Advanced Computational Techniques

Professor Somnath Bhattacharyya
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture 14

Initial Value Problems (Contd)

Well, now we are talking about initial value problem. Now we will continue with that and today in this lecture we will introduce another technique which is a single step and higher order method. So, what we have shown before that if we use multistep methods that more number of previous step solution and also we have introduce the term implicit scheme. So, in those cases if we have a multistep method that is the solution is depending on previous few step solution.

So, then the method can have a improved or higher accuracy and also this implicit method we will have a stability higher order of stability. Now, the multistep method the problem is that it is not self starting.

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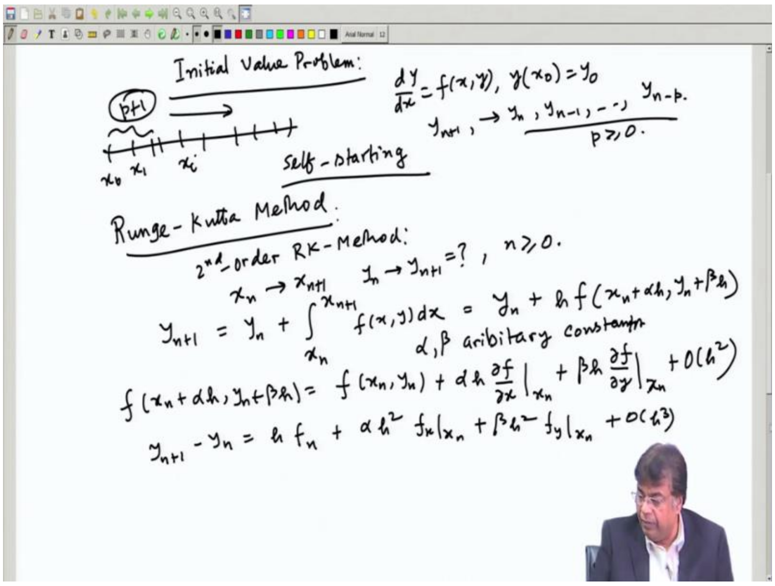

So, that means if you have the equation, the problem is

$$\frac{dy}{dx} = f(x, y) \text{ and } y(x_0) = y_0$$

So, in the multistep method if it is a four-step method or three previous step method so we need to have to get solution at y_{n+1}. So, one need to have the knowledge of solution $y_n, y_{n-1}, \ldots, y_{n-p}$ so which is a (p + 1) step previous step method multistep method where $p \geq 0$.

Euler method, for example, it is a single step method or self-starting method. So, ideal situation will be obviously if we have a single step method which can self-start that means from x_0 then we can obtain the solution then next step x_1 and so on x_i and like this way we have to change our algorithm, for that in the very beginning we solve for some few steps.

So, (p + 1) few steps and then we apply another algorithm so that is not needed for the self-starting method. So, one of the most popular method or most efficient method is the Runge-Kutta method. Now, Runge- Kutta method so there can be a second order Runge- Kutta method or fourth order Runge - Kutta method. So, first we talk about second order Runge - Kutta method RK method.

We simply write instead of saying the big term Runge- Kutta we can call it RK method. So, if we integrate suppose we know the solution at x_n and we would like to go for solution x_{n+1}. So, that means knowledge of y_n is there we need to find out the y_{n+1} equals to how much . So, $n \geq 0$. So, that means it is the self starting from y_0 or x_0 to x_1 and like that way.

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x,y)dx = y_n + h f(x_n + \alpha h, y_n + \beta h)$$

So, if we integrate between x_{n+1} to x_n we can write this as x_n to x_{n+1} $f(x,y)dx$ So, if I apply the integral mean value theorem so what I can write is $y_n + h f(x_n + \alpha h, y_n + \beta h)$ where α, β some unknown parameter arbitrary constants. So, α, β are arbitrary now. If we write this $x_n + \alpha h$, $y_n + \beta h$ if we expand by Taylor series.

$$f(x_n + \alpha h, y_n + \beta h) = f(x_n, y_n) + \alpha h \frac{\partial f x}{\partial x}|x_n + \beta h \frac{\partial f y}{\partial x}|x_n + O(h^2)$$

So, what we have is

$$y_{n+1} - y_n = h f_n + \alpha h^2 f_x | x_n + \beta h^2 f_y | x_n + O(h^3) \qquad \text{- (i)}$$

So, if we substitute and then it is the $O(h^3)$ because h is multiplied. Now, so let us call this as equation (i).

