Advanced Computational Techniques
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Lecture 04
Spline Interpolation

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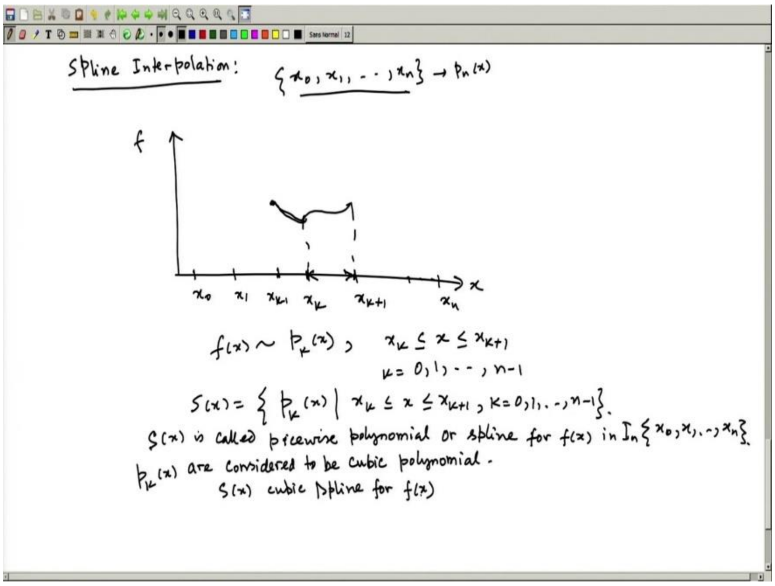

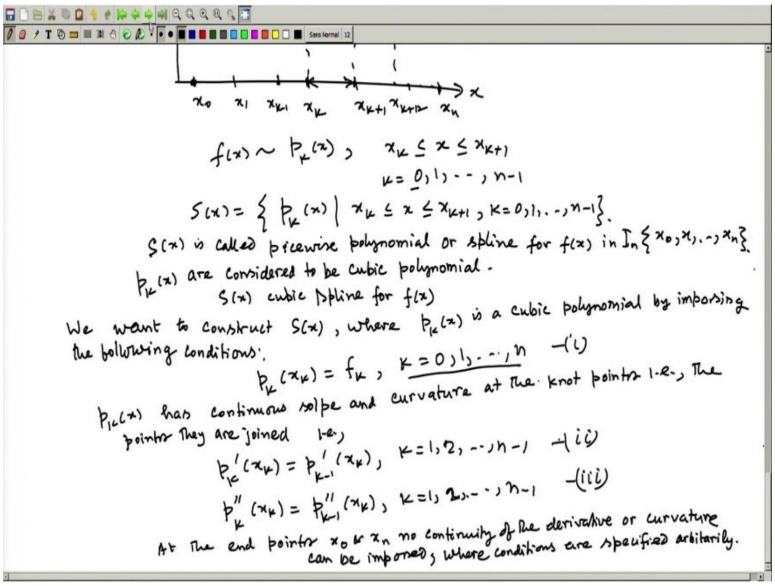

Now, we will talk about Spline Interpolation. Now, so far what we are talking is that you have data points $\{x_0, x_1, \ldots, x_n\}$. So, considering all the set of data points we construct the polynomial either of degree n or in the harmonic polynomial case we construct the polynomial which is interpolating the function as well as its derivative at the same data points. So, that the degree of the hermite polynomial becomes double or higher than the ordinary polynomial itself.

Now, a cubic spline or the spline interpolation, later on we will discuss only on cubic spline. So, before defining the cubic spline let us first discuss what is spline interpolation. Now, under spline interpolation, instead of choosing all the data points, what we will do is we will do a piecewise polynomial interpolation. So, that means, suppose you have the x and you are constructing a function so, let us call this x_0, x_1 like that way x_k, x_{k+1} and so on so say that this is x_n

So, this is your unknown function f, we approximate by a function so and then between x_{k-1} to x_k, we approximate by another function say another function another polynomial. So, that means, a polynomial we define which is interpolating the function at x_k, x_{k+1} and it is approximating the function within this interval sub interval instead of considering a global interval, we take a discrete sub interval in each of which the interpolating polynomial is defined.

