Advanced Computational Techniques
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Lecture 16
Linear Boundary Value Problems (BVP) (Contd.)

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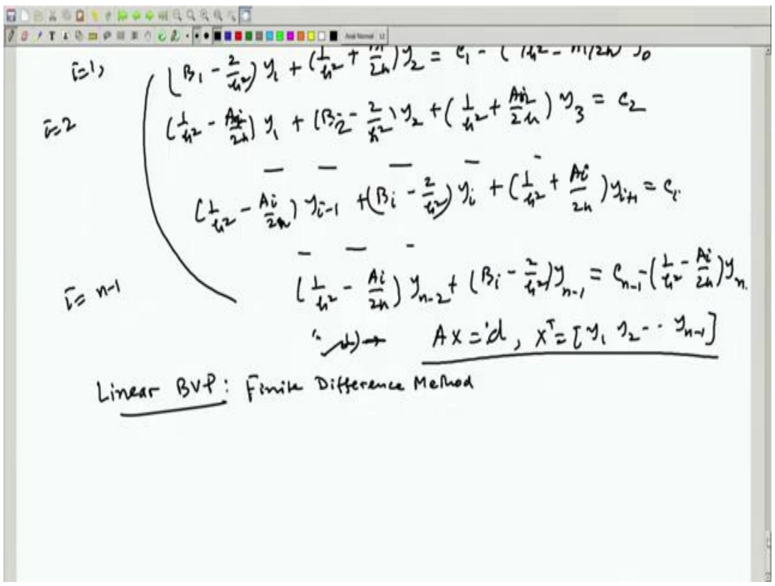

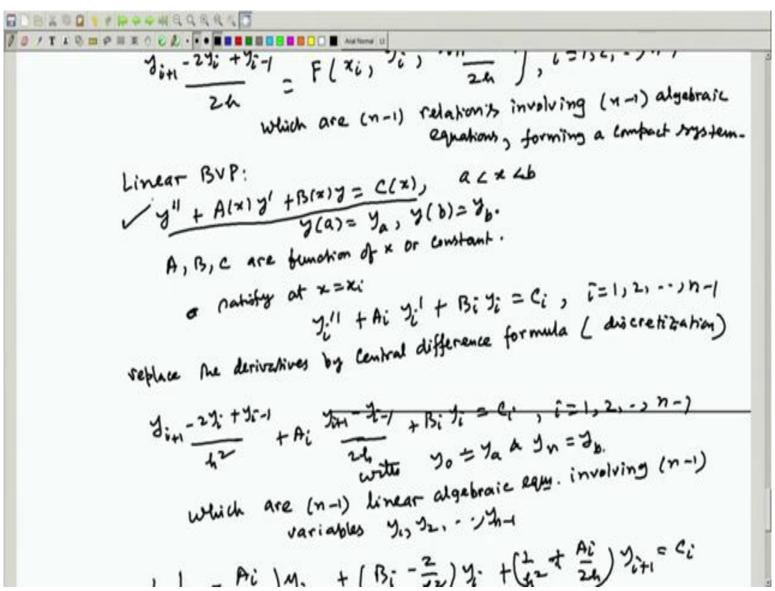

So, we are in the continuation of the linear boundary value problem solving a finite difference method. So, as we have shown that if you have a linear boundary value problem which is given by this manner that means, the coefficients of y and its derivatives are independent of or free from or rather it is only function of x or constant.

So, in that case, if we use the central difference approximation, we get a set of linear algebraic equations given by this manner which is involving (n – 1) equations and (n – 1) variables y_1 , $y_2,\ldots,y_{n-1}$.

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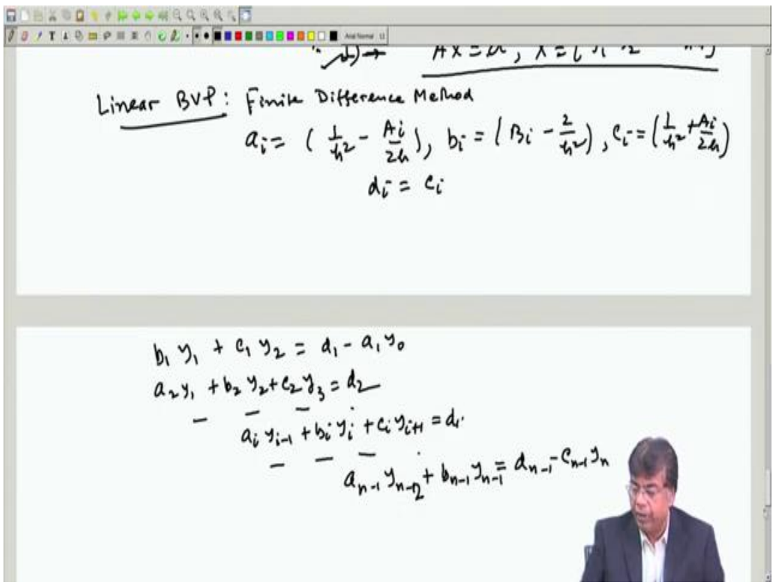

