Applied Linear Algebra in AI and ML
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Lecture No 2
Vector Subspaces

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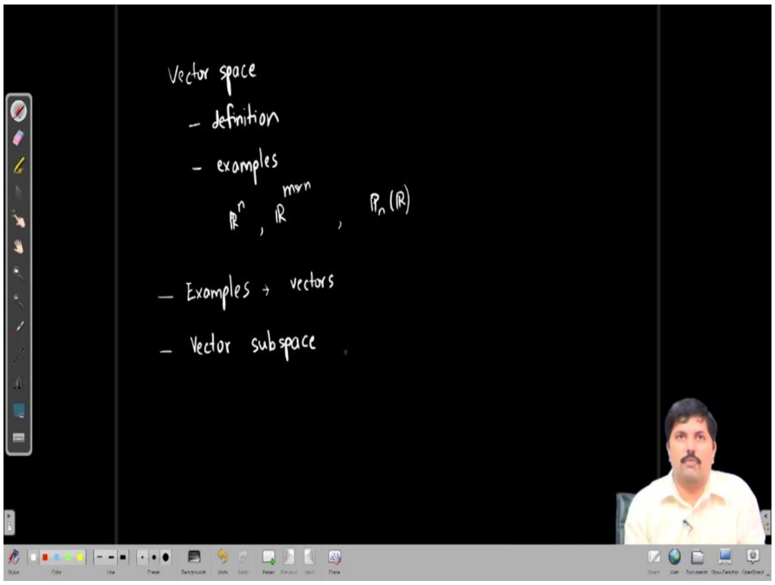

Welcome to the second lecture of the course Applied Linear Algebra for AI and ML. In the last class, we had seen the concept of vector spaces. We had seen the definition of this concept of vector space, and we had seen examples.

The most important examples that we will use throughout this course are basically $\mathbb{R}^n$, the collection of all n tuples, $\mathbb{R}^{m\times n}$, the collection of all matrices of the size $m \times n$, and polynomials of degree at most n with real coefficients. These were the three examples that we studied in this class. In today's class, what we are going to do is we are going to first see the examples, at least a couple of examples, where we encounter these vectors.

So that is the first thing that we want to do today, and the second task that we want to do in today's class is that we want to understand the concept of vector subspace. So, like a vector space that we had defined in the last class, we want to understand what is the vector subspace and try to understand the geometrical interpretation or geometrical meaning of this concept; that is what we want to do in today's class.

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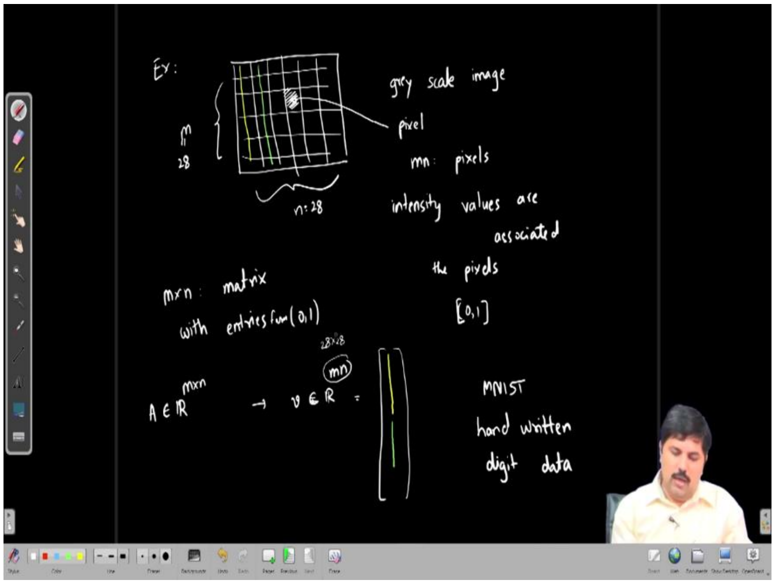

So, before we go any further, let us see where we encounter these vectors looking like $\mathbb{R}^n$. So, suppose we have an example of an image. So typically, the image is an $m \times n$ two-dimensional array, and suppose you are looking at a grayscale image, then this is how typically the grayscale image looks like. So, what happens in a grayscale image? What really happens is I have drawn several pixels.

