1Advanced Computational Techniques
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Lecture 17
Non - Linear (BVP), Iterative Method

The continuation of non-linear boundary value problem so, we are talking about Iterative methods and based on the Newton's linearization technique in which the non-linear discretized equation will be solved in a iterative fashion so at every iteration we upgrade the solution and then we continue the process till we get the convergence.

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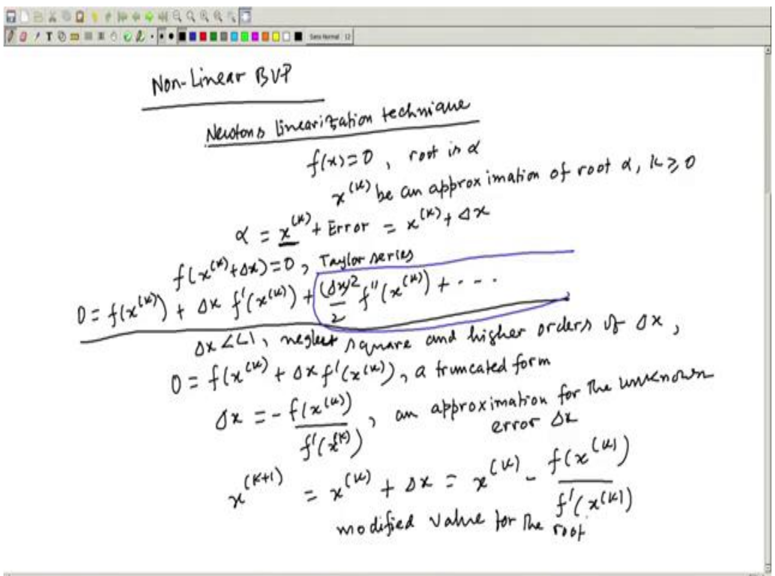

So, basic idea in Newton's linearization technique is the same as the one what we adopt to solve a non-linear algebraic equation at x range any algebraic equation for finding a root of say any equation f (x) = 0.

So, what we do is that if the root is α then we start with an approximate value and $x^{(k)}$ be an approximation of this root α for some k ≥ 0 then what we say that α is can be said as

$$\alpha = x^{(k)} + \text{Error} = x^{(k)} + \Delta x$$

So, that means, it will solve the governing equation

$f(x^{(k)} + \Delta x) = 0$ identically now, if we expand by Taylor series, so what we can write is

$$0 = f(x^{(k)}) + \Delta x f'(x^{(k)}) + \frac{(\Delta x)^2}{2} f''(x^{(k)}) + \ldots \text{ and so on.}$$

Now, if we assume that $\Delta x << 1$ is very small quantity and neglect in this series because now this series is an infinite degree polynomial in Δx and all these $x^{(k)}$, $f'(x^{(k)})$, $f''(x^{(k)})$ are known quantity because $x^{(k)}$ is known value now, with the knowledge of this $x^{(k)}$ one can find out theoretically $f(x^{(k)})$ all its derivatives and an infinite degree polynomial can be constructed.

Now, solving an infinite degree polynomial for the error Δx is an impossible task. So, what we do we now drop these terms which is involving square and higher orders. So, if we now neglect all these the square and higher orders of Δx then we can write these as approximately an approximate equation

$$0 = f(x^{(k)}) + \Delta x f'(x^{(k)})$$

truncated form, from there we find an approximate value of Δx as

$$\Delta x = -\frac{f(x^{(k)})}{f'(x^{(k)})}$$

an approximate value and approximation for the unknown error Δx.

Obviously this is not exactly the same Δx or the error value what we are looking for, this is an approximation. Now, if I add to the previous whatever the guess value or whatever the approximate value already obtained for the root $x^{(k)}$ so then we call this the modified value of the root and this is Δx.

$$x^{(k+1)} = x^{(k)} + \Delta x = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} \qquad - (*)$$

So, that means, this is an modification so, modified approximation ,modified value for the root or a better approximation compared to the $x^{(k)}$.