Now, if there is the pattern for this this linear system. If I define

$a_i = (1/h^2 - A_i/2h)$

$bi = Bi – (2 / h^2)$

$Ci = 1 / h^2 + A_i / 2h$

$d_i = C_i$

Then what we can say that that we can write this equation in this form

$b_1 y_1 + c_1 y_2 = d_1 - a_1 y_1 = \overline{d_1}$

$a_2 a_1 + b_2 y_2 + c_2 y_3 = d_2$

and likewise any arbitrary ith equation

$a_i\, y_{i-1} + b_i\, y_i + c_i\, y_{i+1} = d_i$

And the last one is

$a_{n-1}\, y_{n-2} + b_{n-1}\, y_{n-1} = d_{n-1} - c_{n-1}\, y_n$

because these y_n, y_1 and y_0 are unknown .

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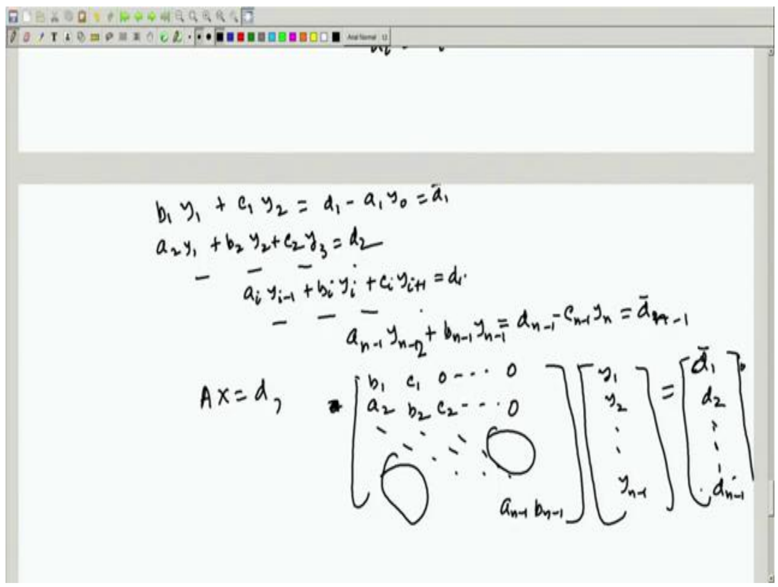

Now, if I introduce a coefficient matrix A X = d, so, this coefficient matrix becomes

$$\begin{bmatrix} b_1 \;\; c_1 \;\; 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n-1} \;\; b_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ \cdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} \overline{d_1} \\ \ldots \\ d_{n-1} \end{bmatrix}$$

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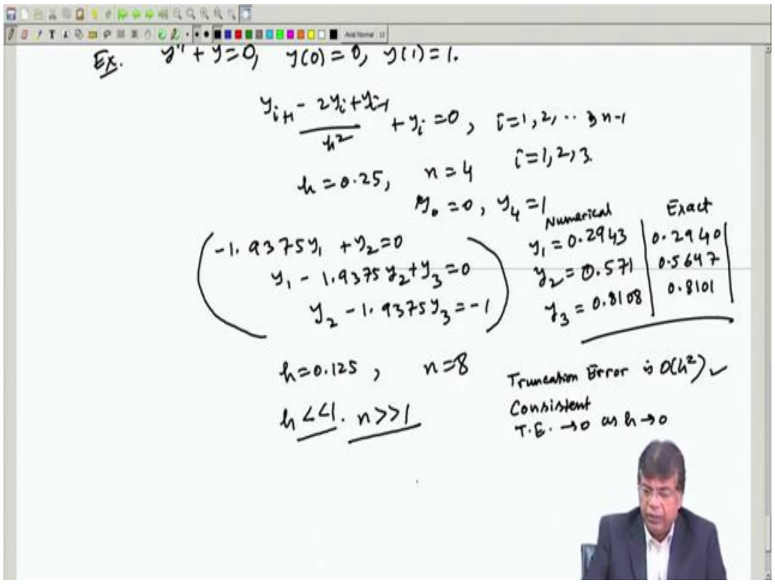