So, suppose there are m number of pixels here and n number of pixels here. So, this one particular rectangle that you see here is a pixel. So, there are a total mn number of pixels in this image. And what happens is in a grayscale image, which is nothing but a black and white image, roughly speaking, intensity values are actually stored in these pixels.

So, there are intensity values which are associated with the pixels. So, what really happens is that based on the darkness or the faintness of the image in that pixel, you will have a value between [0,1]. So, any real number between [0,1] you will have as a value. So, if you really want to see this image as a mathematical object, then you can think of this image as an $m \times n$ matrix.

So, this image is basically an $m \times n$ matrix with entries from $[0,1]$. So, this is a matrix, basically. If you want to convert this matrix into a vector, you can do an operation which is called vectorization. So, this image basically is a matrix in $m \times n$. So, $A \in \mathbb{R}^{m\times n}$; that is what this image is.

So, from this image, you can construct a vector. So, suppose that vector v, then will be a vector of the length mn. It will be a vector in the dimension in $\mathbb{R}^{mn}$, and how will I construct this vector? The way I will construct this vector is I will look at this m number of rows first. I will put those m number of rows here. After I do that, I will look at the next column, and I will put these blue rows here below it.

So, I am doing vectorization column by column. I am stacking one column below the other, and in that way, I am constructing this particular vector which is a vector of length mn clearly. So, this vector is a vector in mn. This vector also represents the grayscale image where the entries are in $[0,1]$. Whether to represent an image as a matrix or whether to represent an image as a vector depends on the application and depends on the further mechanism that we will be dealing with. So, we will see examples of grayscale images. In particular, we will see the MNIST handwritten digit data.

So, they all are 28×28 grayscale images. So, here $m = 28$, and $n = 28$. So, they all will be 28×28 grayscale images, and we are going to make use of these images. We are going to make use of this data in this course further, and then we will decide whether you want to store this as a matrix or a vector, but that is where you will encounter the examples of vectors that $mn = (28).(28)$.

So, that is what is going to happen. So, this is one example where you get the image as a vector.

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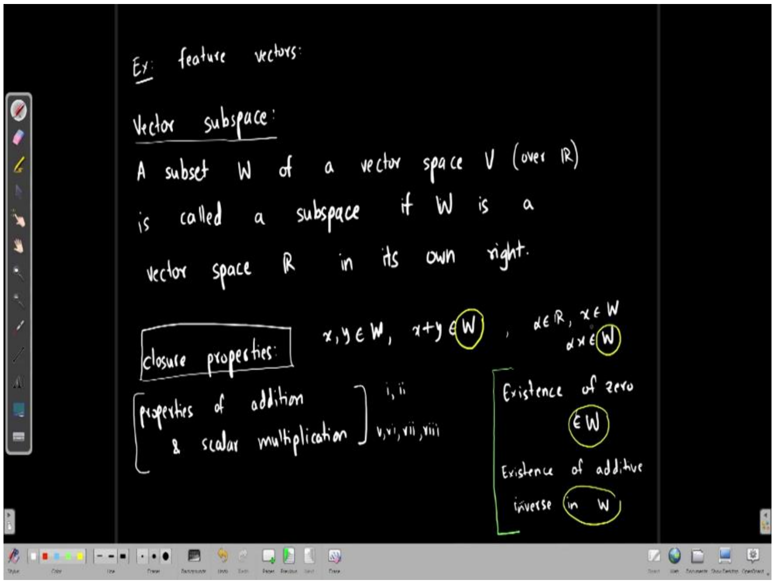

So, another example where the vectors often occur in context with AI and ML is actually feature vectors. So, while you are collecting data for a population, there are certain properties of the population that you want to list component-wise. For example, in a population, you are trying to find out the height, weight and all other components like the blood group of a person, and so on and so forth. So, when those entries will be put per person, then you get feature vectors.