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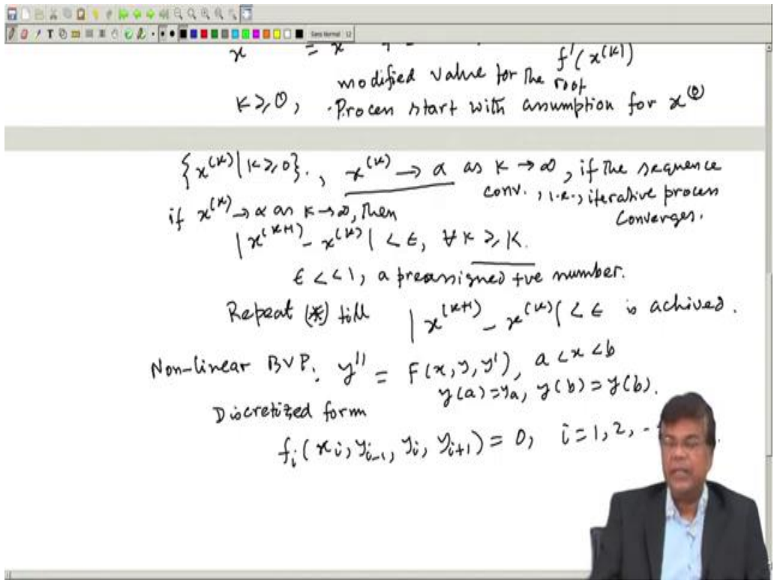

So, what we do is repeat this process so $k \geq 0$, so, and we start the process starts with an assumption for $x^{(0)}$.So, that means, in this process we develop a sequence of iterates $k \geq 0$ and we need that this $x^{(k)} \to \alpha$ as $k \to \infty$, if the sequence converge, if the iterative process converges and in that case we call the sequence converge and that is the iterative process converges.

Now, testing this is not possible because we do not have the idea what is the value of α. So, that is why what we do is we apply the Cauchy's criteria for convergence that is, we know that if it is stated that the sequence converges, then we can say that x_{k+1} if $x_k \to \alpha$ as $k \to \infty$ then

$|x^{(k+1)} - x^{(k)}| < \xi$ for all $k \geq K$.

This is the criteria for Cauchy's principle of convergence so that means, after few iteration leaving aside few iteration.

So, if it the process is in the convergent state, then the difference between the successive iterates are small. So, $\xi << 1$, pre-assigned was positive number through the iteration process so we repeat the process. Repeat (*) till this happens $|x^{(k+1)} - x^{(k)}| < \xi$. W\hen this is achieved then this is the approximate value of the root $x^{(k)}$.

So, this is the basic principle for the Newton method for solving a non-linear algebraic equation the same principle we are applying here. So, when you have a non-linear boundary value problem, so, for a non-linear BVP say, as we discussed in the last class say

$y'' = F(x, y, y')$, $a < x < b$ with condition as $y(a) = y_a$,$y(b) = y_b$ these values are given.

So, if we discretize this equation, we can write the discretized form by central difference scheme. So if I use the central difference scheme, three points we are using .So, in general, we will have a situation like this

$f(x_i, y_{i-1}, y_i, y_{i-1}) = 0$,for $i = 1, 2, \ldots, n - 1$

So, this f_i is non-linear .

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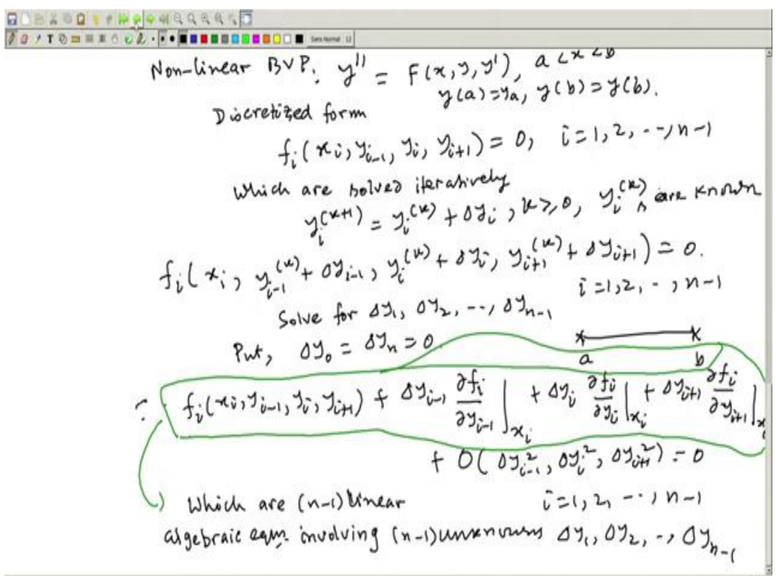

