Advanced Computational Techniques
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Lecture 12
Initial Value Problems (Contd.)

So, we were discussing of solving initial value problem of this type that is conditions is prescribed at a single point and that point we call as the initial point and without ambiguity we can choose that initial point as 0. Now, little bit of theory on existence of the solution that is under what condition this initial value problem will have a solution and if the solution exists, then how we can demand that is a unique solution.

Now, as we said in the beginning that in the numerical analysis, we are guaranteed that solution exists and it is unique and is also well post. Well post means the solution vary continuously with the auxiliary conditions that means the boundary conditions or initial condition whatever is provided. So, solution are continuously depends on those conditions. So, under all this ideal situation we can obtain the numerical solution by using some numerical technique.

Now before that we need to see that under what condition the solution exists and if it exists then unique. Obviously, solution exists means you can integrate. So a function which can be integrated, a function which is a bounded within that interval where we are at the domain of x over which x is varying, so if the function is integrable so that is the function is of bounded variation.

So, that means if you partition the interval say between x is varying say $0 < x < a$, so that means 0 to a. So, y(x) can be written as

$$y(x) = \int_0^a f\big(x, y(x)\big)dx + y_0$$

So, now what if the function f(x) which is an implicit variable for y is depending on x, so basically this is a function of x where y is an unknown function of x. So, if this function is said to be integrable between 0 to a, so that means the solution exist.

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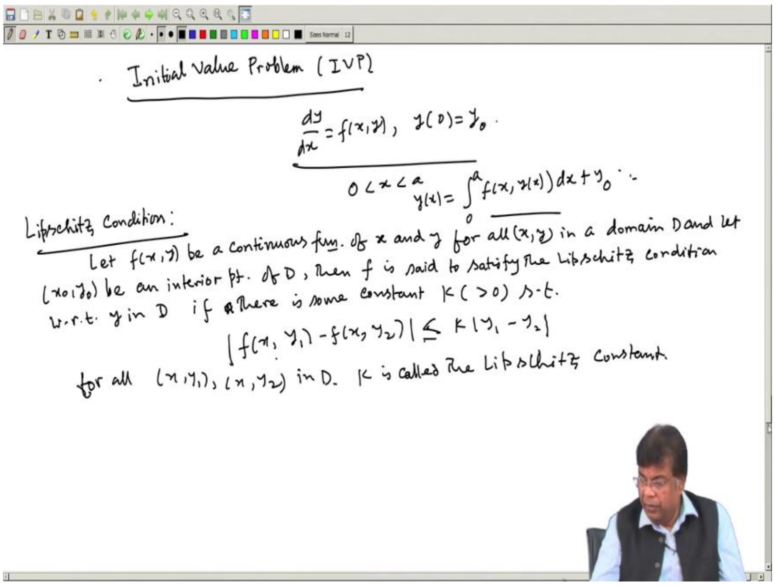