So, these feature vectors, collection of data and all these different examples actually tell us that the vectors are going to be a very important object in the study of AI and ML. So, that is why we are looking at the examples of vectors in $\mathbb{R}^n$ vectors in $\mathbb{R}^{m\times n}$, which are nothing but matrices and polynomials.

So, the polynomial context will come a little later. At that time, I will recall why we are looking at the polynomial example. So, with this background about why we want to study these vectors, let us go back to the main topic of today's lecture, which is vector subspace. So, what we want to do today is we want to understand what is a vector subspace and the examples of vector subspaces and their geometrical meaning.

First, I will write down the definition. So, we have a subset W of a vector space V. So, ideally, whenever I say a vector space V, I have to specify the field of scalars, but in this particular course, unless I state otherwise, the field of scalars is always going to be $\mathbb{R}$. Sometimes I will initially write so that you get used to this, but then slowly I will start skipping this.

So, a subset W of a vector space V over $\mathbb{R}$ is called a subspace if W is a vector space over $\mathbb{R}$ in its own right. So, what do we want? We want to have a subset of V, and what should that subset be? First, a vector subspace is a subset, and it is no ordinary subset; this is a subset that in itself has a vector space structure.

So, if you recall, in the first class, in the definition of vector space, what were the properties that were required? The properties that were required were closure properties, and the other properties were properties of addition and scalar multiplication.

So, the properties of addition and scalar multiplication property- number one, two and properties five, six, seven and eight, these properties are satisfied by every vector in V. So, they will be satisfied by the vectors in W also. So, these properties are not a question. The property in question for W to be a vector space in its own right is the closure property. It should be satisfied. What is the closure property?

The closure property says that if you start with vectors $x, y \in W$, then $x + y \in W$. What we know for sure is that $x + y \in V$ because $W \subseteq V$, and V is a vector space. So, if $x, y \in W$, $x + y \in V$. What we are not sure is whether this belongs to W or not. Because you want W to be a vector space in its own right, you want the closure property to be satisfied by W.

So, this is the first thing. And $\alpha \in \mathbb{R}$, and $x \in W$, you want $\alpha x \in W$. So, here also we know that $\alpha x \in V$ for sure, but whether $\alpha x \in W$ or not is what we are trying to understand to make sure that closure properties are satisfied by W. So, W is closed under addition and scalar multiplication; that is what we want. And the other important properties are property number three and four.

So, what is property three? Property three is telling us the existence of zero. So, the existence of zero is guaranteed in V, but whether the existence of zero is guaranteed in W or not, we do not know. We want that also to work out: the existence of $0 \in W$. So, these are the three properties that we will really want, and what is the fourth property? The fourth property is the existence of additive inverse in W.

So, the existence of additive inverse in W is also not a very big thing. Why is it not a big thing? It is a consequence because if you take $\alpha = -1$, then $-x \in W$; that is what you are saying. So, the scalar multiplication closure is kind of taking care of the existence of additive inverse, and if you know scalar multiplication is taking care of the existence of additive inverse, then you will add the given vector with its additive inverse. The additive closure is taking care that the existence of $0 \in W$.

So, these two properties that we have listed here, the existence of 0 and the existence of additive inverse, actually are the consequence of additive closure and scalar multiplicative closure. So, what we really see here is that if W is a subspace of V, then it is enough to check whether W itself is closed under vector addition and scalar multiplication or not. So, if W itself is closed under vector addition and scalar multiplication, then W will be a vector space in its own right and hence a subspace of V.