So, we solve this system by iterative process, which are solved iteratively at every iteration $y_i^{(k+1)}$ we write as

and then we substitute these and retain only up to the linear order. So, that means, now we substitute this we say that the better approximation for the values we get,

$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i$, $k \geq 0$

$y_i^{(k)'}$ s are known either we are starting from a initial approximation for k =0 or after certain stage of iteration we update this $y_i^{(k)}$ if K is higher than 0.

So, if I substitute, we get a form like this way

$f\,(x_i\,,\, y_{i-1}^{(k)} + \Delta\, y_{i-1}\,,\, y_i^{(k)} + \Delta\, y_i\,,\, y_{i+1}^{(k)} + \Delta\, y_{i+1}\,) = 0$, I =1,2,…, n-1

So, this set of equations for i =1 to n - 1 need to be solve for these unknown $\Delta\, y_1$, $\Delta\, y_2$,…,$\Delta\, y_{n-1}$ now, since the at the two endpoints x_0 and x_n the values are prescribed so we can put since, this is already given so $\Delta\, y_0 = \Delta\, y_n = 0$ can be considered because no iteration is required at the two ends of two endpoints at the two extremes or the two boundary a,b so, there is no iteration involved.

Now, if I expand last class as we said expand by Taylor series and retain only the linear order because we need to solve this system of equation for $\Delta\, y_i$'s and most simple situation is the linear one so if I expand by Taylor series we get

$$f_i\,(x_i\,,\, y_{i-1}\,,\, y_i\,,\, y_{i+1}) + \Delta y_{i-1}\,\frac{\partial f_i}{\partial y_{i-1}}\,|\, x_i + \Delta y_i\,\frac{\partial f_i}{\partial y_i}\,|\, x_i + \Delta y_{i+1}\,\frac{\partial f_i}{\partial y_{i+1}}\,|\, x_i + O\,(\Delta y_{i-1}^2, \Delta y_i^2, \Delta y_{i+1}^2\,) = 0 \qquad - (*)$$

these are the variables because x_i is the independent variable. So, we are treating this f_i as a function of three variables.

So, we can approximately write it for i =1,2,…,n - 1 now, this form set of equations, a linear set of equations which are (n – 1) linear algebraic equation involving (n – 1) unknowns Δy_1 ,$\Delta\, y_2$,…,$\Delta\, y_{n-1}$ .So we solve for these linear equations for Δy_1 and Δy_{n-1}.

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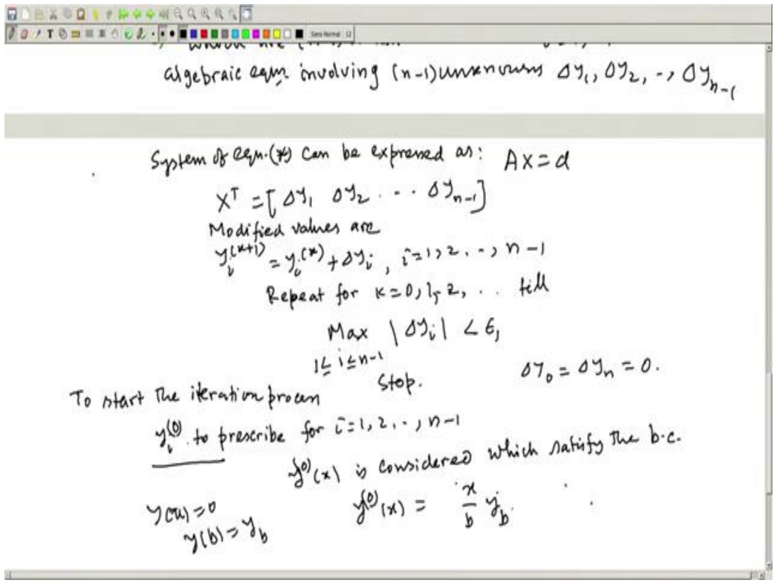

So, if we solve this linear system, let us call this as (*).This will form a tri- diagonal, which can be or (*) can be expressed in a tri-diagonal set of equations, because every equations are involving three variables.