So, A is a tri-diagonal matrix and already we have talked about how to solve a tri-diagonal system that is the Thomas algorithm we already have algorithm to solve AX = d, a direct method. So, through the back substitution so basically in the Thomas algorithm what we do is we convert to a upper triangular matrix rather what we do is we will eliminate all these a_i's replace b_i 's by unit number 1 and C_i will be changed.

And in that process what the solution is determined so basically that is the Thomas algorithm we can use to solve this system of equations, so let us take an example, very simple example, say

$y'' + y = 0,\ y(0) = 0,\ y(1) = 1$

so this is very simple thing. Now, I can write

$(y_{i+1} - 2 y_i + y - 1) / h^2 + y_i = 0,\ \ i = 1, 2, \ldots, n - 1$

the desired grid points.

So, if I choose say h =0.25 so, in that case n = 4. So, that means, we need to solve this i =1,2,3 so if I put the value of i so, $y_0 = 0$ and $y_4 = 1$. So, if I put h =0. 25. So we get the algebraic system as $-1.9375 y_1 + y_2 = 0$

$y_1 - 1.9375 y_2 + y_3 = 0$

$y_2 - 1.9375 y_3 = -1$

because, y_4 is given to - 1.

So, this 3 setup equation can be solved by elimination technique and we find out the solution is $y_1 = 0.2943$, $y_2 = -0.571$, $y_3 = 0.8108$

there is numerical solution and if we compare this is very simple to solve and one can find out the exact solution that comes out to be 0. 2940 , 0. 5697, 0. 8101. So, what do we find that more or less comparable.

Similarly if I reduced the $h = 0.125$, so basically in that case n become larger and what if $n = 8$. So, in that case the number of unknown so, you have 7 equations and 7 unknowns, and one can solve that and we find that it is more close to the exact solution, if we reduce h so as the value of h is reduced, so accuracy will be truncation error is $O(h^2)$. Also, if we see that if we solve by discretization, where the central difference scheme, this is consistent scheme, consistent as

T. E.$\rightarrow 0$, the truncation error, as $h \rightarrow 0$.

So, consistency is satisfied. Also, truncation error is $O(h^2)$ so, obviously, if we choose $h \ll 1$ then the truncation error is comparable one now, so should that say if we use the central difference scheme in order to have to this is of second order accurate we have to choose h sufficiently small.

So, if h sufficiently small means n becomes quite large, so, we have a large system to solve. For large system to solve we need to write a computer algorithm and that algorithm should be based on the Thomas algorithm as described before. Now, so far what we have taken is we have been given the function values or the y value is given at the 2 end points.

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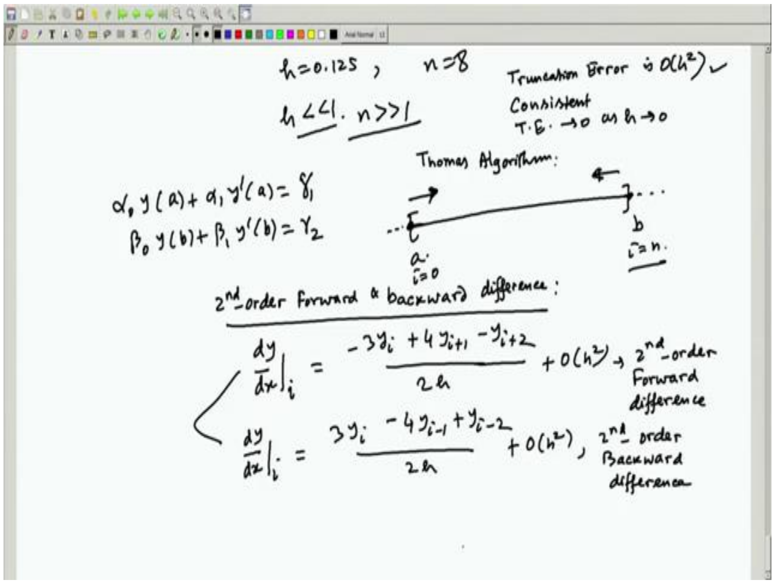

Now, in many cases we may not be that lucky that the only the function values is given so we can have a mixed derivative condition that is

$\alpha_0 y(a) + \alpha_1 y'(a) = \gamma_1$

$\beta_0 y(b) + \beta_1 y'(b) = \gamma_2$

So, in those cases what should be the technique . So, we cannot write y (a) directly evaluate .So, what should be the technique to use.