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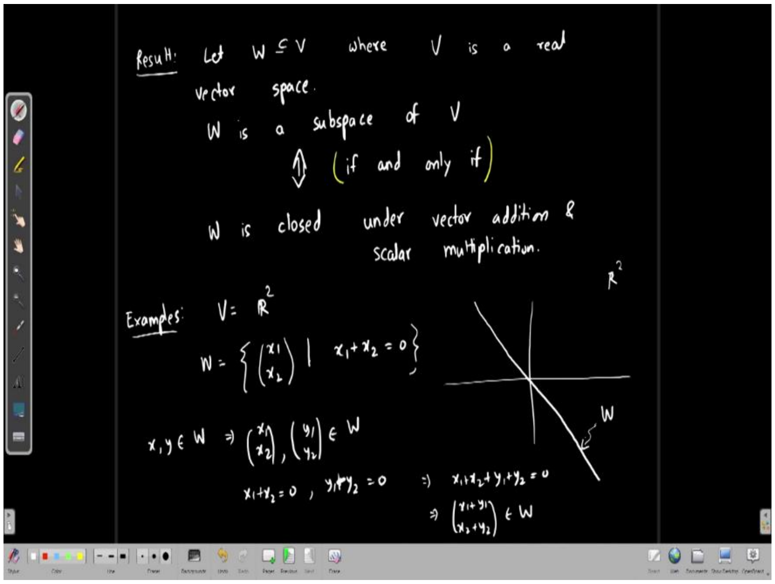

So, that is what we observe here; that is what we write down. So, one of the important results that we can write down here is that let $W \subseteq V$, where V is a real vector space, then the discussion that we just had ensures that W is a subspace of V if and only if; this if and only if is this double implication. I will put this in brackets. So, these two things mean the same thing.

So, W is a subspace if and only if W is closed under vector addition and scalar multiplication. So, how will we check, given in an example, whether W is a subspace or not? We check for closer closure properties of W with respect to addition and scalar multiplication. So, let us see a few examples of subspaces of vector space. Let $V = \mathbb{R}^2$.

So, my vector space is $\mathbb{R}^2$, and let us take W to be equal to the set. So, what is the set at x? So, $W = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 + x_2 = 0\right\}$. So, now let us try to see this through a picture. So, this is our plane $\mathbb{R}^2$, and $x_1 + x_2 = 0$. So, this is the line $x_1 + x_2 = 0$. This is set W, correct? So, this set geometrically is actually a line passing through the origin.

So, all of us see that $W \subseteq \mathbb{R}^2$. Now, we want to see whether W, in its own right, is a vector space or not. So, what do we want to do? We want to check the closure properties. Let us consider $x, y \in W$. So, that means that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W$. So, what does that mean? That means $x_1 + x_2 = 0$, and $y_1 + y_2 = 0$. These are the two properties that x and y vectors satisfy. Then clearly, what does that mean?

That means that $x_1 + x_2 + y_1 + y_2 = 0$, which means that the vector $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in W$. That means W is closed under vector addition. Similarly, for scalar multiplication also, what can you see? For scalar multiplication, you can very easily see if $x_1 + x_2 = 0$, that means $\alpha(x_1 + x_2) = 0$. So that will ensure that W is closed under scalar multiplication.

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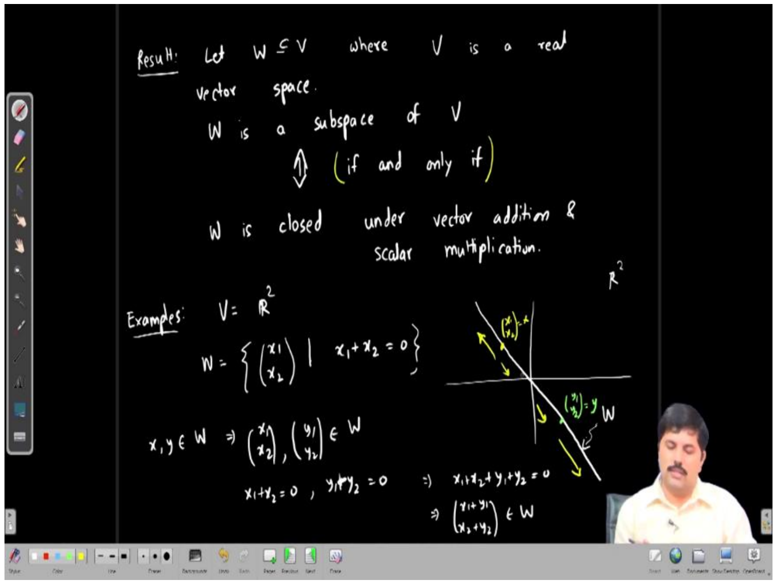