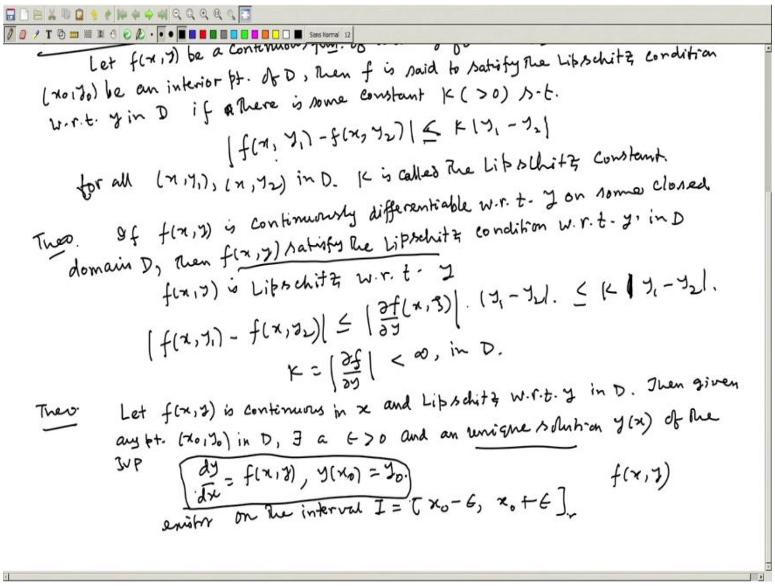

Now what we need to know that under what condition the solution is unique. So, for that there is one called Lipschitz condition. So, this Lipschitz condition can be stated in this way. let f (x, y) be a continuous function of x and y for all (x, y) in a domain D and let (x_0, y_0) be an interior point of D, then f is said to satisfy the Lipschitz condition with respect to y in D if there is some constant K (>0) exist such that

$$| f (x, y_1)- f (x, y_2) | \leq K |y_1 - y_2|$$

where for all (x, y_1) and (x, y_2) in D, K is called the Lipschitz constant.

So, this implies that if the difference between y_1 and y_2 are making small as we are contracting this point (x, y_1) and (x, y_2) we are contracting, so the difference between the function value (x, y_1) and (x, y_2) is also becoming and the proportionately small or contracted. Now if we can show that if f (x, y) is continuously differentiable with respect to y on some closed domain D then f (x, y) satisfies the Lipschitz condition with respect to y in D or in other words this can be said that f (x, y) is Lipschitz with respect to y.

So, this very simple to show that

$$| f (x, y_1)- f (x\ y_2) | \leq |\frac{\partial f}{\partial y}(x,\xi) | (y_1 -y_2) \leq k (y_1 -y_2)$$

(if I apply the mean value theorem). Now if it is continuously differentiable so there is a bound, this is $y_1 -y_2$. So, we can choose k = $|\frac{\partial f}{\partial y}|$ in the domain. So, if I call this $|\frac{\partial f}{\partial y}|<\infty$, say bounded in D.

So, if the function f is continuously differentiable with respect to y on some closed interval, closed domain D then f (x, y) satisfy the Lipschitz condition and then now how it is related to the different solution of the differential equation that is given by this theorem. Let f (x, y) is Lipschitz condition is something higher than the continuity. So, if it is a function is continuous that guarantee that it is integrable so that means you have a solution exist. But to have the solution unique what you need to have is the Lipschitz condition to satisfy within the domain. That is what this next theorem says.

Let f (x, y) is continuous in x and Lipschitz with respect to y in D. Then given any point (x_0, y_0) in D, there exist a $\xi>0$ and unique solution y(x) of the IVP

$$\frac{dy}{dx} = f (x, y)$$

we call the initial point as $y(x_0) = y_0$ on the interval $I = (x_0- \xi, x_0+ \xi)$ suitably chosen and unique solution of y of the IVP exist in the interval in the neighbourhood of x_0.

So, this is what the uniqueness of the solution is guaranteed. So, if the function f (x, y) is continuous with respect to x and it is Lipschitz with respect to y then we can say that the given initial value problem have a unique solution at the neighbourhood of x_0 in which these conditions are satisfied.

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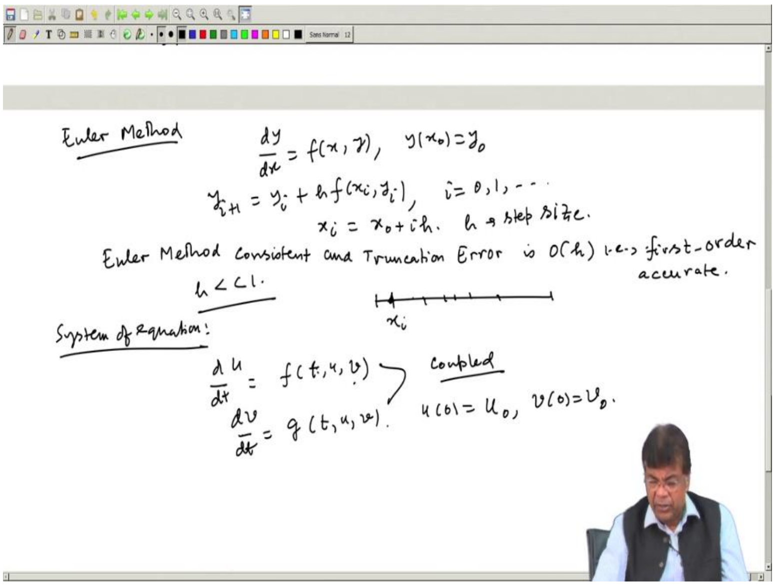

