Advanced Computational Techniques
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Lecture no. 20
Non - Linear Advection – Diffusion Equation

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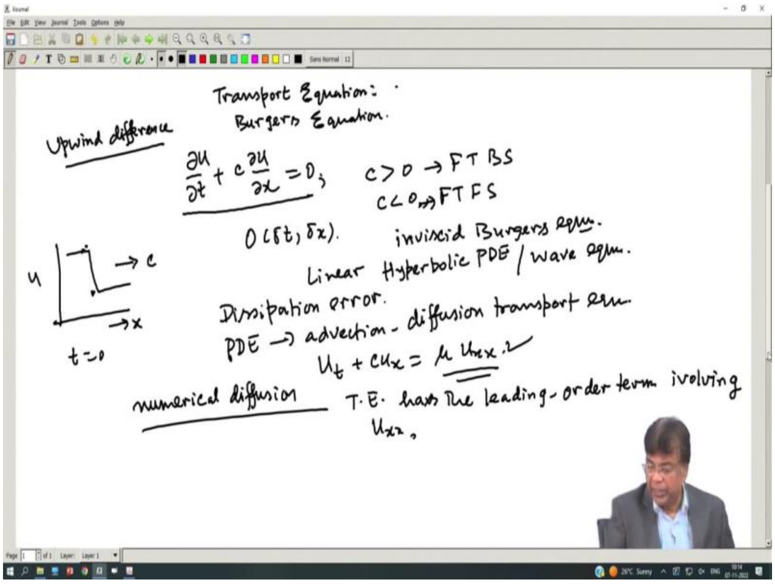

Well we now talk about the Burgers equation that is what we are continuing the upwind scheme for burgers equation, we are talking about the inviscid burgers equation so, that means, we have the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

this is the equation upwind scheme which we have discussed, which was the first order upwind difference scheme.

So, that means, if $c > 0$ then we use FTBS if $c < 0$ then in that case we have to use FTFS this is the upwind scheme, but which is also first order $0(\delta t, \delta x)$

Now, one important aspect in solving this kind of inviscid burgers equation. Either way, you can call that it is also a hyperbolic PDE. Hyperbolic PDE are linear PDE of course, linear or referred as the wave equation.

Last class I have discussed this wave equation because any pattern which forms at time t=0 it propagates without deformation. So, that is the wave equation. Now, that wave equation has a disadvantage is that solving these that if any sharpness appears so that means, if any discontinuity of the solution appears for example, shockwave in gas dynamics, if there is any shockwave appears so that means, there is a jump discontinuity all of a sudden, I mean in any region there is a change in velocity, pressure, temperature. So, that is what is called the shockwave.

Now, that propagates with the constant speed along with the velocity c, because without any deformation, so, if you have initially say the solution is something like this, x at t=0 there is a step change, step jump is occurring at these two points so what happens is that the step form will propagate with a velocity c in the time period or in the time step.

So, that means, it will go in if we have an animated picture what we will find that without deformation it will move to the positive direction if $c > 0$ negative direction if $c < 0$, so, that is the reason it is referred as the wave equation. This is the difficulty. Now numerically capturing this solution, if there is any sharpness occurs that resolution of the numerical scheme for wave equation is very important. And for that, what we now check the dissipation error, so, I will not go much into details distribution error.

So, basically, if we consider the PDE which is advection diffusion equation type, advection diffusion type problem, diffusion transport equation say

$$u_t + cu_x = \mu u_{xx}$$

If there is second order derivative present so whatever it is that if there is any sharpness occurs in the solution so that smeared out that means that sharpness get flattened as time progresses.

So, that is the advantage of the diffusion term the presence of this diffusion term. So, now in the PDE, the word PDE we have, we do not have any second order terms. Now, what do you want to do is that through numerical diffusion the different scheme should be in such a way that the error term, the truncation error or a leading order term involving second derivatives u_{xx} .

So, that means we need to check that the numerical scheme should be in such a way that the truncation error should involve u_{xx} terms.