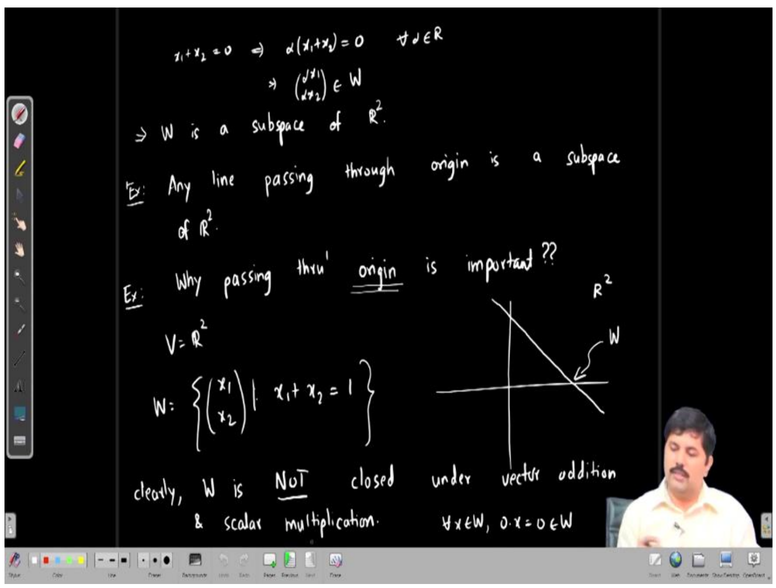

So, $x_1 + x_2 = 0$ will imply that $\alpha(x_1 + x_2) = 0, \forall\, \alpha \in \mathbb{R}$. This will mean that $\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} \in W$. So, W is closed under scalar multiplication. So, once we see that these two properties are satisfied, this implies that W is a subspace of Vor $\mathbb{R}^2$ in this particular case.

So, if we look at this particular example, is there anything specific about this line W which is passing through origin $x_1 + x_2 = 0$? Is that the only line that is important? If I take any line which passes through the origin, what is going to happen? The same calculation is going to work.

There is no difference in this algebraic calculation. What is really happening here, if you see carefully, is that you have a vector here $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

So, this is your vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and suppose this is another vector $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. You can see that you take a scalar multiple of this vector. You multiply α to x. What is really going to happen? You are going to take $\alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$. So, you are still going to be on this line. Either in this part, if $\alpha > 1$, or in this part, if $0 < \alpha < 1$, and this part when $-1 < \alpha < 0$ and this when $\alpha < -1$.

So, that is what is going to happen. So, the scalar multiplication of x is always going to keep you on this line, and moreover, if you add these two vectors, x and y, you still are going to be on this line. So, you always are going to be on this line. Take any two vectors, and add them; you are on the line. Take any vector, and take scalar multiplication; you are still on this line. That is the geometry of closure of addition and scalar multiplication. So, that is what is really happening here. So, I can write a generalized statement here.

So, what is the generalized statement here that I can write? I can say that any line passing through the origin is a subspace of $\mathbb{R}^2$. So, I am saying passing through the origin. So why passing through origin is important? Let us take another example. So, I will take an example. Again, $V = \mathbb{R}^2$, and I will take $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 + x_2 = 1 \right\}$.

So, now clearly, if you see the diagram, what is really happening? This is our $\mathbb{R}^2$, and $x_1 + x_2 = 1$. I am just having some intercepts. So, this is the line that I am speaking of. So, this is my W now; the line $x_1 + x_2 = 1$. And now, is this set closed under vector addition? So clearly, W is not closed under vector addition and scalar multiplication.