Now, advantage of such polynomial representation or piecewise polynomial representation are manifold. One of the biggest advantage is that the function derivative so when I represent the function f (x) by a polynomial say $\boldsymbol{p_k\,(x)}$ which is valid globally, say, for example, so, there is a fluctuation of this $\boldsymbol{p_k\,(x)}$ will create the derivative of this function a huge difference. So, instead of that what we do is we present this function f (x) by $\boldsymbol{p_k\,(x)}$ in the sub interval say

$\boldsymbol{x_k \leq x \leq x_{k+1}}$ $_{,\mathbf{k=0,1,\ldots,n-1}}$.

So, in each sub interval we represent this function by $\boldsymbol{p_k\,(x)}$, so then and these functions represent a set of function now we have constructed

$$\mathbf{S(x)} = \{\boldsymbol{p_k\,(x)\; x_k \leq x \leq x_{k+1}}\,_{,\mathbf{k=0,1,\ldots,n-1}}\}$$

So, this set of function S(x) is called piecewise polynomial or spline. Then this is the one spline for f(x) in $I_n$which is containing the points $\{x_0, x_1, \ldots., x_n\}$, and if in most cases these $\boldsymbol{p_k\,(x)}$ are considered to be cubic polynomial. In that case S(x) is called the cubic spline, spline for f(x).

Spline word is coming because these polynomials are knot at these points x_k, except $x_{0,}x_n$at these two open points. So, remaining all other points $\{x_1, x_2, \ldots., x_{n-1}\}$these splines are noted. So, now we would like to construct S(x) where $\boldsymbol{p_k\,(x)}$ is a cubic polynomial by imposing this following condition. Now, first of all this $p_k(x)$ are interpolating the function f(x), so, first of all

$$p_k(x_k) = f_k, k = 0, 1, \dots, n$$

then we impose the condition that this $p_k(x)$ has continuous slope and curvature at the knot points. So, whichever way we come, so, that is the point where they are joined, continuous slope which is governed by the first derivative, so, that means

$$p'_k(x_k) = p'_{k-1}(x_k), k = 1, 2, .. n-1$$

$$p''_k(x_k) = p''_{k-1}(x_k), k = 1, 2, \dots n-1$$

So, these are the conditions. So, now, this condition is of course valid for all k and these conditions are at the node points only so, they have these small pieces, which have the same slope and same curvature when we approach from either side. So, that means, there is no discontinuity in the slope and curvature between say $x_k, x_{k+1,} x_{k+2}$. So, whichever way you come from this side or this side you have the same slope or same curvature for interpolating the spline function spline interpolation $p_k(x)$.

Now, at the end points x_0 and x_n no continuity of the derivative or it can be imposed or curvature can be imposed as it is now where conditions are specified arbitrarily.

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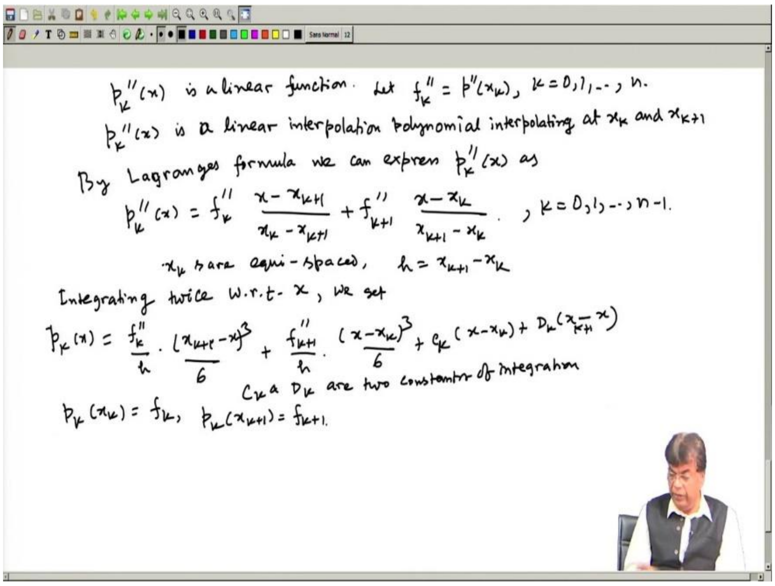