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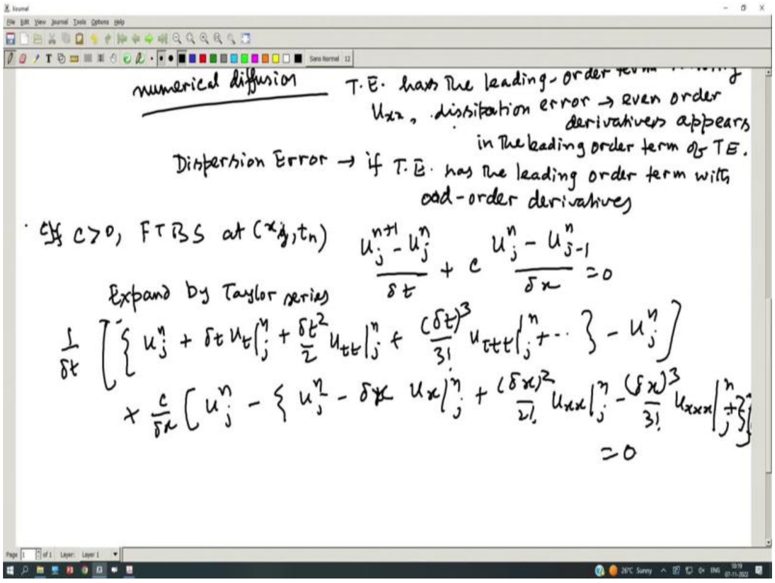

So that is called the dissipation error. So this dissipation error is most welcome. Compared to the dispersion error if we have dispersion error of even order, order derivatives so in general there is the dissipation order means even order derivatives involved, not involve, appears in the leading order term of truncation error.

If this is the one then it is referred as the dissipated scheme or dissipation error. Now, on the contrary we call it as dispersion error, if odd order, the same thing if t has the leading order term or odd order derivatives. Now, it is intuitively appear that when you take a one sided difference either forward or backward so there will be a second order dissipation, a second order term in the truncation error and if it is central so it will be third order so on to do to check that.

So, what we do is we have, say, $c > 0$ then we know this FTBS so if $c > 0$ we have already checked that FTBS is the one which is if $c > 0$ then we can use the FTBS and at (x_j, t_n) so, we have the discretization

$$\frac{{u_j}^{n+1} - {u_j}^{n}}{\delta t} + \mathrm{c}\frac{{u_j}^{n} - {u_{j-1}}^{n}}{\delta x} = 0$$

Now, to get the truncation error explained by Taylor series we get

$$\frac{1}{\delta t}\left[\left\{u_j^n + \delta t\, u_t|_j^u + \frac{\delta t^2}{2}u_{tt}|_j^n + \frac{(\delta t^3)}{3!}u_{ttt}|_j^n + \cdots\right\} - u_j^n\right] + \frac{c}{\delta x}\left[u_j^n - \left\{u_j^n - \delta x\, u_x|_j^n + \frac{(\delta x)^2}{2!}u_{xx}|_j^n + \frac{(\delta x)^3}{3!}u_{xxx}|_j^n + \cdots\right\}\right] = 0$$

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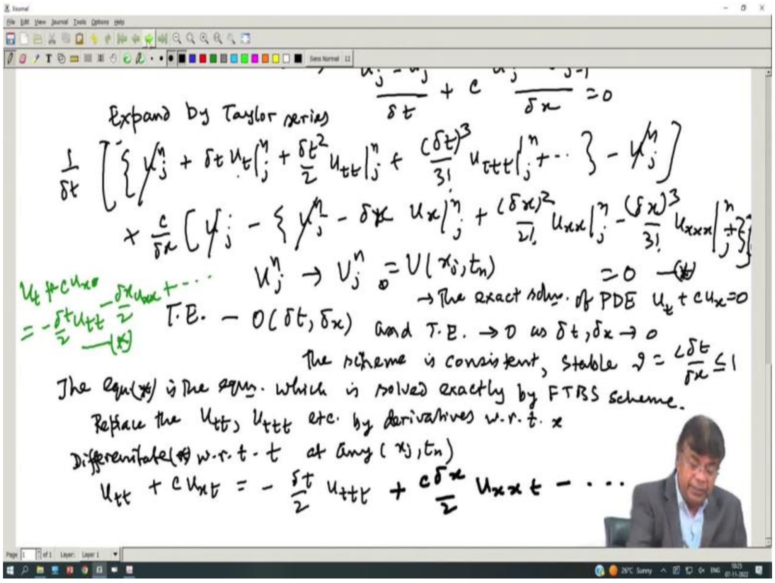