Now, another thing is that at the two end points. So, i is this is the grid points i starting from here, i ending at here so at the two end points, we cannot apply central difference formula so, we can only use the forward difference at this first point and the end point we can use only the backward difference that is going back we cannot have a central difference because we do not have any grid points outside.

So, that is one of the difficulty when we have a derivative boundary condition involved. Now, one remedy is that one can derive this formula as symmetric second order for forward and backward difference formula so far what we have derived is the forward or backward difference formula are all first order approximation.

So, now, the situation is such that since we are at the edge that is i = 0, we cannot consider i lower than that. So, you have to take forward points ahead of this i = 0. And the same way the points before this that is proceedings points now, because we cannot go beyond this point i = n.

So, one of the formula is

$\frac{dy}{dx}|_{xi} = (- 3\ y_i + 4\ y_{i+1} - y_{i+2}) / (2\ h) + O(h^2)$

So, this is a second order forward difference formula which can be used at the end points.

Similarly, we can consider we can derive the second order backward difference formula by this $\frac{dy}{dx}$ $|_{xi} = (3\ y_i - 4\ y_{i-1} + y_{i-2}) / 2\ h + O(h^2)$

so in this case this is second order backward difference formula. These two are useful when there is derivative boundary condition is involved.

So, because we cannot contaminate if we use in discretizing the derivatives by central difference formula which is second order accurate so one cannot use the first order forward difference or backward difference for the end conditions. So, order of accuracy cannot be uniform.

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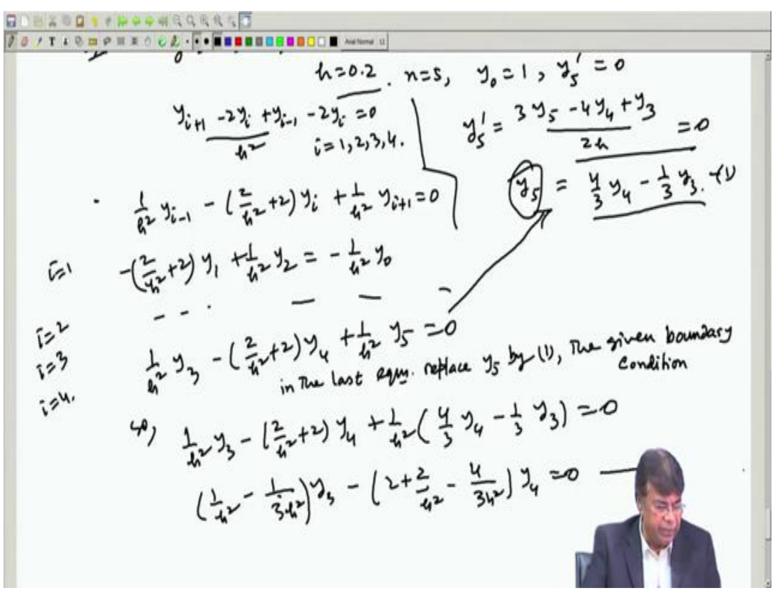

For example consider this problem y ″- 2 y =0, y (0) = 1 y ′(1) =0 and h = 0.

So, in this case what we find that we have to use the forward difference formula. So, we use y so h =0.2 so that means n =5, so, at y_0= 1 and y_5' = 0. So, what we do we discretize this equation

$\frac{y_{i+1} + y_{i-1} - 2\,y_i}{h^2}$ - 2 y_i =0 we have to write i =1,2,3,4 and y_5' = 0.

So, now, if I write y_5' that is backward difference formula so

$y_5' = (3\,y_5 - 4\,y_4 + y_3) / 2h = 0$

So, from here from this relation, we get a representation of y_5 which is

$y_5 = 4/3\,y_4 - 1/3\,y_3$

Now, when we write this equation so, i =1 ,2,…

Let us first collect this

$$\frac{1}{h^2}y_{i-1} - \left(\frac{2}{h^2} + 2\right)y_i + \frac{1}{h^2}y_{i+1} = 0$$

this is the discretized equation.