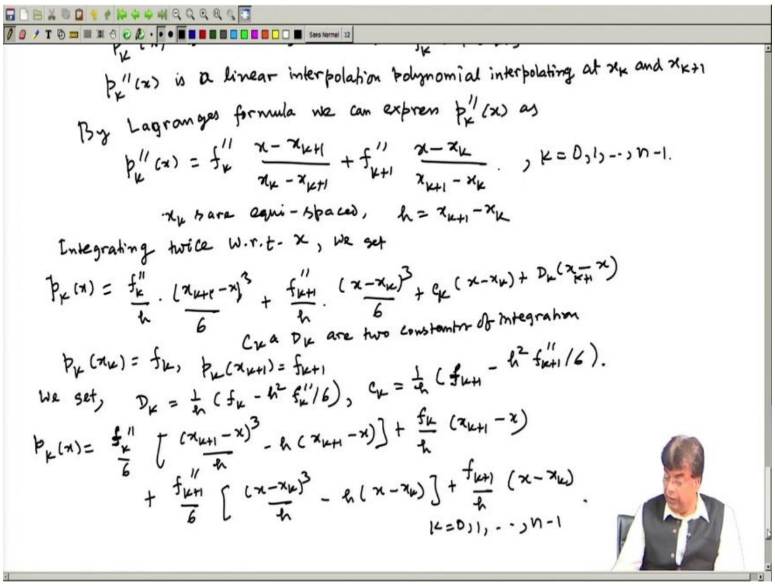

Now, with that what we do is that we now construct the form of the $p_k(x)$. Now, obviously, $p_k(x)$ is a cubic polynomial. So, that means $p_k''(x)$ is a linear function. let

$f''_k = p_k''(x_k), k = 0, 1, \dots n$.

Now, what we can say that $p_k''(x)$ is a linear interpolation polynomial interpolating at x_k, x_{k+1}. So, what we do is we can write by like Lagrange's formula we can express this $p_k''(x)$ which is linear as

$$p_k''(x) = f''_k \frac{x-x_{k+1}}{x_k-x_{k+1}} + f''_{k+1} \frac{x-x_k}{x_{k+1}-x_k} \quad \text{k=0,1,...n-1}$$

Now, this is cubic this is a linear one, but we need a cubic one. So, if we assume that these x_k's are equi-spaced, so, let

$h = x_{k+1} - x_k$

Now you integrate twice with respect to x we get

$$p_k(x) = \frac{f''_k}{h} \frac{(x-x_{k+1})^3}{6} + \frac{f''_{k+1}}{h} \frac{(x-x_k)^3}{6} + e_k(x-x_k) + D_k(x_{k+1}-x)$$

Now, what we have, the conditions to obtain the integrating constant. Now, we have

$p_k(x_k) = f_k, \; p_k(x_{k+1}) = f_{k+1}$

you impose this condition. If we impose this condition these two conditions what we get? We get

$D_k = \frac{1}{h}(f_k - h^2 f''_k/6)$, $e_k = \frac{1}{h}(f_{k+1} - h^2 f''_{k+1}/6)$,

if we now substitute I get $p_k(x)$. Now $p_k(x)$ if I substitute so that involves two unknown. these f''_k, f''_{k+1} these are not given but we have considered that this is the value of the second derivatives which is not prescribed, but we have expressed $p_k(x)$ in terms of the second derivatives conditions.

So, if I now substitute I get a form like this

$$p_k(x) = f''_k/6\left[\frac{(x_{k+1}-x)^3}{h} - h(x_{k+1}-x)\right] + \frac{f_k}{h}(x_{k+1}-x) + f''_{k+1}/6\left[\frac{(x-x_k)^3}{h} - h(x-x_k)\right] + \frac{f_{k+1}}{h}(x-x_k)$$

K=0,1,....n-1

So, this is the polynomial.

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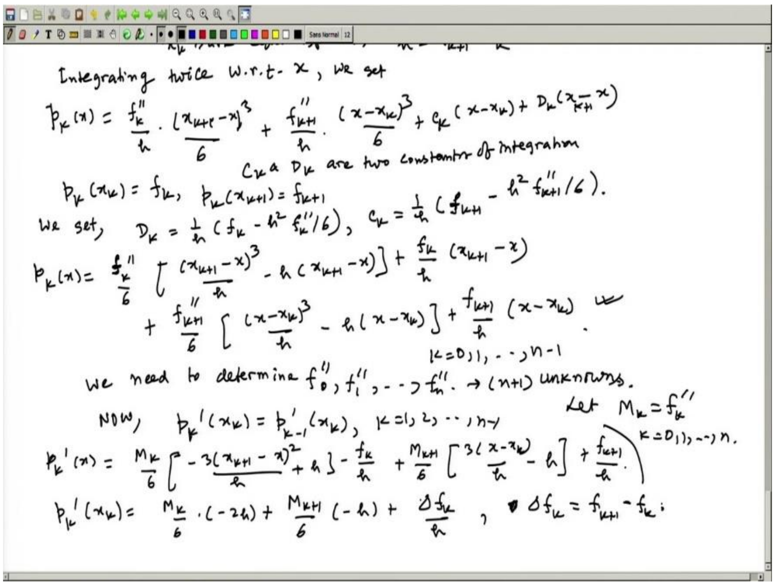