So, this is the expansion of the now, how we define the truncation error is that when I replace these u_j^n by the exact solution of the differential equation that is PDE then whatever the residue that is called that truncation error. So, what we can find from here that if I replace this u_j^n of course, it is get canceled.

If I replace this u_j^n by U_j^n, is the exact solution of the PDE so we call us (x_j, t_n) the exact solution of the PDE. So what we find that it is very evident that truncation error is $0(\delta t, \delta x)$

so whatever the residue left is because $u_t + cu_x = 0$ because it is the exact solution so what is left is the u_{tt} so δt so and this shows that t is $0(\delta t, \delta x)$ and all they find that this t→0 as $\delta t, \delta x \to 0$.

So, that means the scheme is consistent and what we have proved that it is stable previously we have shown that stable for

$\vartheta = c\delta t(\delta x) \leq 0$

Now, prior to that what we have to see that the dissipative nature of the scheme.

Now, this is we can call that the numerical scheme. If I do not replace this u by the exact solution of the PDE then the equation which is solved exactly by the FTBS scheme. Now FTBS scheme, whatever the finite difference.

Now, what we do now, we replace u_{ttt}, all the time derivative by derivatives with respect to x. And to do that what we do is time relevant with respect to x so and to do that what we do is suppose if I differentiate with respect to t okay so we get is if I get u_{tt} at j, so at any point that with respect to t at (x_j, t_n).

So what you get is u_{tt}, we are differentiating with respect to t.

$$u_{tt} + cu_{xt} = -\frac{\delta t}{2} u_{ttt} + \frac{c\delta x}{2} u_{xxt} - \cdots$$

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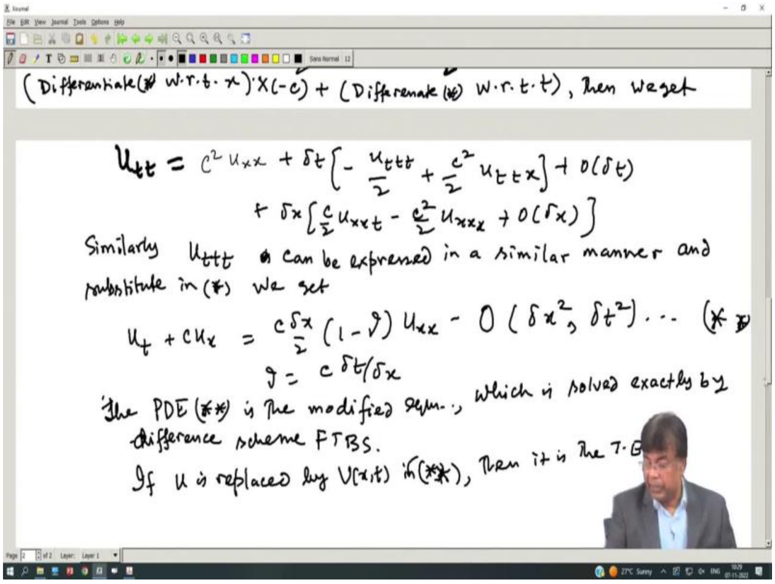

So, likewise what I do is from here if we replace the u_t so u_{tt} can be written as

$$u_{tt} = c^2 u_{xx} + \delta[-\frac{u_{ttt}}{2} + \frac{c^2}{2} u_{ttx}] + 0(\delta t) + \delta x \left[\frac{c}{2} u_{xxt} - \frac{c^2}{2} u_{xxx} + 0(\delta x)\right] \ldots (*)$$

So if I go on like this way, we replace u_{tt} . If I replace u_{ttt} what I find is the modified equation as

$$u_t + cu_x = c\frac{\delta x}{2}(1 - \vartheta)u_{xx} - 0(\delta x^2, \delta t^2) \ldots .(**)$$

so this is the modified equation which has infinite number of terms involved which is solved exactly by the difference scheme FTBS, now one FTBS.

