Advanced Computational Techniques
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Lecture no. 19
Hyperbolic (PDE)

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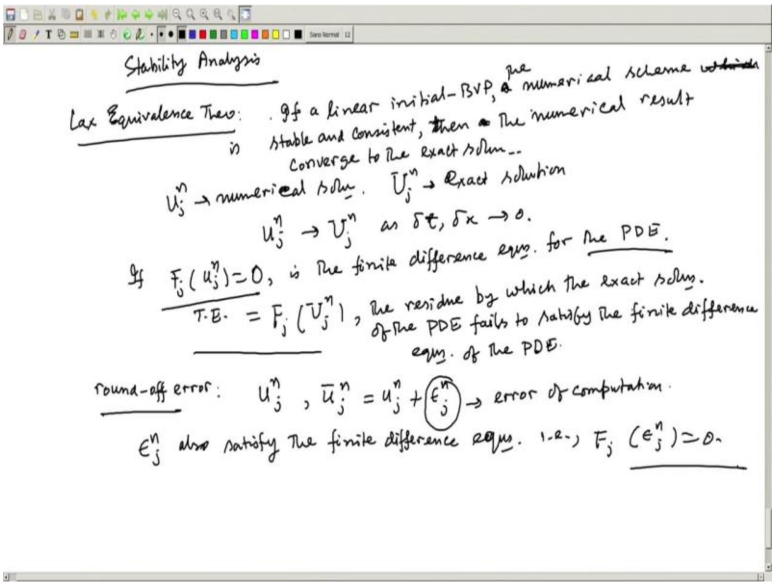

Now, so, what we said in the previous one that if it is a Lax equivalence theorem that for a linear this is very important to know that it should be linear for a linear initial boundary value problem BVP a numerical scheme which is stable and consistent yields a, then a converged solution then if any numerical semantic I should write in this if a numerical scheme if a linear initial boundary value problem, the corresponding numerical scheme is stable and consistent then the numerical result converge to the exact solution.

So, basically if I call u_j^n is the numerical solution and U_j^n is the analytic solution or exact solution which is not known. So, what we need is this $u_j^n \rightarrow U_j^n$ as $\frac{\delta y}{\delta x} \rightarrow 0$. So, that is very important to establish that, note that accuracy is one thing, but, whether the numerical solution converges to the exact solution testing that is very important.

Otherwise, the solution what we are obtaining by a numerical scheme may lead to a solution of some other PDE so a correspondence 1 to 1 correspondence between the numerical scheme and the partial differential equation for which we are expressed the numerical scheme is very important to establish. Now, truncation error as if we say that if we denote say

$F_j\,(u_j^n) = 0$ is the finite difference equation for the PDE.

Let us say for the PDE whatever the PDE so then truncation error we define as

T.E. = $F_j(U_j^n)$

So, that means the residue by which the exact solution of the PDE fails to satisfy the finite difference equation of the PDE. Corresponding to this PDE we have certain time level n. This is the finite difference equation which we are solving for the numerical solution.

So, the truncation error is the one which by the way when I replaced the numerical solution by the exact solution of the PDE so whatever the residue amount that represents the truncation error, is very important to know. So, that gives the merit of the numerical scheme the second order accurate, first order accurate so that the merits of how accurate we can expect accuracy of the solution that is measured by the truncation error.

So, always we have to determine the order of the truncation error now, when we are exactly implementing the numerical scheme, so, what do we find that because of this round off error that is most important, so, one of the error is the round off error and several other situation. So, instead of finding the u_j^n we are obtaining a solution $\bar{u}_j^n$ so that is

$$\bar{u}_j^n = u_j^n + \xi_j^n$$

So, this is the error arise of competition so, when we are implementing the numerical scheme, so, the error is committed and that error that can be most part of the error can be by the round off error. So, it denotes this as ξ_j^n. Now, since we are talking about a linear system so what we can say that this ξ_j^n also satisfies the finite difference equation so that means, that also satisfies this $F_j(\xi_j^n) = 0$, a linear finite difference equation because we are assuming it is a linear equation.