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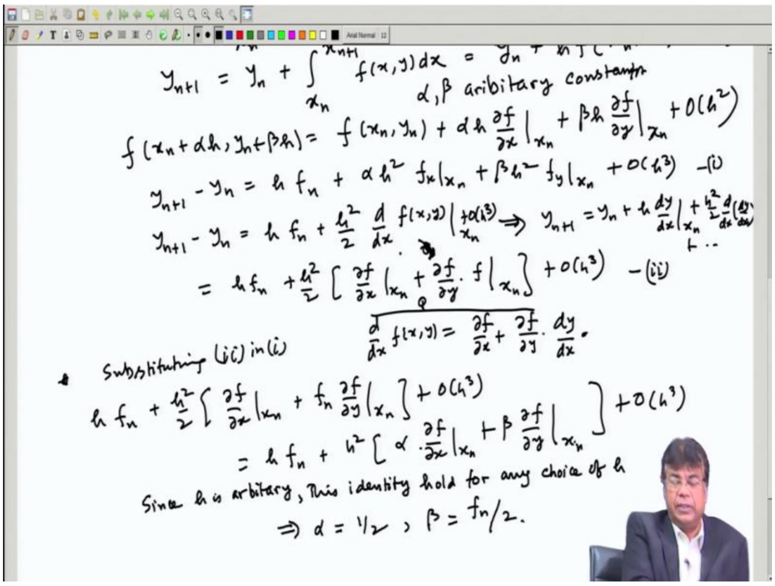

Now we can write as

$y_{n+1} - y_n = hf_n + \frac{h^2}{2}\frac{df(x,y)}{dx}\,|_{x_n} + O(h^3)$

So, order of h^3 remaining higher order terms. So, this we can equate now this is a implicit variable.

$y_{n+1} - y_n = (h^2/2)\, d^2f/dx^2 = (\partial f/\partial x)|_{xn} + (\partial f/\partial y)\,(dy/dx)|_{xn} + O(h^3)$

$\rightarrow y_{n+1} = y_n + h\,(dy/dx)|_{xn} + (h^2/2)\,(d/dx)(dy/dx) + O(h^3)$ -(ii)

Now this is the equation (ii). I hope this is clear, not exactly. So, what we did here

$\frac{df}{dx}|_{(x,y)} = \partial f/\partial x + (\partial f/\partial y)\,(dy/dx)$

So, this d y /d x is we are replacing by f. So, this is the basically this term is here is used. So, now what we can say substituting (ii) into (i) i.e. if I substitute (ii) into this equation (i) so what I get is h f_n ,we get

$hf_n + (h^2/2)(\partial f/\partial x)|_{xn} + f_n(\partial f/\partial y)|_{xn} + O(h^3) = hf_n + h^2\alpha(\partial f/\partial x)|_{xn} + \beta(h^2/2)(\partial f/\partial y)|_{xn} + O(h^3)$

Now, this is an identity since h is arbitrary chosen. So, this identity hold for any choice of h.

$\alpha = \frac{1}{2}$, $\beta = f_n/2$

So, this thing should happen. So, since this is an identity this term is for any choice of h.

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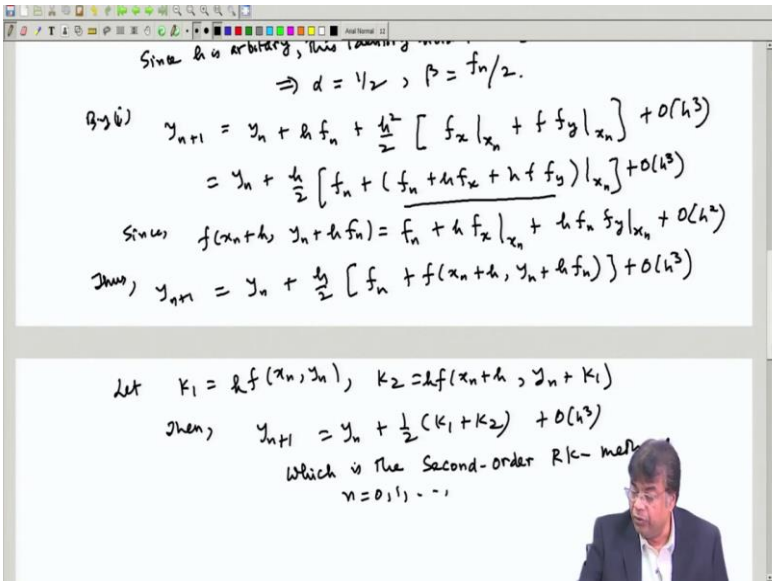