Now, if u is replaced by $U(x,t)$ in (*) which is this exact solution of (**), then it is the truncation error. So, these are all at the tier point n,j and of course, our discretization is independent of n, j.

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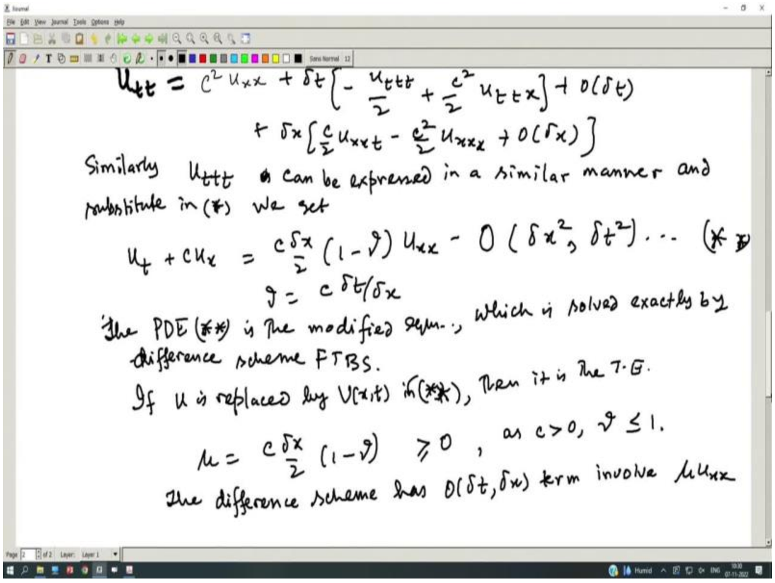

So, if I call this

$\mu = c\frac{\delta x}{2}(1 - \vartheta) \ \geq 0, as\ c > 0, \vartheta \leq 1$

So, that means the difference scheme as the order $0(\delta t, \delta x)$ term involves μu_{xx} so that's why it is called the dissipative scheme.

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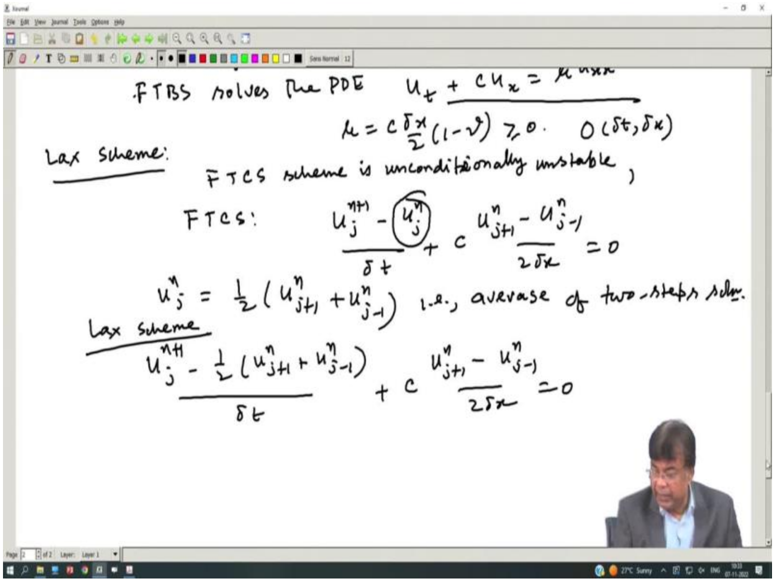

So that means, basically what we are solving is we are solving the FTBS that solves the PDE

$$u_t + cu_x = \mu u_{xx}$$

where $\mu = c\frac{\delta x}{2}(1 - \vartheta) \geq 0$

As you could see this is consistent stable, but it is of first order in $\delta x, \delta t$.