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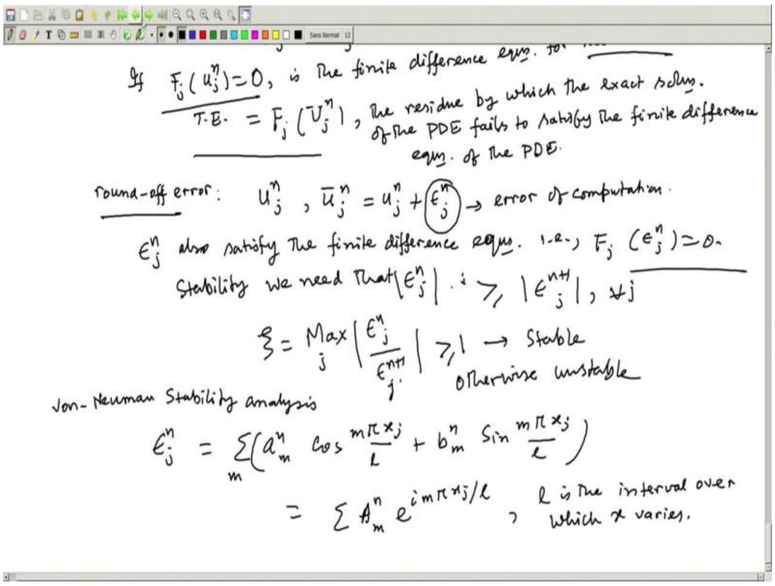

Now, there are several ways to test that support stability what we need for stability we need that

$$|\xi_j^n| > |\xi_j^{n+1}| \quad \forall j$$

So, as time progresses and the next step and onwards. So, this should be decay or remain constant so may call an amplification factor, factors as

$$\xi = Max_j \left| \frac{\xi_j^n}{\xi_j^{n+1}} \right| \geq 1$$

the amplification factor ≥1 so if this is the 1 then it is stable otherwise unstable as simple as that.

Now, join the stability analysis basically what we need to do is? To check whether the error is growing or decaying or it is remaining the same. So, if it is decaying or remaining same it is a stable numerical scheme. Now to check that this von-Neumann stability analysis that is the most popular one are very simple one, so what is done is this ξ_j^n is expressed by a Fourier series.

So, because this is a discrete function and it is bounded we are talking about ξ_j^n it is not a unbounded value and it is valid only on the discrete points so if it is bounded within certain interval and then we can say is the function of bounded variation so we can say that it is integrable, if it is integrable we can express in a form of a Fourier series so

$\xi_j^n = \sum_m a_m^n \cos\frac{m\pi x_j}{l} + b_m^n \sin\frac{m\pi x_j}{l}$

So, this can be written in a complex form as

$\xi_j^n = \sum A_m^n e^{i\frac{m\pi x_j}{l}}$, l is the interval over which x varies.

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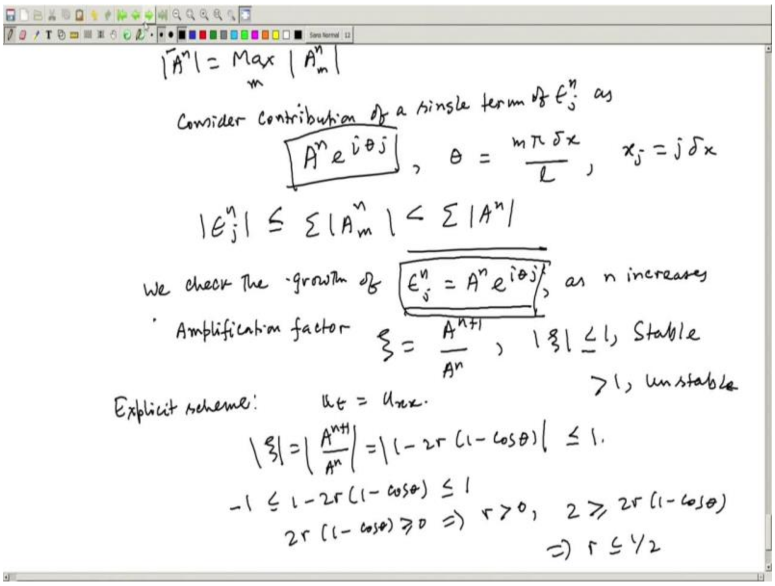

Now, so, this is a way an amplitude is n now, if I choose

$|A_n| = Max_m \, |A_m^n|$

Now, thing is this ξ_j^n satisfy a linear equation linear finite difference equation. So, if I want to know the contribution ξ_j^n so which is a summation of all these individual terms. So, instead of taking the full summation I can consider the single term and we consider term which is having the maximum amplitude that is maximum value as A_n.