So, if I put i =1 I get

$$-\left(\frac{2}{h^2} + 2\right)y_1 + \frac{1}{h^2}y_2 = -\frac{1}{h^2}y_0$$

then we can write other equations now, for i=4

$$\frac{1}{h^2}y_3 - \left(\frac{2}{h^2} + 2\right)y_4 + \frac{1}{h^2}y_5 = 0$$

Now, alike the previous case why 5 is not given a single value I mean a numerical, but y_5 is given , what I do in this last equation replace y_5 by given boundary condition what I get is

$$\frac{1}{h^2}y_3 - \left(\frac{2}{h^2}+2\right)y_4 + \frac{1}{h^2}\left(\frac{4}{3}y_4 - \frac{1}{3}y_3\right) = 0$$

Or, $$\left(\frac{1}{h^2} - \frac{1}{3h^2}\right)y_3 - \left(\frac{2}{h^2} + 2 - \frac{4}{3h^2}\right)y_4 = 0$$

So, this will be the last equation. So, number of equations the number of variables remain same so these kind of tricks one has to apply where you have derivative boundary conditions. So, that means, the function value is not given but the function value y_5 for example can be expressed in terms of y_4 and y_3 the points prior to that.

Similarly, if the boundary condition at the other end is prescribed instead of condition at the first one is prescribed so in that case also we can adopt the same thing.

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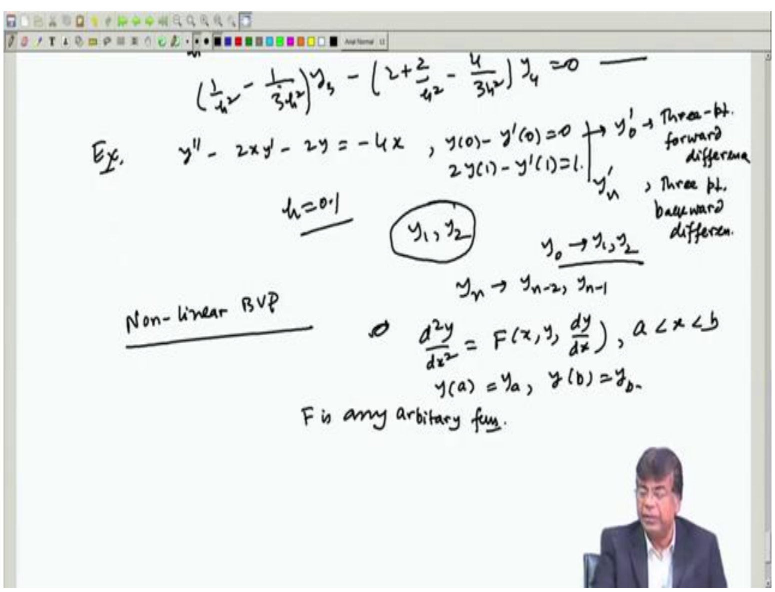

So, similarly one can solve this problem

$y'' - 2x y' - 2y = -4x$ and $y(0) - y'(0) = 0$, $2y(1) - y'(1) = 1$.

So, this involves the derivative boundary conditions. So, if we choose h =0. 1 or whatever so what do you have to do is here $y'(0)$ replaced by forward difference formula, three point forward difference formula and $y'(1)$ will replace by that is $y'(n)$ by three point backward difference formula.

So, wherever the y_0 is involved that equation is containing y_1, y_2 so when you are replacing y_0 in terms of y_1, y_2. So, that means, you are not introducing more number of variables, the same number of variables you are using. Similarly, the y_n', y_n is involved in the last equation, where y_{n-2}, y_{n-1} are present.

So, if we apply the same principle, we get the system with n - 1 variables will be involved there. So, that way one can proceed the same system remain compact. So, that means, we are not using anything extra variables or extra effort.

Now, our next topic is nonlinear boundary algebra. So, far we restricted our attention to special form of the boundary value problem that the coefficients of the derivatives are independent of y or its derivative. Now, all of us we cannot say that our given equation is linear we can have several nonlinear situation, so in that case any nonlinear system or nonlinear equation, it does not have any pattern so anything which is not following to the previous one is a nonlinear one.