Now, to improve the accuracy, there is one method that is called the Lax scheme so what we do is the Euler scheme whatever we have discussed before which was FTCS that means forward time and central space is unstable scheme, is unconditionally unstable. So, that is why we do a modification, as then what we do is we replace is

$$\frac{u_j^{n+1}-u_j^n}{\delta t}+c\,\frac{u_{j+1}^n-u_{j-1}^n}{2\delta x}=0$$

Now, this u^n is unstable. So, for that

$$u_j^n=\frac{1}{2}(u_{j+1}^n+u_{j-1}^n)$$

If we do this modification that is average of two steps solutions then, we find the numerical scheme become stable so, this is the called the Lax scheme.

$$\frac{u_j^{n+1}-(\frac{1}{2}u_{j+1}^n+u_{j-1}^n)}{\delta t}+\mathrm{c}\,\frac{u_{j+1}^n-u_{j-1}^n}{2\delta x}=0$$

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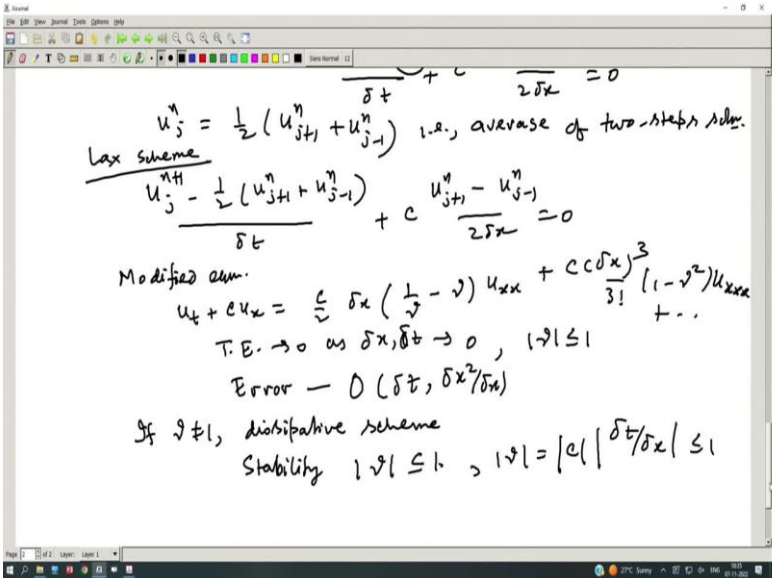

So, what you can see that the truncation error or the modified equation

$$u_t + cu_x = \frac{c}{2}\delta x\left(\frac{1}{\vartheta} - \vartheta\right)u_{xx} + c\frac{(\delta x)^3}{3!}(1 - \vartheta^2)u_{xxx} + \cdots$$

So obviously T.E.→0 as $\delta x, \delta t \to 0$. So, and also $|\vartheta| \leq 1$ for stability and what we find that error is $0\left(\delta t, \frac{\delta x^2}{\delta x}\right)$ so not much improvement in the order of accuracy. So, if $\vartheta \neq 1$ this is dissipative scheme, what we can do for stability that $|\vartheta| \leq 1$. So,

$$|\vartheta| = |c|\left|\frac{\delta t}{\delta x}\right| \leq 1$$

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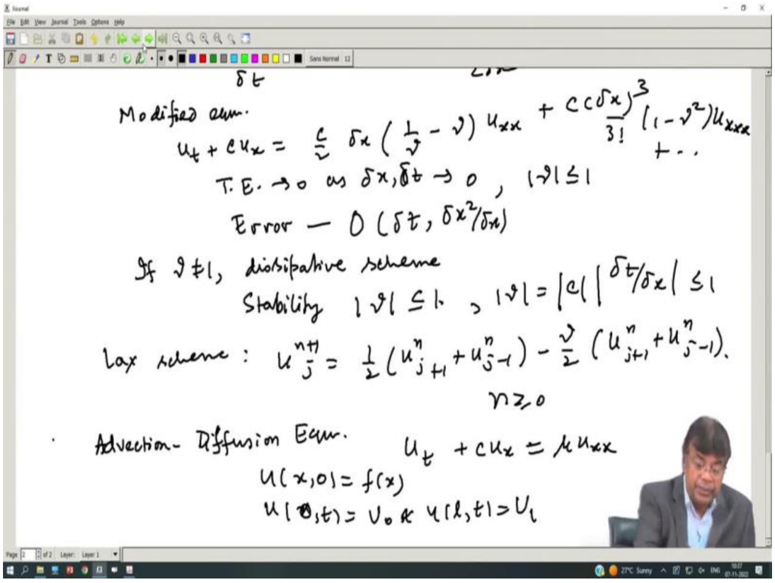