So, if this is the one so we consider we consider contribution of a single term of $\xi_j^n = A^n e^{i\theta j}$,

let us define $\Theta = \frac{m\pi\delta x}{l}$ because we are writing as $x_j = j\delta x$, so A_n, I, Θ_j. So, if I find out that, so, $|\xi_j^n| \leq \sum |A_m^n| < \sum |A^n|$, maximum value we have taken.

So, one of the term we choose and we see that whether these terms so that means, what we do is we check the growth we check the growth of A^n so, let us call this as the we can denote this as the error ξ_j^n = A ^{n} $e^{i\theta j}$, grow up each time or maybe let us call this a ξ_j^n is the error so that is the summation of that growth with successive time or as n increases.

And we define an amplification factor we define an amplification factor

$$\xi = \frac{A^{n+1}}{A^n}$$

no need to be modulus here. So, if | ξ | ≤ 1 stable. So, that means the growth is decaying and it is unstable. So, that means the scheme is not reliable, if we consider the growth is enhancing .

So what is the working procedure that the ξ_j^n satisfies the finite difference equation because it is the linear equation and ξ_j^n is expressed as a summation of individual term A ^{n} $e^{i\theta j}$. So, that means, each of the term also satisfy the sum equation and the summation can be taken out and so, that means, the finite difference equation now, we replace by the finite difference equation at time level in our what is obtaining at n + 1 or deleting n by this term and see the growth of A.

So for example, if we have the explicit scheme say so, what we have is $u_t = u_{xx}$ so, I define find that | ξ | = | $\frac{A^{n+1}}{A^n}$ | = | (1 - 2 r) (1 - cos Θ) | ≤ 1this can be checked and what do you find that this should be ≤ 1.

So, that means - 1 ≤ (1 - 2 r) (1 - cos Θ) ≤ 1. So, that means 2 r (1 - cos Θ) ≥ 0 ⇒ r > 0 and also another condition we get 2 ≥ 2r (1 - cos Θ) the other inequality so, this implies r ≤ 1/2. So, that implies that the numerical scheme is conditionally stable. So, this is how the stability is justified.

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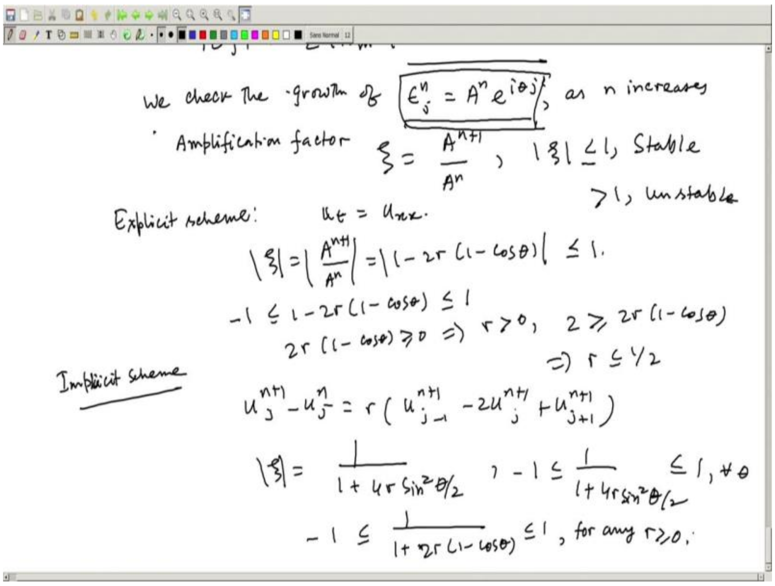

Similarly, if we go for the implicit scheme so implicit scheme we have already said so, which was

$$u_j^{n+1} - u_j^{n} = r\left(u_{j-1}^{n+1} - 2\, u_j^{n+1} + u_{j+1}^{n+1} \right)$$

So, if I substitute this, we get

$$|\xi| = \frac{1.}{1+4r\sin 2_{\theta/2}} \Rightarrow -1 \le \frac{1.}{1+4r\sin 2_{\theta/2}} \le 1, \forall \Theta$$

has to happen and what we find that this

$$-1 \le \frac{1.}{1+2r(1-\cos\,\Theta)} \le 1\ ,\ r \ge 0$$

and this is happening for any choice for any $r \ge 0$, this is happening. So this is how the numerical scheme are checked for stability.