So, from (i) what we get is

$$y_{n+1} = y_n + h f_n + (h^2/2) [f_x|_{x_n} + f_y|_{x_n}] + O(h^3)$$

$$y_{n+1} = y_n + (h/2) [f_n + (f_n + h f_x + h f f_y)|_{x_n}] + O(h^3)$$

Now, since if

$$f(x_n + h, y_n + h f_n) = f_n + h f_x|_{x_n} + h f_n f_y|_{x_n} + O(h^3)$$

This is the Taylor series expansion if we have. So, thus what we find that

$$y_{n+1} = y_n + (h/2) [f_n + f(x_n + h, y_n + h f_n)] + O(h^3)$$

So, this is becoming $O(h^3)$ .So, when I replace this expression by this relation so it is order h^2 and outside h is multiplied.

So, we can call at order h cube. So, now let me call this k_1 =h f (x_n, y_n) and k_2 =f (x_{n+h}, $y_n + k_1$) so either way. So, when what I find is

$$y_{n+1} = y_n + (k_1 + k_2)/2$$

which is so this is order h cube accurate. So, which is the second order Runge Kutta method, RK method.

And this is self starting for valid for n =0, 1, 2 and higher accurate. So, it goes in a two step manner.

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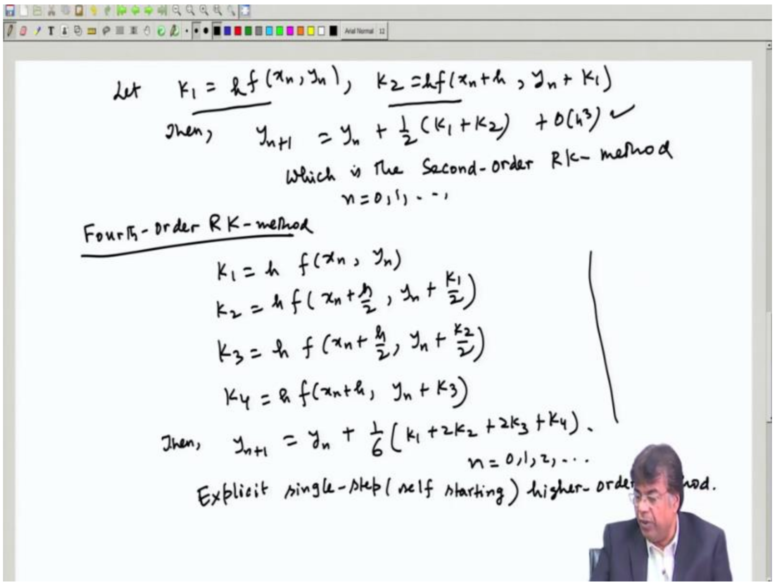

First, you find out this k_1 then next step is k_2 and then next we find out y_{n+1} given by this formula. So, this is the derivation of the second order Runge Kutta method. Now, same way we can generalize two number of points and get a little higher order Runge Kutta method. This is what is referred as the fourth order Runge Kutta method.

So the same way we generalize this technique whatever we have adopted for this case and we get a fourth order Runge Kutta method. So, what is fourth order Runge Kutta method is the formula is we have to obtain this in a four stage like we are finding k_1, k_2. So, the same thing to have a higher accuracy. So, first we find

$k_1 = h\,f(x_n, y_n)$

$k_2 = h\,f(x_n + h/2, y_n + k_1/2)$

$k_3 = h\,f(x_n + h/2, y_n + k_2/2)$

$k_4 = h\,f(x_n + h, y_n + k_3)$

So, once this four steps are obtained then

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and this is what is referred as the fourth order Runge Kutta method and the derivation is quite lengthy. So, any book or board or any other book can be followed for this derivation.