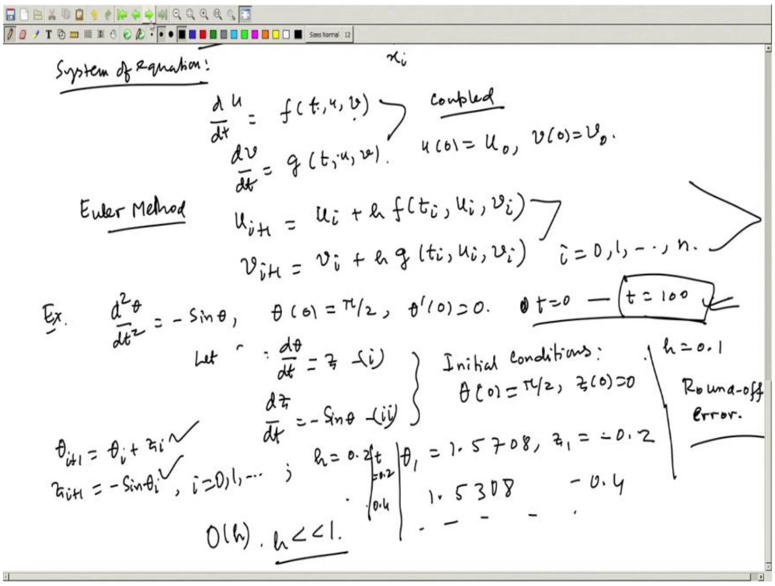

Now in the last class we have discussed about this Euler method, the simpler method for solving an initial value problem. So, $\frac{dy}{dx}$= f (x, y) and if we call $y(x_0) = y_0$ instead of 0, let us call this x_0 is the initial point. So, the Euler method is $y_{i+1} = y_i + hf(x_i, y_i)$ where i = 0,1,2 ,... n. Whatever the point you need and x_i we are defining as $x_i = x_0 + i h$, h is the step size. So, it can be uniform or non-uniform. And what we have shown that this Euler method is consistent and the truncation error is O(h), that is first order accurate. So, what it means that ($h<<1$) has to be, for Euler method, very small in order to have a good approximation.

Now we talk about system of equations. The consistency is very important to show for the convergence of the solution. Basically what we need that the solution whatever the numerically obtained solution should converge to the analytic solution or the exact solution of the differential equation as h is sufficiently small. So, for that what we need is the equation to be consistent and the numerical computation should be stable, stable means if we give a little bit of perturbation.

So, that perturbation should not grow as we progress in the time or greed domain. As we say at certain stage x_i, a small perturbation to the solution or some small error, small very infinitely small error is committed at x_i. So, obviously it will perturb the solution in the subsequent steps. So, now a stability analysis is to check whether that error whatever has been incorporated whether it grows with the subsequent steps as i increase, or it decays or remain constant. So, if it grows then the method is unstable, if it is decay or remain constant founded then it is stable method. So, a consistent and stable lead to a convergence of the solution.

Now if we talk about system of equations say if we have, say,

$\frac{du}{dt}$ = f (t, u, v),

say, not x, let us talk about t, is the independent variable and

$\frac{dv}{dt}$ = g (t, u, v)

And the initial condition is

u (0) = u_0, v (0) = v_0.

So, this is a system of equation so these two equations are coupled.