So, system of equation (*) can be expressed as AX = d for some, $X^T = [\Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}]$ these variables so if I now solve this I get this x and then next approximation modified values are

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i, \ i = 1, 2, \ldots, n - 1$$

which is already obtained and the error we add for i =1 to n - 1.

So, the same principle repeat for k =0,1,2,… etc till we get that maximum of all, say, $\max\limits_{1 \le i \le n-1} |\Delta y_i| < \xi$.

So, ξ is a pre assigned positive number. We repeat the process till we get the convergence criteria then stop. So, then we achieve the convergence, this is the basic principle for solving the non-linear boundary value problem.

So, as we said here we have taken $\Delta y_0 = \Delta y_n = 0$ to start the iteration so we need to start the iteration process we need a guess value for x_0, we need $y_i^{(0)}$ to prescribe for i =1, 2,…, n - 1, so, that is k = 0 need to be prescribed. Now, this method does not say that what should be the case value for $y_i^{(0)}$.

So, there can be several ways. Suppose I know the solution of the linear part of the boundary value problem. So, some approximate solution is already known so that can be taken as the initial approximation or initial guess for this y_i and through that one can start the process or so, in general what we do is we take as a function form in such a way $y^{(0)}(x)$ is considered that is minimum which satisfy the boundary conditions.

So, $y^{(0)}(x)$ can be taken as a linear interpolation between these two, for example, I can take as no linear interpolation means this will be not the case so one can take in several ways so one of these is in such a way that we have the

$y^{(0)}(x) = (x – a) (b – x) y_b$.

So, that means if I put a this is becoming y we can take it as y_b say for example, if we have say y(a) for this is a very simple way so say y(a) = 0 and y(b) = y_b say some value so I can just take $y^{(0)}(x)= \frac{x}{b}\ y_b$. So, that is one of the simple way of considering. So x = 0 it is the same way we can proceed for non-zero situation.

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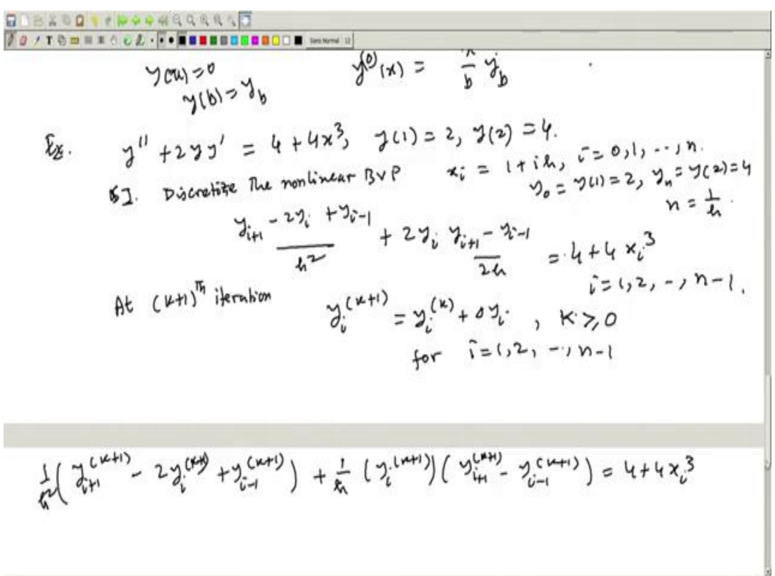

Now, let us take one example of non-linear boundary value problem .