So, passing through the origin was really very important. So, why passing through origin is really very important? You can see that scalar multiplication closure is what we require. So, for every $x \in W$, $0.x = 0 \in W$. That means that if you want W to be a subspace, 0 be better inside W.

That is not guaranteeing that W is a subspace, but if 0 is not there, then you clearly know that W is not a subspace.

So, in this particular set, you can see that $0 \notin W$. So, there is no additive identity present in this set. So, this particular set is not a subspace. So, lines look like they are subspaces, but they have to satisfy a stringent property that they have to pass through the origin always.

So, if the lines are passing through the origin, then they are subspaces. The lines which do not pass through the origin are not the subspaces, but they are almost like subspaces. They just got translated away from the origin. So, we will have a special name for this particular set. We will call these sets affine spaces.

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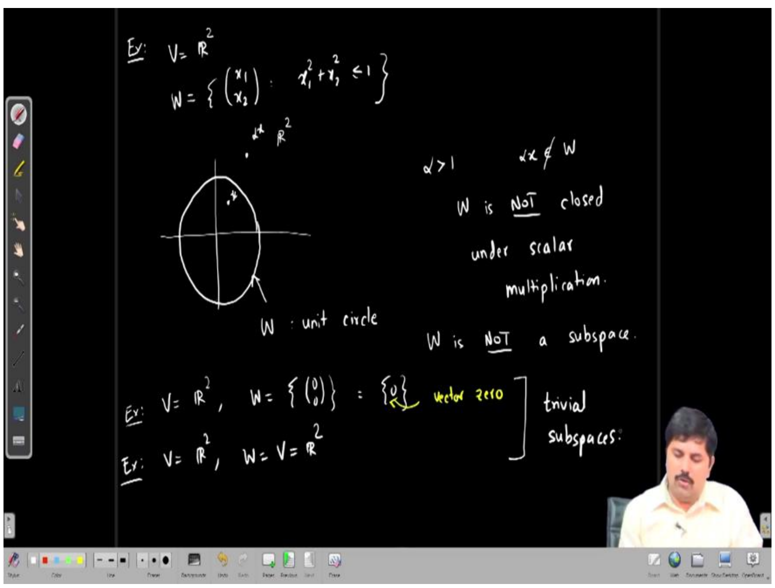

Let us take another example. So, another example is again $V = \mathbb{R}^2$, and I take set $W = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1^2 + x_2^2 \leq 1\right\}$. So, what is the geometry here? The geometry here is as follows: I am having the plane $\mathbb{R}^2$ here, and I am having the unit circle.

So, this particular set is W is a unit circle, and what is the region that I am considering? The region I am considering is inside the circle and the boundary of the circle. So, now the question is: Is the closure satisfied? Is the scalar multiplication satisfied? So, look at this particular vector. So, suppose this is the vector x.

So, this is the vector x that I have. Suppose I take $\alpha > 1$, then what is going to happen to αx? αx is going to be somewhere here. This is my point αx. That means $\alpha x \notin W$. That means W is not closed under scalar multiplication, correct? So, if W is not closed under scalar multiplication, then W is not a subspace.

So, W is not a subspace. Let us take a few more examples. Again, $V = \mathbb{R}^2$, and I take $W = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} = \mathbf{0}$. So, this $\mathbf{0}$ is a vector 0. So, is W a subspace here? W is trivially a subspace. It has only 0 elements. So, $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\alpha . \mathbf{0} = \mathbf{0}$.

So, this set is trivially closed under addition and scalar multiplication. So, this is a trivial example of a vector subspace. Similarly, I take $V = \mathbb{R}^2$, and $W = V = \mathbb{R}^2$. If I take this, then this also, because $V = \mathbb{R}^2$ is a subspace, trivially is a subspace. So, these two examples are actually examples of trivial subspaces.