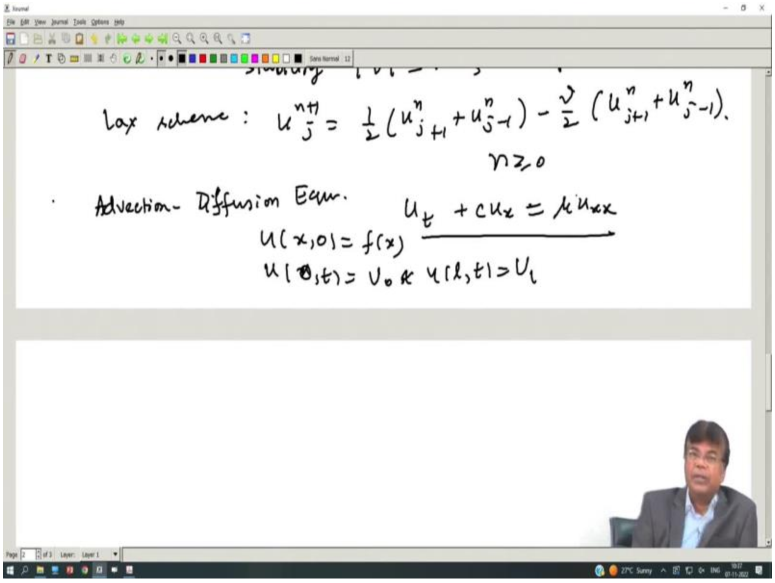

Now, so, that means the Lax scheme is

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - \vartheta \frac{u_{j+1}^n + u_{j-1}^n}{2}$$

From there several other numerical scheme are developed, for example, the Lax–Wendroff scheme, then MacCormack scheme and all these things, due to the time constraint we may not discuss this kind of scheme and what we talk about the linear one also we may not be able to do.

So, what we do is another very simple situation. So, that means, when you have a situation like this kind of equation, say the nonlinear Burgers equation or nonlinear equation or advection diffusion equation because this was the advection equation. So, we started with this advection diffusion equation so, that means either

$$u_t + cu_x = \mu u_{xx}$$

the PDE is given by this way and you have

$u(x, 0) = f(x)$,   this is the initial condition.

and we need only one initial condition

$$u(0, t) = u_0 \quad u(l, t) = u_1$$

Sothis kind of equation can be solved very easily by Crank Nicholson scheme.

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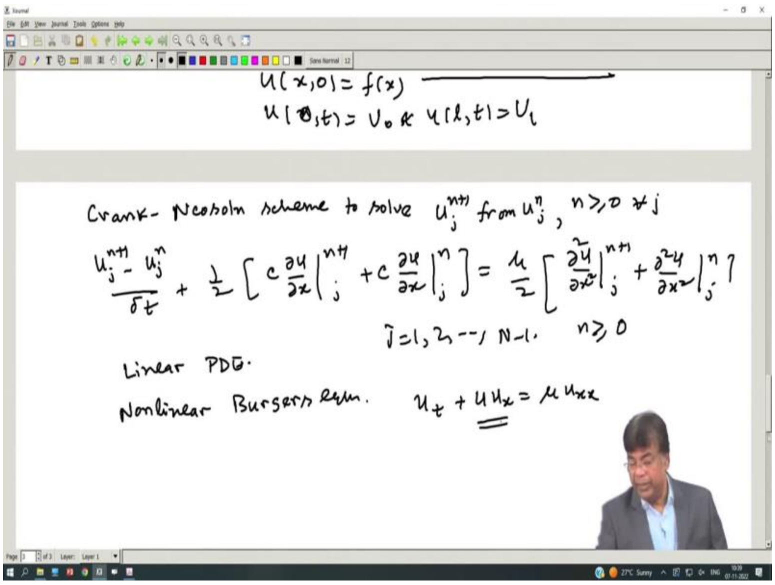