$y'' + 2yy' = 4 + 4x^3$ and y (1) =2 , y (2) =4 this is the boundary condition is prescribed like this way. So, first step, discretized this equation discretized the non-linear BVP straight away so, if I discretized this

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{y_{i+1} - y_{i-1}}{2h} = 4 + x_i^3 \ , \ i = 1,2,3,\ldots,n-1$$

So, first of all I choose this $x_i = 1 + ih$, $i = 0,1,2,\ldots,n$. So, what we have given is

$$y_0 = y(1) = 2 \quad , \ y_n = y(2) = 4 \text{ and } n = \frac{1}{h}$$

that is the integer n up to so you choose either n or h and find out the step size if I define so accordingly the number of grid points is defined by this manner. So, this is $\frac{y_{i+1} - y_{i-1}}{2h}$ so, this is for $i = 1,2,3,\ldots,n-1$. So, this is the discretized equation and because of this term you have non linearity. So, how we proceed is that at (k + 1) iteration level we substitute so when we are at iteration so either $k \geq 0$

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i \ , \ k \geq 0$$

So, that means, we have already obtained either we start from k = 0 that is the initial approximation or we are at certain stage after a few iterations so this is all $i = 1,2,\ldots,n-1$.

Now, if I substitute in the governing equation, so,that means

$$\frac{1}{h^2} \left(y_{i+1}^{(k+1)} - 2y_i^{(k+1)} - y_{i-1}^{(k+1)} \right) + \frac{1}{h} \left(y_i^{(k+1)} \right) \left(y_{i+1}^{(k+1)} - y_{i-1}^{(k+1)} \right) = 4 + 4x_i$$

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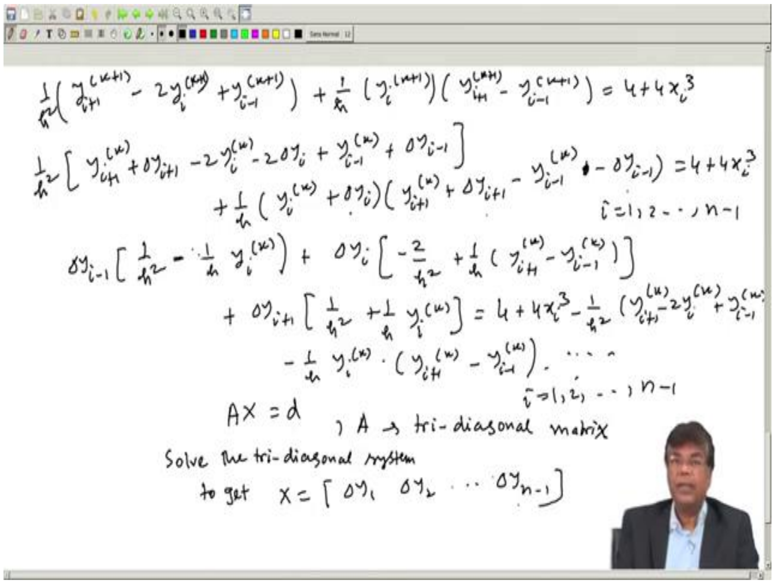

If I approximate this manner,

$\frac{1}{h^2}(y_{i+1}^{(k)} + \Delta y_{i+1} - 2y_i^{(k)} - 2\Delta y_i + y_{i-1}^{(k)} \Delta y_{i-1}) + \frac{1}{h} (y_i^{(k)} + \Delta y_i) (y_{i+1}^{(k)} + \Delta y_{i+1} - y_{i-1}^{(k)} - \Delta y_{i-1}) = 4 + 4x_i^3$,$i = 1,2,\ldots,n - 1$.

Now, instead of this, Taylor series expansion and all these things, we can straight away do the algebra from here you expand and drop all the square and higher orders, written only up to linear order terms.