So, I am not doing this way the same I am not deriving this fourth order Runge Kutta method here, but in the similar way one can generalize and obtain the fourth order Runge Kutta method and this is we can use from n =0, 1, 2,…, etc. So, obviously this is a explicit single step or self starting higher order method. So, that is why all the good points are there in this Runge Kutta method.

So, that is why this Runge Kutta method is most popular in solving the differential equation initial value problem in particular only thing is that it is little laborious because compared to say Milne's predictor corrector method or Adams Moulton Predictor Corrector method. So, there the advantage is that you can have a higher accuracy and the number of steps are lesser compared to this with the computation time is comparatively low.

But only thing is that like Adams Moulton method or Milne’s predictor corrector method you are not self starting. So, one need to know the few step solution before hand. So, in some cases when one has to go for a large time.

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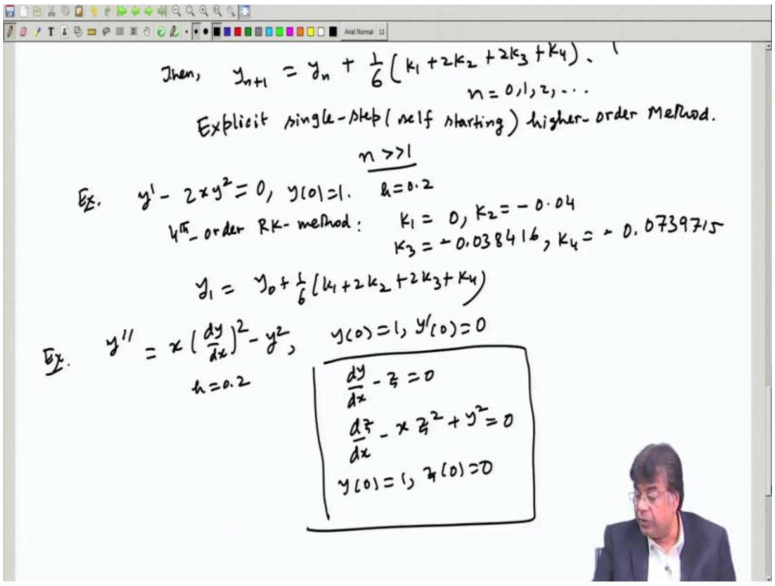

So, when n is very, very large in that case what is done is few steps one can obtain by Runge Kutta method and then this predictor corrector method are used. So, let us apply to an example say a very simple

Example-
$y' - 2xy^2 = 0$, $y(0) = 1$ if we choose $h = 0.2$

So, if we use he fourth order Runge-Kutta method RK method so we get the solution as

$k_1 = 0$, $k_2 = 0.04$, $k_3 = 0.038416$ and $k_4 = -0.0739715_2$ in a very first step I am talking about.

So, this is how the Runge-Kutta method goes and in that process one can find out the solution as the

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Now, as we have shown in the previous case that the Runge-Kutta method or this initial value problem can be used for any order higher order equation ODE also.

For example if we have a equation like $y'' = x y'^2 - y^2$.

so that is in other words this is like given by

$$y'' = x\left(\frac{dy}{dx}\right)^2 - y^2, y(0) = 1, y'(0) = 0 \qquad h = 0.2$$

and your given condition is $y(0) = 1$, $y'(0) = 0$. This is the given condition and I am choosing $h = 0.2$ we need to solve by Runge-Kutta method. So, what we do is we write

$$\frac{dy}{dx} - z = 0$$

that is one equation and then

$$\frac{dz}{dx} - xz^2 + y^2 = 0$$

These are the two equations and conditions are given as $y(0) = 1,\ z(0) = 0$

So, these two coupled sets of equation to be solved. So, one can use if we want to use a Runge-Kutta method so what we do is.