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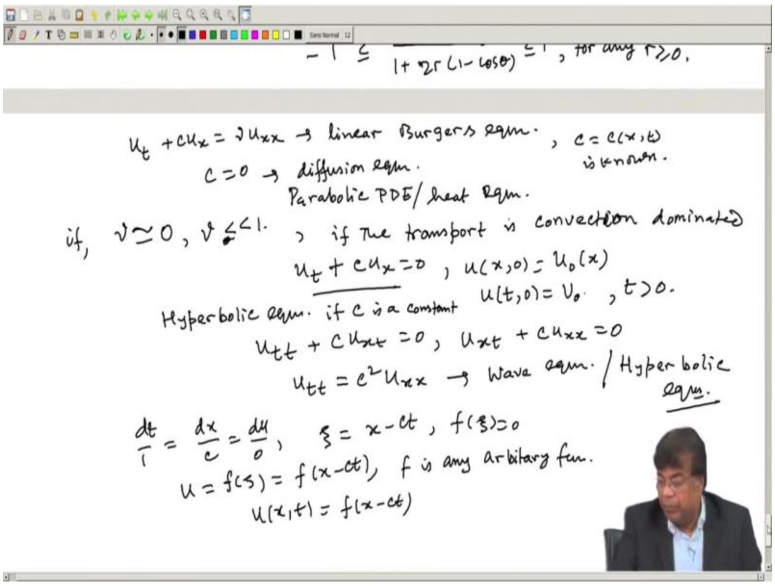

Now, we talk about the other extreme. So far in the advection, diffusion equation. So, that means, we started with this equation

$u_t + c\, u_x = \mu\, u_{xx}$ which is the linear Burgers equation. So, now, there we have considered linear of course, c is known see some $c = c(x,t)$ is a known constant. So, the previous case what we have taken is $c = 0$ so, that was a pure diffusion equation, division dominated equation is also called the parabolic PDE or heat equation.

Now, if the other case if the division is 0 if $\vartheta \approx 0$ okay or $\vartheta << 1$ that is if the transport is convection dominated that means,

$$u_t + c\, u_x = 0$$

this very looks like a simple equation. Initial condition is $u(x,0) = u_0(x)$ and what we need is only a single boundary condition say some situations $u(t,0) = \vartheta_0$ for $t > 0$.

Now, these equations are also referred as hyperbolic equation why? So if I differentiate with respect to t so, c if I take constant if c is constant to $c\, u_{xt} = 0$ and similarly, if I take differentiate with respect to x equals 0

$$u_{tt} + cu_{xt} = 0\ ,\ u_{xt} + cu_{xx} = 0$$

so if I now, eliminate u_{xt}, what I find is if I multiply by these and

$$u_{tt} = c^2\, u_{xx}$$

so, this is a wave equation or hyperbolic equation.

So, we are not going into details of this. Only thing is that what we see that the way hyperbolic equations and these equations are very important particularly for studying the any advection process like gas dynamics that is where the diffusion is very small. So, that means, well away from the boundary, where the viscosity effect is very low so then any transport process is mostly governed by the advection pure advection process.

Now, there is a difficulty in this if you do little analysis. So, the solution of this advection equation, if I write, one can write that auxiliary equation

$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}$, ξ = x - ct , f (ξ) = 0

and from there what we can write if ξ if I say the integral is x - ct. So, then what do I find that f (ξ) = 0 so, that means, the solution is so, solution can be written as u is a function of any arbitrary constant ξ .

u = f (ξ) = f (x – ct) , where f is any arbitrary function

what does it say that u though is a function of x and t but it is behaving like x – ct

u (x,t) = f (x – ct)