Now, we need to determine this f''_k or whether $f''_0, f''_1, \ldots, f''_n$ this is known, then the polynomial form is known. So, that means there are n p+1 unknown. Now, what are the new condition, what are the conditions we have not used? One condition we have not used the continuity of the slope. Now, it is given that

$p'_k(x_k) = p'_{k-1}(x_k)$, k =1,2,....,n-1 if we use,

$$p_k'(x) = \frac{M_k}{6}\left[-3\frac{(x_{k+1}-x)^2}{h}+h\right] - \frac{f_k}{h} + \frac{M_{k+1}}{6}\left[\frac{3(x-x_k)}{h} - h\right] + \frac{f_{k+1}}{h}$$

whatever algebra I have done so, same thing can be verified.

So, this is one condition and also we can write

$$p_k'(x_k) = \frac{M_k}{6}(-2h) + \frac{M_{k+1}}{6}(-h) + \frac{\Delta f_k}{h}, \qquad \Delta f_k = f_{k+1} - f_k \text{...}\mathbf{(a)}$$

So, this is the first order forward difference.

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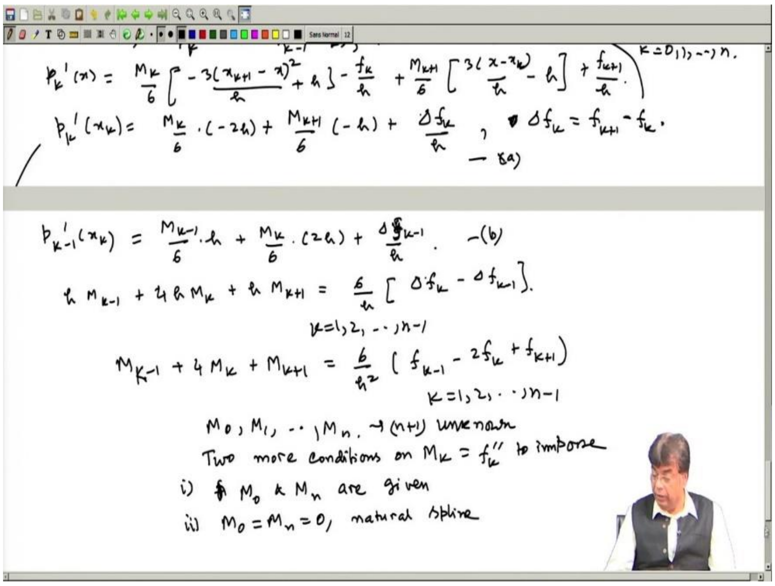