So, if I write a system of differential equation $\frac{d^2y}{dx^2} = F(x, y, \frac{dy}{dx})$ with $a < x < b$.

Let us consider function value is prescribed $y(b) = y_b$.

So, f is any arbitrary function then this system is nonlinear one. Now, there is problem in solving non-linear system, thing is that when you discretized a nonlinear system, we get a nonlinear algebraic equation.

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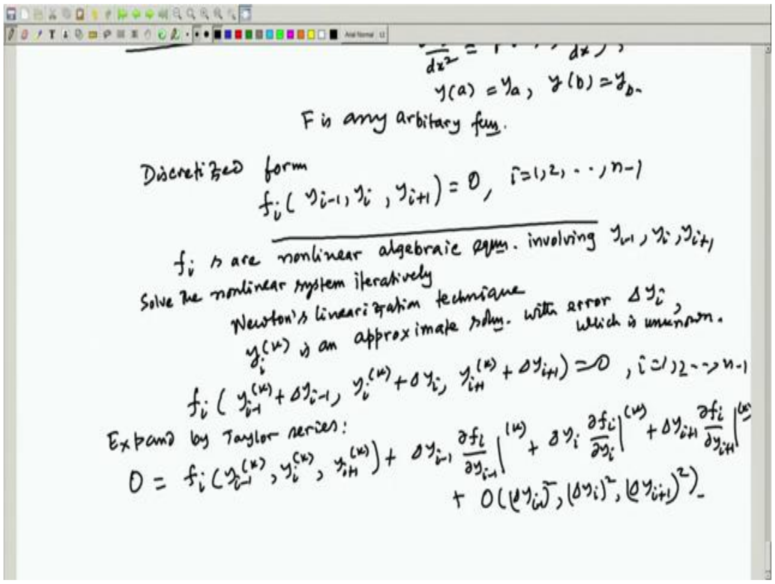

So, if I discretized, form will be something like

$f_i(y_{i-1}, y_i, y_{i+1}) = 0$, i =1,2,…, n - 1

then the same procedure if I use that is y′ and y″, yi′, yi″ if I replace by central difference formula and we proceed the same algebra whatever we have done so we find that there is n - 1 nonlinear algebraic equation where fi are nonlinear algebraic equation involving the variables y_{i-1} ,y_i , y_{i+1}.

So, whenever there is non-linearity so what we do is we solve the nonlinear system iteratively so you will choose that one of the simple iterative procedure is Newton Raphson iteration procedure or it is also referred as the Newton linearization technique. Very simple what we do is suppose ${y_i}^{(k)}$ is an approximate solution and with error with some error say let us call Δy_i.

So, that means

$$f_i\left({y_{i-1}}^{(k)} + \Delta y_{i-1}\, , {y_i}^{(k)} + \Delta y_i\, , {y_{i+1}}^{(k)} + \Delta y_{i+1}\right) = 0$$

satisfy exactly which error with error y which is unknown which are unknown. Now, so we have for i =1,2,…, n - 1 system of equations. Now, if I expand this by Taylor series form so, this $y_{i-1}{}^{(k)}$all this superscript index with k unknown it is already obtained by some other manner or some approximate solutions are already assigned.

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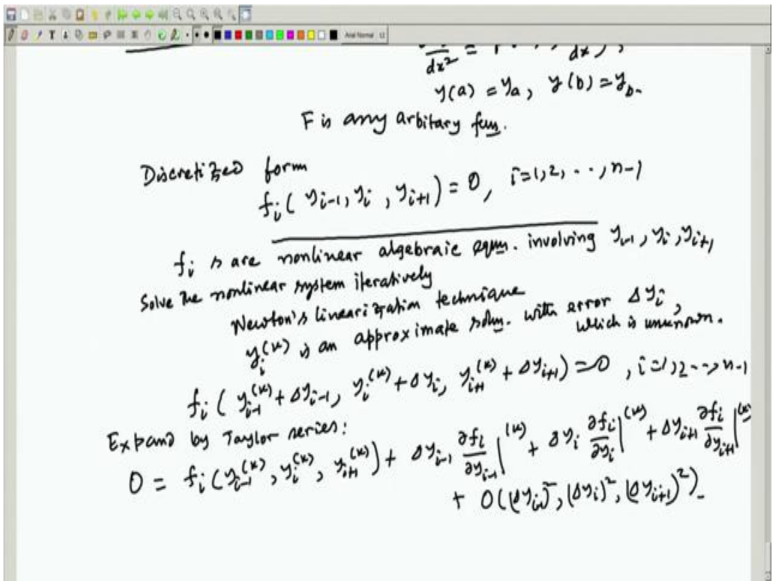

