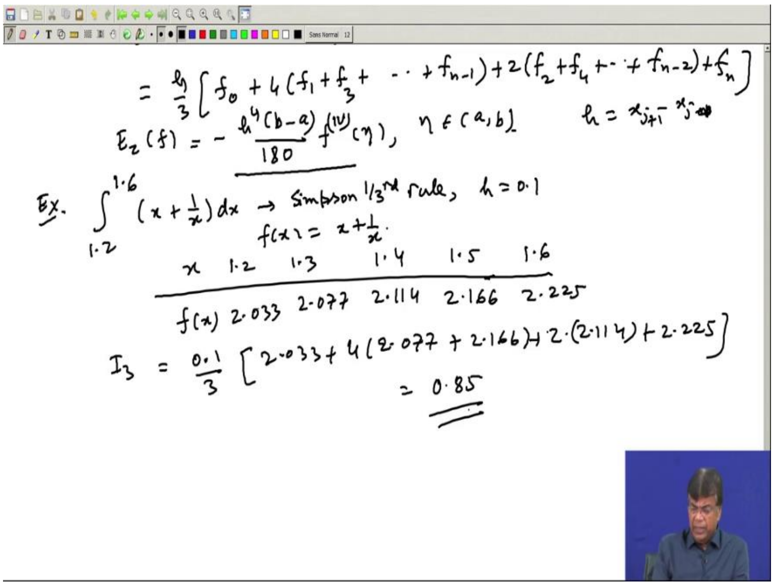

So, one of the example just so, and complete it here, so this is the function so, if I want to do by Simpson one third rule

$$\int_{1.2}^{1.6} \left(x + \frac{1}{x}\right) dx$$

$$h = 0.1\,, \qquad f(x) = x + \frac{1}{x}$$

X	1.2	1.3	1.4	1.5	1.6
F(x)	2.033	2.077	2.114	2.166	2.225

we can simply substitute the value of x and get this formula, this values, so now, if I perform the numerical integration

$$I_3 = \frac{0.1}{3}[2.033 + 4(2.077 + 2.166) + 2(2.114) + 2.225]\text{=0.85}$$

So, between the two grid points or node points the difference spacing between the two points. And if you compare with the exact solution, you will find that it is quite close compared to the trapezoidal formula. So, with that we stop first on the numerical integration. Thank you.