So, if I do that, and equate the coefficients of Δy_{i-1} from the first term it is coming to be 1 -, because here only this Δy_i 's is the error and which are unknown so if I retain only the square and higher order terms are deleted and retain only the linear order terms so $1/h^2$ and from here, what I find that Δy_{i-1} is multiplying with because Δy_i ,Δy_{i-1} need to be neglected because that is a higher order so what we get from here is

$\Delta y_{i-1} [\frac{1}{h^2} - \frac{1}{h} (y_i^{(k)})] + \Delta y_{i+1} [\frac{-2}{h^2} + \frac{1}{h} (y_{i+1}^{(k)} - y_{i-1}^{(k)})] + \Delta y_{i+1} [\frac{1}{h^2} + \frac{1}{h} y_i^{(k)}] = 4 + 4x_i^3$

this Δy_i s is multiplying with this terms. Now, the cross product term that is $\Delta y_i \Delta y_{i+1}$, $\Delta y_i \Delta y_{i-1}$ are neglected.

these are all superscript k .So, whatever all these terms known terms we send to the right side so, that means, always superscript k can be sent to the right side because they are known

So, I think this is all the terms so this is $k \geq 0$ and for any $k \geq 0$ of course, we have said this before so, this is happening for $i = 1,2,\ldots, n - 1$ so this $(n – 1)$ system of equations so solve.

So, this can be written as $AX = d$ forms a tri-diagonal system which can be solved by Thomas algorithm and once we solve the matrix system, the tridiagonal system and we get if we solve this then to get X this variable $X = [\Delta y_1 , \ \Delta y_2 ,\ldots,\Delta y_{n-1}]$.

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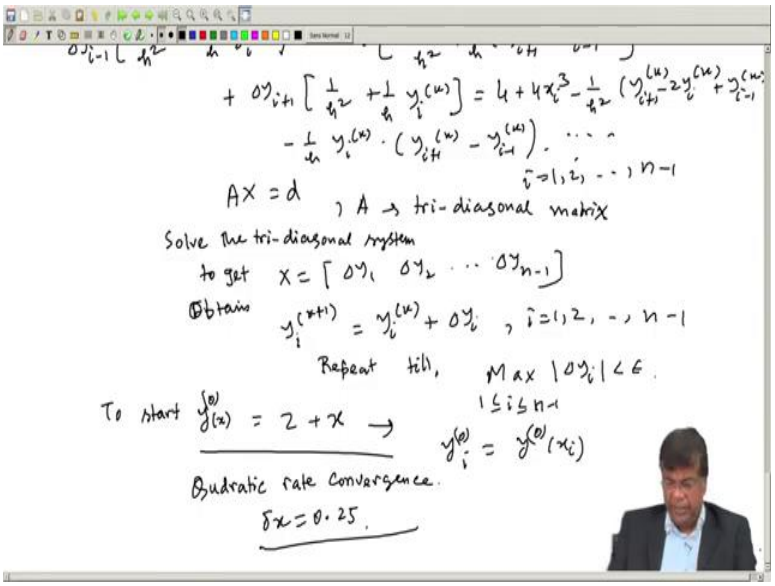

So, at this stage we obtain this. Once I get this x then we find out the

$$y_i^{(k+1)} = y_i^{(k)} + \Delta y_i \text{ , } i = 1,2,\ldots,n - 1$$

for all these $i = 1,2,\ldots,n - 1$ and we repeat till Δy_i maximum over all i because all these we have a set of equations set of variables so till we get

$$\max_{1 \leq i \leq n-1} | \Delta y_i | < \xi$$

so still this process is done.

So, repeat this to start the method as we said we have to have $y_i^{(0)}$ So, maybe we can take as say

$y_x^{(0)} = 2 + x$ because $x = 0$ this is 2 and $x = 2$ this is becoming 4. So, one of the choice is given by this way. So, this satisfies.

So, what is important thing to remember is that this $y_i^{(0)}$ .Now, because in the algorithm if we want to write a computer algorithm, so, if we want to give for step by step all the data points so then we have to fit the values in a tabular form. So, that is a difficult task.

So, instead if we give a analytic form like this way and then we can calculate all these things so $y_i^{(0)} = y_0(x_i)$. So, this is the easy way to fit the data points now, if this when the iteration process converge for any choice of the initial approximation, another advantage is for the quadratic rate of convergence for the Newton's method, that means, the error reduces quadratically.

So, what we can see that we are reaching the approximate value so if the error reduces in a quadratic fashion it will reduce in a very fast manner so, for example, if I choose $\Delta x = 0.25$ so one can solve this system of equations by the same manner. I think this is about the non-linear boundary value problem. So, this is the way we go for the two point boundary value problem in the iterative fashion. Thank you.