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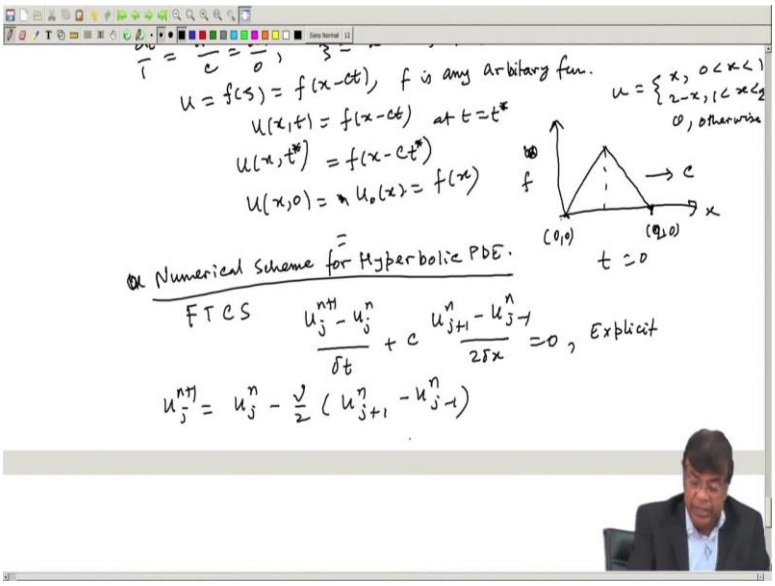

So that means, whatever the solution at certain t^*, so, at $t = t^*$

$u(x,t^*) = f(x – ct^*)$

so, that means u(x,0) which is given to some value some form what if some forms whatever we have written u(x,0) = u_0 (x) = f (x)

So, that means, we know the form of the f (x) and so, say at t = 0 this is your u, u is this is x is some, some functions like this. So, say

$$u = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

So, this is the one this is (2, 0), (0, 0). So, what we will find that at time progresses since f is also behaving the same way so as time progresses it will be just a function of (x – ct). So, without any deformation, this will propagate with a constant velocity c so that is the reason it is called the wave equation.

Now, solving this way we question numerically it's a challenge and so, there are several ways to solve. So, we have this equation, numerical scheme for hyperbolic equation so, if we use the FTCS can be used very simply. So, that means, you have

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\delta x} = 0$$

this is the FT forward time ,this is the explicit and it has to be explicit because we do not have the knowledge of u at the other end.

So, that is why is need to be kept as explicit scheme. If this is the case very simple FTCS central difference scheme. So, $u_j^{n+1} = u_j^n - \frac{\vartheta}{2}(u_{j+1}^n - u_{j-1}^n)$

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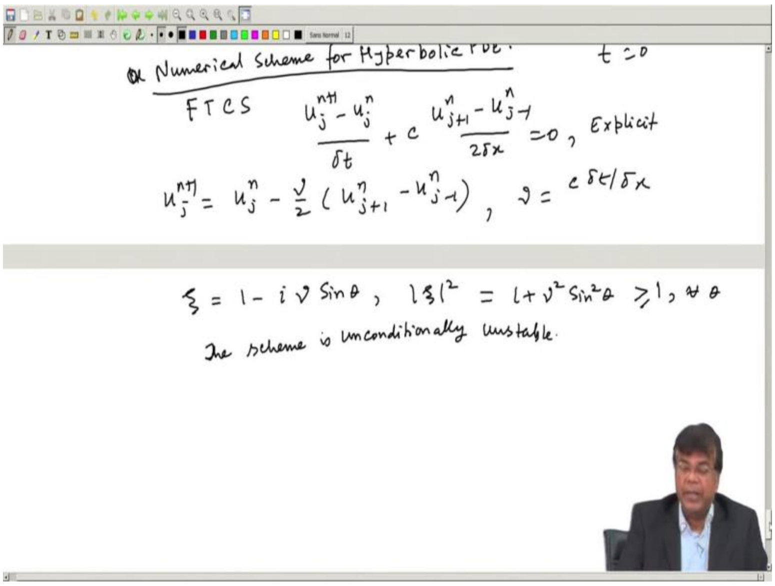

So, now, if I check for stability so, what we find that

$\xi = 1 - i\,\vartheta \sin \Theta\,,\ |\,\xi\,|^2 = 1 + \vartheta^2 \sin^2 \Theta \ \geq 1, \forall\, \Theta.$

$\vartheta = c\,\frac{\delta t}{\delta x}$

So, that means, this scheme is unconditionally unstable.

So, no compromise of any it is that any numerical value of ϑ for which it is stable that is not possible. So, FTCS is straight away unstable. So, one cannot have a solution by FTCS.