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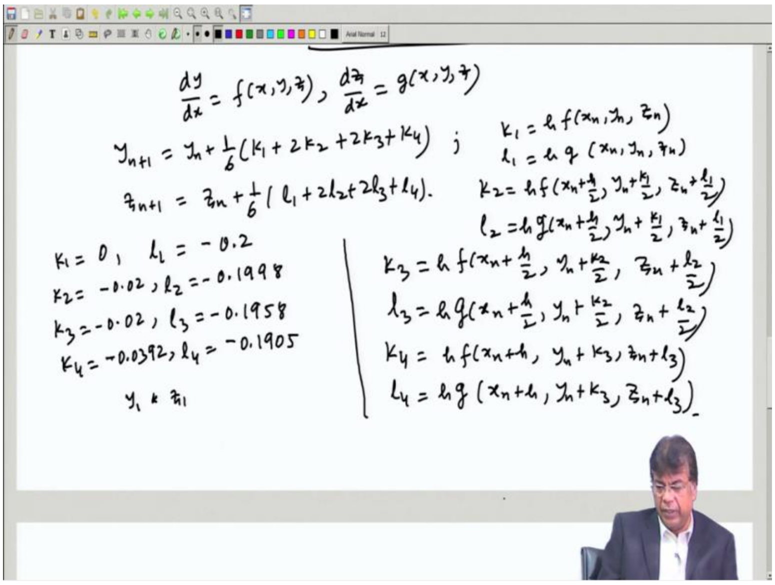

So, what we define this way that if I call

$\frac{dy}{dx} = f(x, y, z)$ and $\frac{dz}{dx} = g(x, y, z)$

So, then one can write the Runge Kutta method correspondingly as

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ and $z_{n+1} = z_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ where we can write this generalizing in the same way one can write the k_1 where $k_1 = h f(x_n, y_n, z_n)$

Then $l_1 = h g(x_n, y_n, z_n)$ then one can write

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right)$$

and

$$l_2 = hg\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}\right)$$

sorry this is g. Now see here when I calculate k_2 then we need not only k_1 we need l_1 as well. So, similarly l_2 we need to calculate, have to have k_1 and l_1. So, that means this will go in a sequential fashion.

First you have to find out k_1 then l_1 then k_2, l_2 we cannot have k_1, k_2, k_3, k_4 separately then l_1, l_2, l_3 separately because these two equations are coupled. Similarly, k_3 we can find out as

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right)$$

$$l_3 = hg\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3, z_n + l_3)$$

$$l_4 = hg(x_n + h, y_n + k_3, z_n + l_3)$$

So, if we do this perform this computations what we can for this case we can find out I am not doing the whole calculation this k_1 comes out to be $k_1 = 0$, $l_1 = -0.2$,$k_2 = -0.02$, $l_2 = -0.1998$, $k_3 = 0.02$,$l_3 = -0.1958$, $k_4 = -0.0392$, $l_4 = -0.1905$
and in that process we obtain the next values.

So, this is how the procedure for the Runge-Kutta method goes. So, obviously this Runge-Kutta method is a good choice, but only thing is that it involves few steps calculations and particularly it becomes complicated when you have a higher orders. So, when you have a large order system so in that case you will have a number of this coefficients are quite large.

And in that case the computations become too many. So, that is one of the drawback for the Runge-Kutta method. So, as I said before that this starting point one can obtain by the Milne's predictor corrector method and then one can proceed to solve by the starting points one can compute by some Runge-Kutta method and then one can proceed to solve the rest of the points by Milne's predictor corrector method or many other methods.