So, if I expand this by Taylor series so what I get this is 0, left side is 0 because it satisfied identically. So this is

$$0 = f_i\left(y_{i-1}{}^{(k)}, y_i{}^{(k)}\,, y_{i+1}{}^{(k)}\right) + \Delta y_{i-1}\frac{\partial f_i}{\partial y_{i-1}}|^{(k)} + \Delta y_i\frac{\partial f_i}{\partial y_i}|^{(k)} + \Delta y_{i+1}\frac{\partial f_i}{\partial y_{i+1}}|^{(k)} +$$
$$O((\Delta y_{i-1})^2, (\Delta y_i)^2, (\Delta y_{i+1})^2$$

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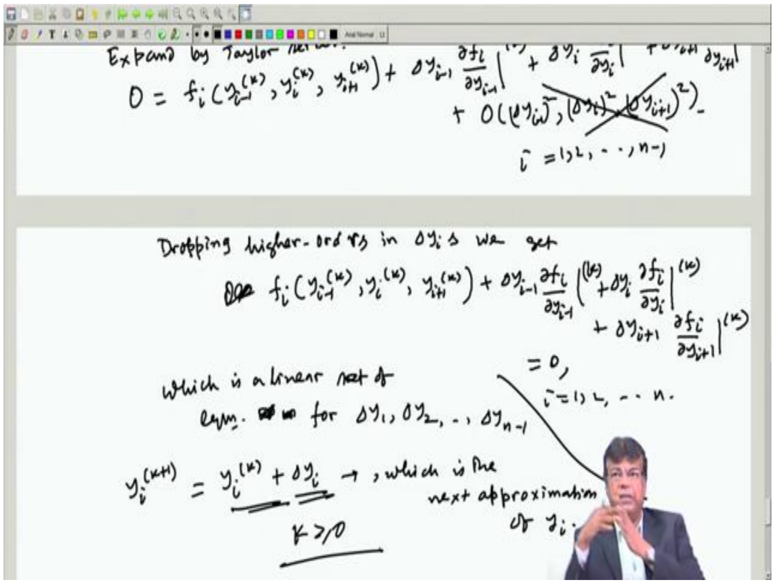

Now, we do a little trick. Because we have to obtain these Δy_i for

i =1,2,..,n - 1 that means there are n - 1 equations involving n - 1 unknowns .

Now, to solve this nonlinear one we cannot solve, so what I do we drop this quadratic terms so dropping higher orders or orders in $\Delta y_i's$ we get

$$0 = f_i\left({y_{i-1}}^{(k)}, {y_i}^{(k)}, {y_{i+1}}^{(k)}\right) + \Delta y_{i-1}\frac{\partial f_i}{\partial y_{i-1}}|^{(k)} + \Delta y_i\frac{\partial f_i}{\partial y_i}|^{(k)} + \Delta y_{i+1}\frac{\partial f_i}{\partial y_{i+1}}|^{(k)}$$

Now, I can happily solve this. This is a linear system which set of equations with unknown $\Delta y_1, \Delta y_{2,}$.., Δy_{n-1}which I can solve. So, basically, I can solve this and that Δy_i is whatever we obtain is not the exactly the same error as we are looking for so their solution.

So if I now add this ${y_i}^{(k)}$ to this Δy_i whatever we obtain from here so this we call as the next better approximation, because this Δy_i is not the exactly the same as we are looking for, that is not exactly the same value as it is an approximation of the error and so, if I add to the existing value so we get a modified form so which is the next approximation or better approximation of y_i. So, this is how the iterative procedure will continue when we have nonlinear system of equations.

So, that means, we have to solve linear system iteratively. Every iteration we solve this for Δy_i then get the modified form of y_i ,that we call as $y_i^{(k+1)}$so we have complete this, then we go for k + 2. So, that means again we repeat the process. So repeat the process till we get a required convergence.

So, this is for the nonlinear system or nonlinear boundary value problem we handle how to tackle when you have a nonlinear module problem is given but we have super restricted to second order situation. So, whenever there is nonlinear we have to solve in an iterative manner that means iterative means same process, same step you have to repeat quite number of times till we get a convergence. Thank you.