So, that means you cannot isolate one from the other. So, solution of u depends on solution of v and solution of v depends on solution of u. So, this two cannot be solved in an isolated manner. If we use the Euler method, so one can write as

$u_{i+1} = u_i + h f(t_i, u_i, v_i)$ and $v_{i+1} = vi + h g(t_i, u_i, b_i)$.

So, i = 0, 1, ..., n. So, in many case whether we have system of equations or we reduce to a system of equations, a higher order initial value problem. So, in that case we can solve by a coupled set of equation like this. For example, if we have a equation

$\frac{d^2\theta}{dt^2} = - \sin \Theta$ and $\Theta(0) = \frac{\pi}{2}$ and $\Theta'(0) = 0$

So, two conditions are prescribed at 0 by this manner. So, now what we do is let

$z = \frac{d\theta}{dt}$

Another equation becomes

$\frac{dz}{dt}$ = - sinΘ.

So, these two coupled equation with initial condition are Θ(0) = π/2 and Θ'(0) = 0.

So, the two initial conditions are given. Now, obviously I can write the Euler method as

$\Theta_{i+1} = \Theta_i + z_i$ and $z_{i+1} = -\sin\Theta_i$.

So, i = 0, 1, 2,.. whatever the point we need. So, what we have here is $\Theta(0) = \frac{\pi}{2}$.

So, if I choose say h = 0.2 and then what I find that Θ_1= 1.5708 and z_1 = - 0. 2. And similarly, h = 0.4 next step not h = 0. 4, h = 0.2 if we choose so that means t = 0.2.

We are starting from 0 then t = 0.4, we can find out this as 1.5308 ,- 0.4 and so on. So, that is how the set of equations if you have a system of equations that means we have to solve the system in a coupled manner and we cannot isolate. We can isolate only thing is that if you are lucky that u is only depending on t and u.

And v is only depending on so that means these two system are isolated but here or in this case, we have a coupled set of equations. So, that means solution of one equation is depending on the solution of the other equations. So, this to be solved by coupled manner.

Now only all these things are good thing for Euler method only things are very big drawback is order h. So, in that case since it is a first order accurate, so to have a sufficient accuracy what we need is h to be sufficiently small. Now if h is small, the number of computations become very high.

For example, if I want to find out from Θ 0 from t = 0, it is started and you want to go say t = 100 or something. So, we have to go for a large time solution at the t called 100 interval. So, 0 to 100 interval so what is the solution. So, in that case suppose if I choose h = 0.1, so we have to go for 1000 steps integration.

So, which becomes anonymous and also another problem is that see every time we are doing some calculations at every steps. So, there is a something called the round of error occurs, round of error. So, this cumulative effect of round of error may not become negligible at after certain stage and in that process this method can become unstable. So, that is why the first order accurate method is not good enough and what we need to do is higher order methods and also more stable method.

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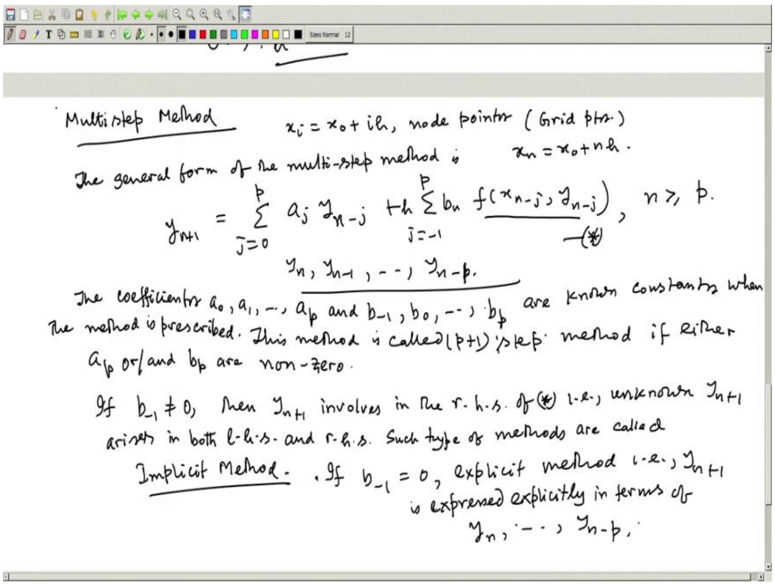