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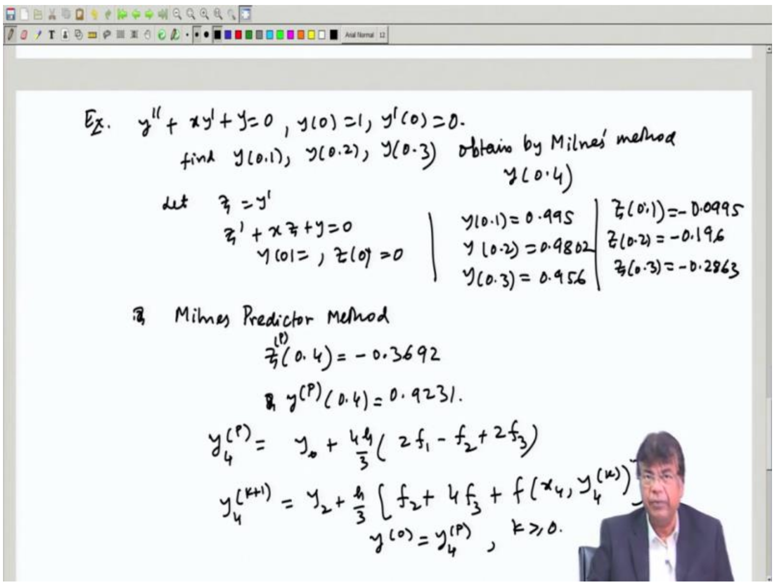

For example, if you have a problem say

$y'' + x\, y' + y = 0$

and you have

$y(0) = 1$ and $y'(0) = 0.$

Find $y(0.1), y(0.2), y(0.3)$ $y(0.4)$. So, once we have these three point solution to obtain by Milne's method $y(0.4)$. So, what we do is suppose we have the solutions as given. So first of all we have to reduce to a first order set.

So, this is $z = y' z' + xz + y = 0$ $, y(0) = z(0) = 0$

So, by some method so suppose you get the solution you will get a solution by Runge-Kutta method for example as $y(0.2) = 0.9802$, $y(0.3) = 0.956$, $z(0.1) = 0.0995$, $z(0.2) = 0.196$, $z(0.3) = 0.2863$

So, this is the starting three points. So, once we have that then we can find out the Milne's predictor corrector method to evaluate this values.

So,if we apply the Milne's predictor formula so we get Milne's predictor formula method gives so this is we call as the $z^{(P)}(0.4) = 0.3692$, $y^{(P)}(0.4) = 0.9231$ and once we get this then we will go to this corrector formula and obtain this better mention. So, that y_4 basically what we do is predictor is

$$y_4^{(P)} = y + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

So, this is the way we have to go and the corrector one $y_4^{(k+1)}$ as we have derived before. This formula we are using. So, this is

$$y_4^{(k+1)} = y_2 + \frac{h}{3}(f_2 - 4f_3 + f(x_4, y_4(x))$$

and what is k? $y(0) = y_4^{(p)}$ and $k \geq 0$. The same way we can solve this remaining other second order problems and repeat the process.

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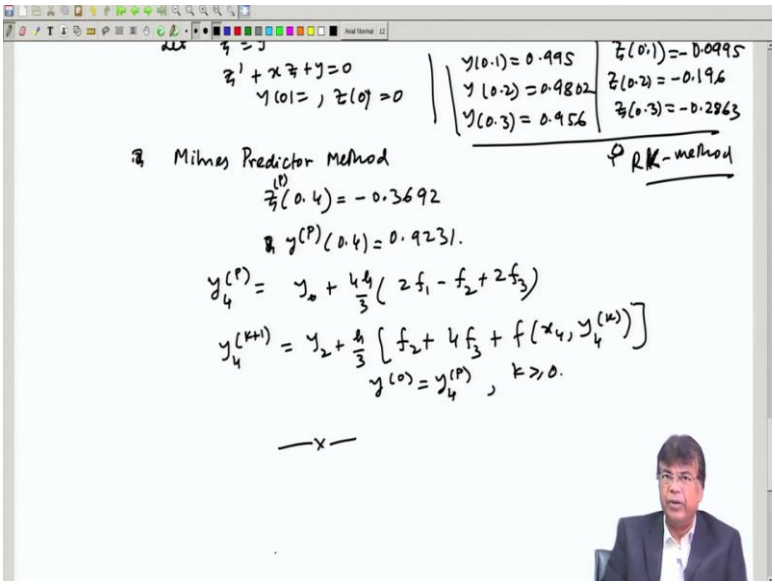

So, once we have these three points what is the precisely I wanted to say for this three we can use the Runge-Kutta method RK method and once we have that then the Milne's predictor corrector method for example this is more simpler than using the Runge-Kutta method and this is for professional computations for large time step solutions, this is the way one proceed. So, that is all about the IVP and next we will talk about the boundary value problem in the next class. Thank you.