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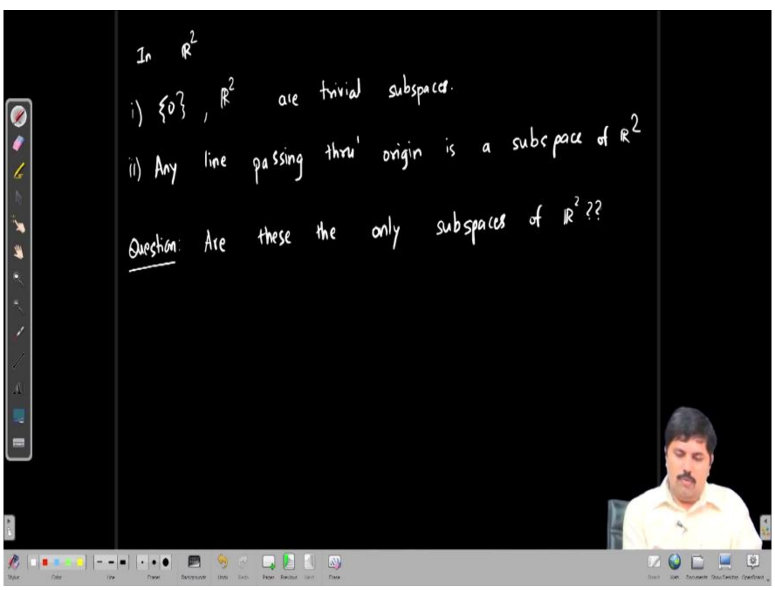

So, what we have seen so far in $\mathbb{R}^2$,? In $\mathbb{R}^2$, we have seen that $\mathbf{0}$ and $\mathbb{R}^2$ itself are trivial subspaces. Any line passing through the origin is a subspace of $\mathbb{R}^2$, correct? The question that we want to ask here is the following:

Are these the only subspaces of $\mathbb{R}^2$, or does there exist a subspace that I have not enlisted yet? So, that is the question that we want to answer, and possibly that is the question we will be able to answer in the next lecture.

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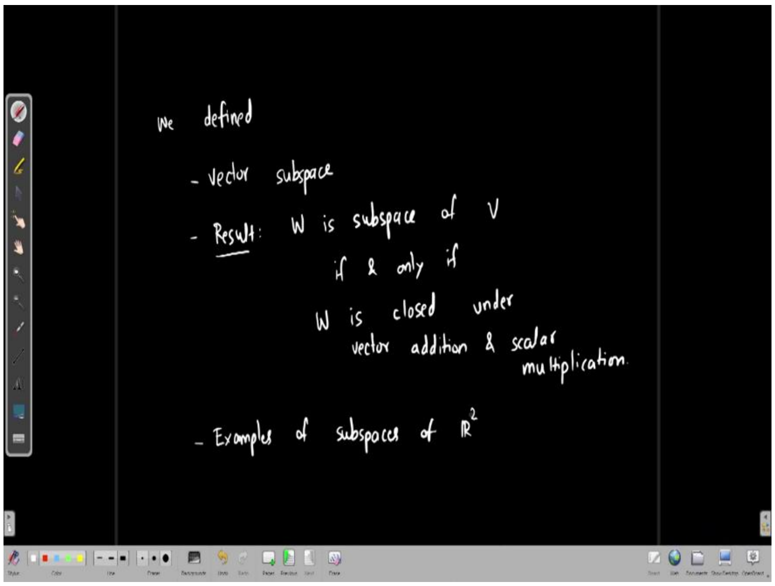

So, summarizing what we have done today, we defined the vector subspace of a given vector space, and then we had a result. We proved a theorem. We had an argument given in the class where we showed that W is a subspace of V if and only if W is closed under vector addition and scalar multiplication, and the third point was that we saw several examples of subspaces of $\mathbb{R}^2$.

So, in the next class, we will continue with these examples of subspaces of $\mathbb{R}^n$, and we will answer this question, especially in the case of $\mathbb{R}^2$, whether the listed subspaces are the only subspaces of $\mathbb{R}^2$ or not. Thank you.