So, what we will find that Crank Nicholson scheme to solve u_j^{n+1} from u_j^n for n≥0 for all j

$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left[c \frac{\partial u}{\partial x}\Big|_j^{n+1} + c \frac{\partial u}{\partial x}\Big|_j^n\right] = \frac{\mu}{2}\left[\frac{\partial^2 u}{\partial x^2}\Big|_j^{n+1} + \frac{\partial^2 u}{\partial x^2}\Big|_j^n\right]$, j =1, 2,…, N-1, n ≥0.

This is the Crank Nicholson scheme that can be used and which can be found to be a stable numerical scheme, unconditionally stable because this is involving a dissipation term or the diffusion term here. So, this can be solved in a fashion like this way.

Now, this is linear PDE, so if you have this nonlinear Burgers equation. So, that means, if you have $u_t + uu_x = \mu u_{xx}$ so non-linearity appears.

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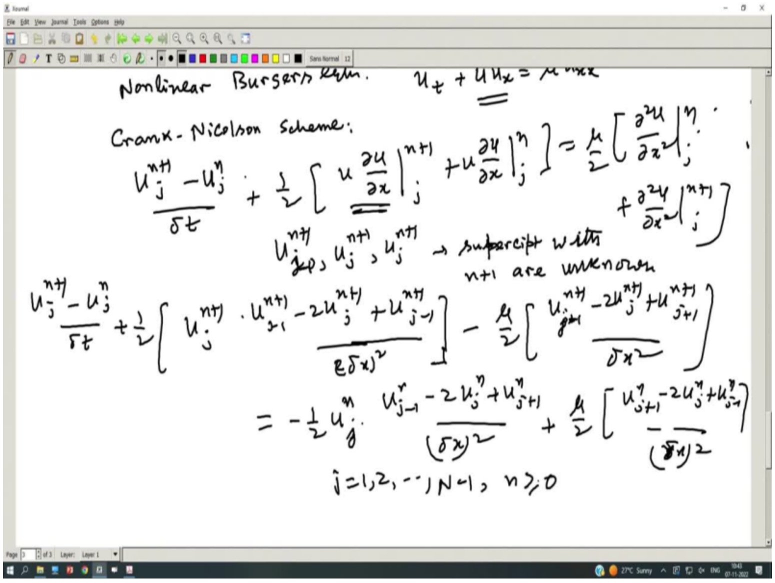

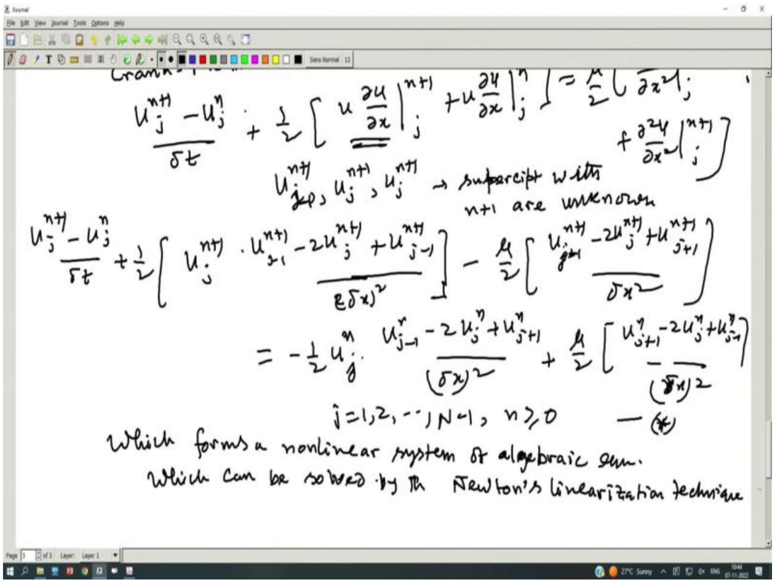

If I apply Crank Nicholson scheme for the nonlinear Burgers equation so what I get is