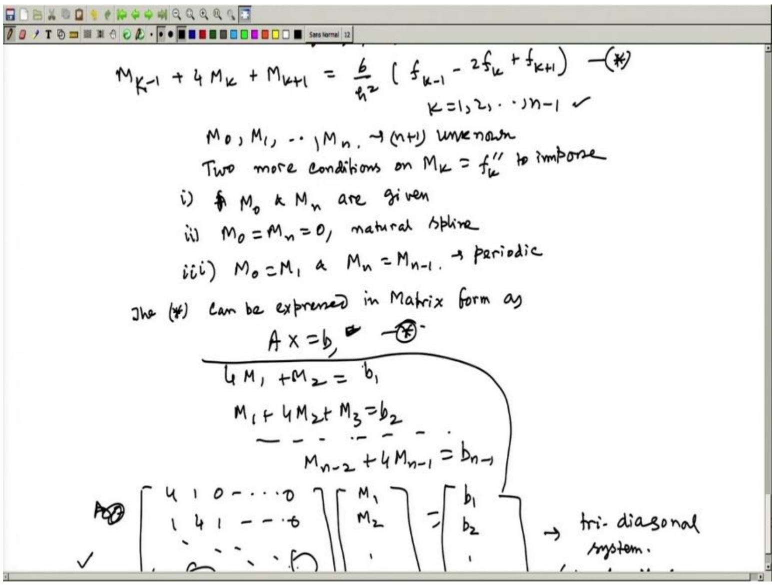

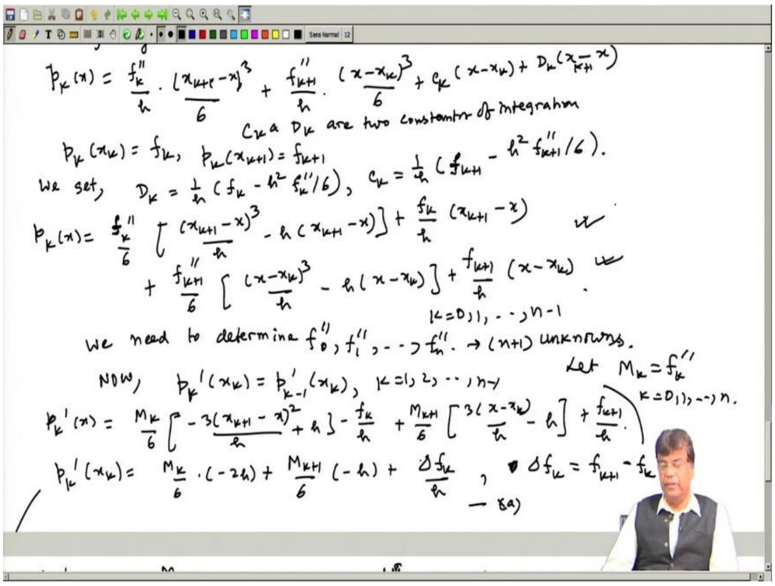

Similarly, one can find out $p'_{k-1}(x_k)$ as

$p'_{k-1}(x_k) = \frac{M_{k-1}}{6}.h + \frac{M_k}{6}.(2h) + \frac{\Delta f_{k-1}}{h}$...**(b)**

So, these two are same.

h $M_{k-1} + 4hM_k + hM_{k+1} = \frac{6}{h}(\Delta f_k - \Delta f_{k-1})$

k=1,2,....,n-1

I should write f_k because that is the notation we are using here. So, that gives

$M_{k-1} + 4M_k + M_{k+1} = 6/h^2\ (f_{k-1} - 2f_k + f_{k+1})$

k =1,2,....,n-1

So, this gives a set of relation between M_k and the linear algebraic equation for M_k's. Now, what we have is unknowns $M_{0,}M_{1,..,}M_n$ and so, n+1 unknown and for these n+1 unknowns, we have n-1 equations. So, two more conditions are to be imposed, conditions on M_k which is nothing but second derivative to impose. So, these two conditions, one can considered that $M_0$and M_n are prescribed at some given value that is the spline is taking a constant slope as we approach at the two ends.

Another condition can be $M_0 = M_n = 0$, this is called the natural spline. Another is also called the periodic spline. So, that means, what you can have $M_0 = M_1$ and $M_n = M_{n-1}$ this is called the periodic. So, in any case, we have then a tridiagonal system of n-1 equations here with n -

1 unknowns. So, because $M_{0,}M_n$ are prescribed values. So, this form can be expressed in matrix form as

$$Ax = b$$

where A is this matrix. If I put the value of k because M_0 is known, so, it is no longer unknown. So, if I put the values so, what we have

$$4M_1 + M_2 = b_1$$

Similarly, next k = 2,

$M_1 + 4M_2 + M_3 = b_2$

like that, and the last one will be k =n-1. So, what we get is

$$4M_{n-1} + M_{n-2} = b_{n-1}$$

So, this forms a linear system so, now which I can put in a form as a matrix A

$$\begin{matrix} 4 & 1 & 0 & \ldots & 0 \\ 1 & 4 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix}$$

which is a tridiagonal system, which can be solved to get solve for M_k's.

So, that is how we solve this, that means to know the spline interpolation, first we have to solve this trigonal system Ax=b. This system need to be solved. Once I solve this along with the given conditions either of this and then we come to form of the polynomial is given by $p_k(x)$, so which is valid between k= 0 to n-1. So, other polynomial cubic spline is constant. So, that is how the cubic spline formation occurs. Thank you.