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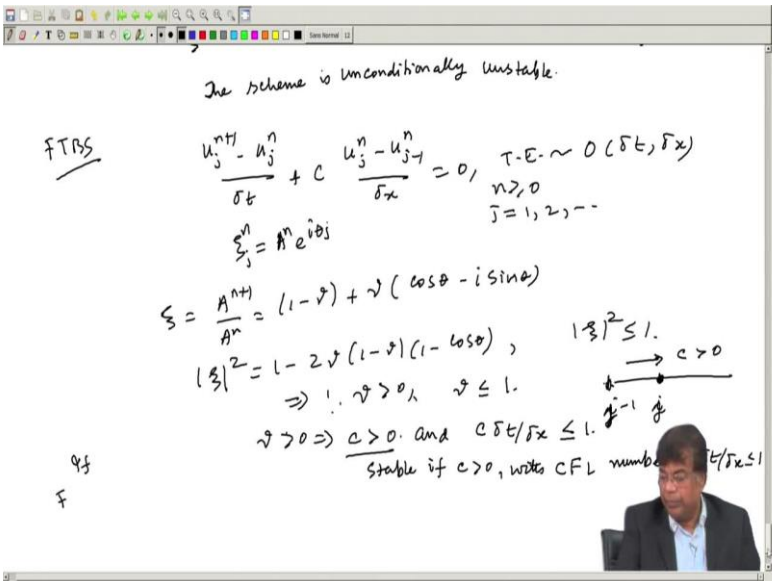

Then what we do is we try for FTBS so FTBS means, we have to have FT forward time, because we are going from n to n + 1 and this is

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\,\frac{u_j^n - u_{j-1}^n}{\delta x} = 0$$

and obviously, the truncation error is first order T.E. ~ O(δt,δx) and for n ≥ 0 we can proceed from j =1,2,… to whatever the we feel like.

So, if this is the case so again if we do by von Neumann stability analysis and ξ_j^n = A ^{n} $e^{i\theta j}$, if I substitute then what they find that $\xi = \frac{A^{n+1}}{A^n}$ = (1 – ϑ) + ϑ (cos Θ - i sin Θ). So, what we get is | ξ |^2 =1 - 2 ϑ (1 - ϑ) (1 - cos Θ)

So, what I need is | ξ |^2 ≤ 1. So, what do you find that this will happen this implies that | ϑ |≤ 1 so that is stable another thing is that this has to be also positive because the other condition so ϑ > 0 and ϑ ≤ 1.

So, that means ϑ > 0 ⇒ c > 0 and the conditional stable so, that means, c $\frac{\delta t}{\delta x}$ ≤ 1 so this came is stable if c > 0 with CFL number this is also called the CFL number, Courant Friedrichs Lewy's number, c $\frac{\delta t}{\delta x}$ ≤ 1.

Now, if c < 0 then we have to if so, this numerical scheme FTBS is stable provided c > 0. So, that means it is propagating in the forward direction. So, that means, here at any level n so here using this is i, i sorry this is i – 1, i so that means, we are going in the forward ahead of so at the step sorry not i, j at the step j we are using the solution from the previous step j - 1.

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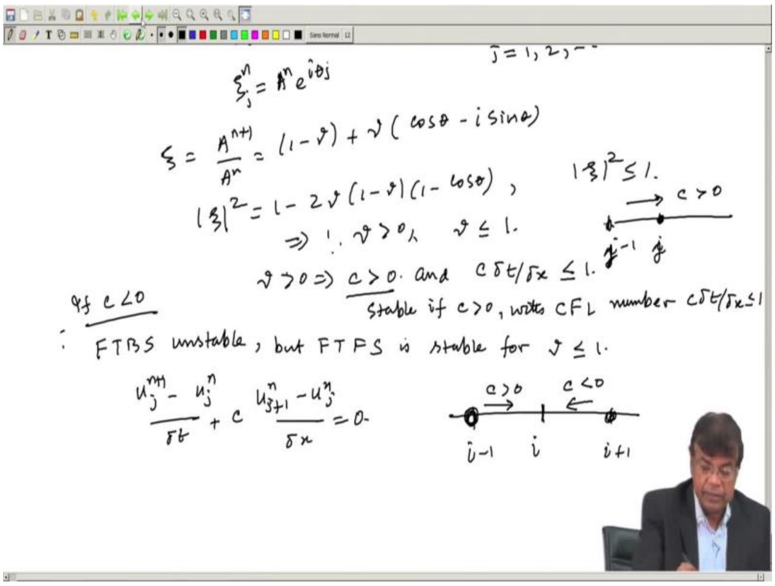