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left[u\frac{\partial u}{\partial x}|_j^{n+1} + u\frac{\partial u}{\partial x}|_j^n\right] = \frac{\mu}{2}\left[\frac{\partial^2 u}{\partial x^2}|_j^{n+1} + \frac{\partial^2 u}{\partial x^2}|_j^n\right]$$

Now, the terms u_{j-1}^{n+1}, u_j^{n+1} are unknown, all these with superscript n +1 are unknown. So we can use the Newton's linearization technique. So, before that if I discretize this

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + \frac{1}{2}\left[u_j^{n+1} . \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\delta x)^2}\right] - \frac{\mu}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\delta x)^2}\right]$$
$$= -\frac{1}{2}\left[u_j^n . \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right] + \frac{\mu}{2}\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\delta x)^2}\right]$$

j=1,2,…,n-1, n≥ 0

So, in this form a nonlinear system of algebraic equation can be solved by Newton's linearization technique, so, that means, an iterative procedure has to be adopt.

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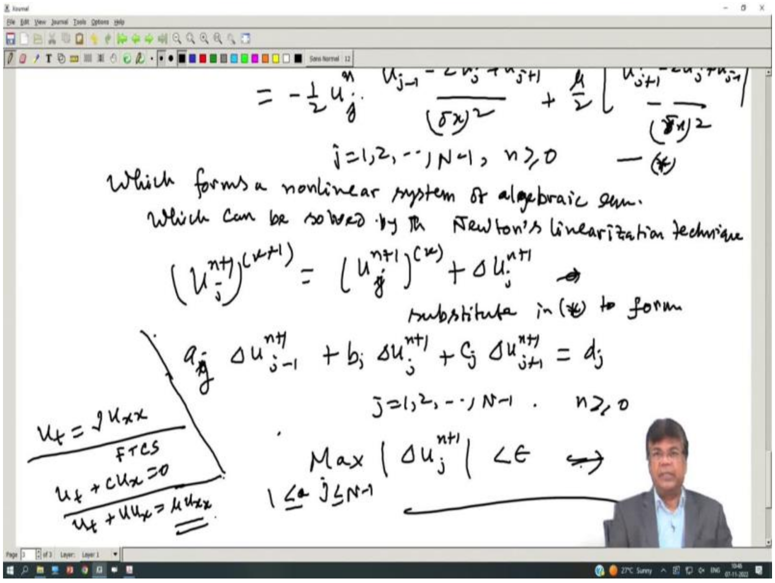

So, what we do is

$(u_j{}^{n+1})^{k+1} = (u_j{}^{n+1})^k + \Delta u_j{}^{n+1}$

So if I substitute in (*) and to get form because square and higher orders are neglected to form a system as

$\Delta u_{j-1}{}^{n+1} + b_j \Delta u_j{}^{n+1} + c_j \Delta u_{j+1}{}^{n+1} = d_j$, j=1, 2,.., N-1 and we start by iteration for n≥0.

So, again the initial has to be prescribed and we proceed till we get

$\max_{1 \le j \le N-1} |\Delta u_j{}^{n+1}| < \in$

this iteration process repeats till we get this convergence. So, this is how the nonlinear advection diffusion equations, so whenever there is a diffusion term we can safely go by the Crank Nicholson scheme.

So, there are three things we have discussed, one with diffusion, purely diffusion, that means, you have $u_t = \vartheta u_{xx}$ so we use the Crank Nicholson scheme or any other implicit scheme or central difference scheme. So, FTCS is the most proper. Then another one is $u_t + cu_x = 0$ this kind of equation where we have the way one sided so that was upwind and another thing is this kind of equation if the diffusion term is present so we can again use a central difference type scheme. With that I stop here for the PDE part. Thank you.