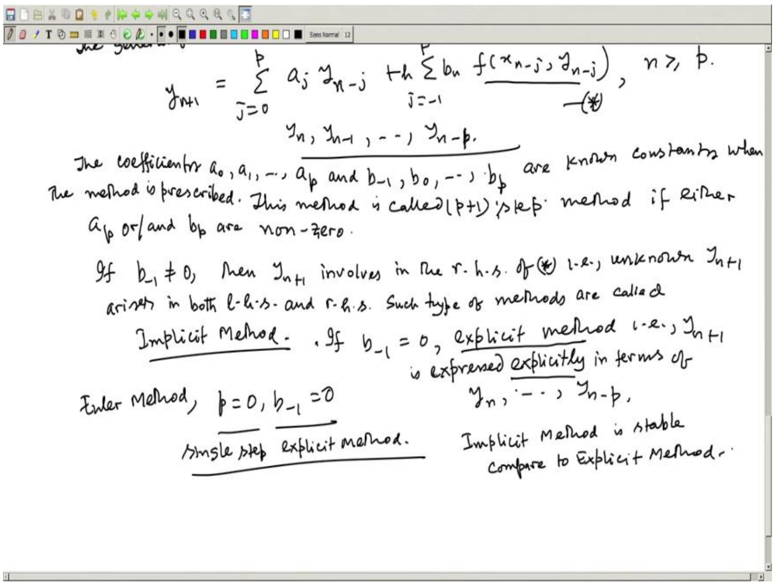

Let us define in general the form of the method as multi-step method. Let us define. Let us denote $x_i = x_0 + i\,h$ as grid points or note points for i = 0 onwards is also grid point, we can define. Normally grid points are referred when x is more than 1 component.

So, if there are x, y or 3 components or more than 1 component so that is referred as the grid points and by and large definition is the same. So, the general form of the multi-step method is $y_{n+1}=\sum_{j=0}^{p} a_j\, y_{n-j}$ + h $\sum_{j=0}^{p}$ y, b_j f(x_{n-j}, y_{n-j}) say.

So, that means we have $x_n = x_0 + nh$. So, we know the solution for $n \geq p$. So, that means, we know the solution for $y_n, y_{n-1},.., y_{n-p}$. So, that means p previous steps solutions are used in this case. And another thing is that, so, whenever we say this is a method so that means the coefficients a_0, a_1,..., a_p and b_{-1}, b_0,.., b_p are known constant. So, when we know a method so known constants are given constant when the method is prescribed.

This method is called (p + 1) previous steps method or previous one step method. If either a_p or and b_p are non-zero. So, that means we are using the solution from y_1, y_n to y_{n-p}, this p+ 1 previous steps method solutions we are using if this coefficient a_p or b_p or both are non-zero.

Another thing is that what we find from here that what we find that if $b_{-1} \neq 0$ so that means we have, then y_{n+1} involves in the RHS. This is RHS of (*). And that is the unknown y_{n+1} which arises in both LHS and RHS. Such types of method are called implicit method.

So, that means here we are not able to express the unknown exactly in terms of the known quantities, known values or known functions. So, if b_{-1} is non-zero, so then this f is involving y_{n+1}. So, this is an implicit method and if of course $b_{-1} = 0$ then it is an explicit method. So, that is y_{n+1} is expressed explicitly in terms of y_n to y_{n-p}.

So, then it is called the explicit method. So, for example Euler method, Euler method was a single step explicit method. So, that means p = 0, $b_{-1} = 0$. So, single step explicit method.

Now, we will talk about derivations of some methods and I will give some derivation as the home task in the subsequent lecture which leads to a higher order method and why this implicit method, though it is a complicated one but why we prefer this implicit method over the explicit method is because of the stability compared to explicit method. So, that is the reason we prefer for the explicit method. So, we will talk about in this in the next class.