Now, if c < 0, so, that means, the information is coming from the other side. So, in that case FTBS is unstable. But, FTFS is stable is conditionally stable, stable for how we say that $\vartheta \leq 1$. So, that means FTBS means

$$\frac{u_j^{n+1} - u_j^n}{\delta t} + c\,\frac{u_j^n - u_{j-1}^n}{\delta x} = 0$$

So, this came is stable forward difference.

So, that means, what we have here? This is i, i - 1 this is i + 1. So, if c > 0 then we take this way, we are using the value of i - 1 and if c < 0, then we are using the value as this so this procedure.

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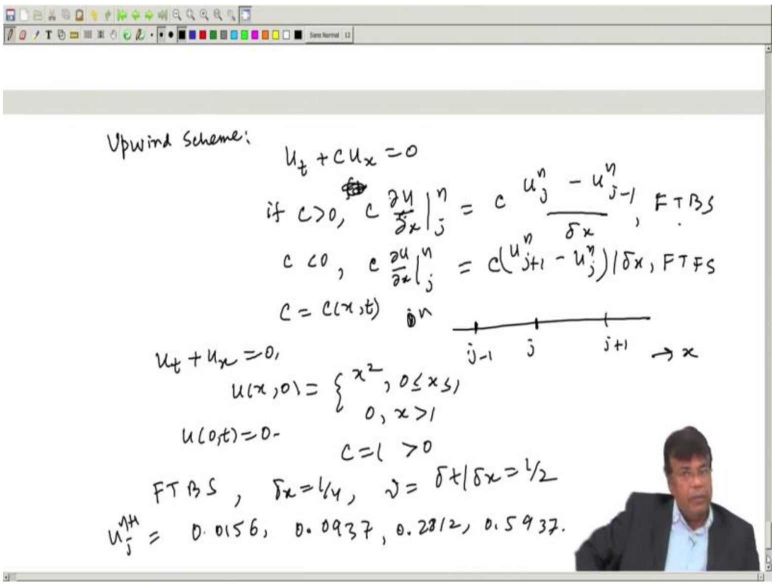

So, that means, what we can conclude here is the this is called the upwind scheme. So, that means

$$u_t + c\,u_x = 0$$

stable upwind scheme so if then, if c > 0 then you write so, then you discretize

$c > 0$, $c\,\frac{\partial y}{\partial x}\Big|_j^n = c\,\frac{(u_j^n - u_{j-1}^n)}{\delta x}$, that means, FTBS

and if c < 0 then in that case you have to consider for the stable scheme

$c < 0$, $c\,\frac{\partial y}{\partial x}\Big|_j^n = c\,\frac{(u_{j+1}^n - u_j^n)}{\delta x}$,if it is a FTFS.

So, that means, depending on the c or this is also called the wind direction. So, know which direction the wind is flowing. Now, c can be a function of c =f(x,t). So, it can change the value c at different at a particular n so it can depend, vary the sign can change for c as we progress in x, So, if we find that c > 0 we discretize this by backward difference if it is c < 0 we discretize by forward difference, that is all is the for stability.

So, for example, say $u_t + u_x = 0$ and you have $u(x,0) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$ and u(0,t) = 0 should be as the c =1 > 0. So, we have to use a FTBS if I choose δx = 1/4 and the Courant number as $\vartheta = \delta t / \delta x = 1/2$.

So, if I choose that way, so, what do I find that this the solution is one can find out the solution as u_j^{n+1} = 0.0156, 0.0937, 0.2812, 0.5937 so, these are all the first order accurate numerical scheme, but one need to be very careful about how the discretization is met. That means, we have to discretize in such a way that the direction of the propagation so if c is positive we take backward and if t is negative we take them forward difference.

So that is how the hyperbolic system, hyperbolic type of equation has to be discretized. So, any central dividend scheme that is higher order lead to a numerical instability so that converge the error can be improved truncation error can be improved by taking several other type of numerical scheme some modification of that, but for the first order this is the situation that we have to go by upwind manner. So, that is it about what today we will meet next time. Thank you.