Advanced Computational Techniques
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Lecture – 11
Initial Value Problems (IVP)

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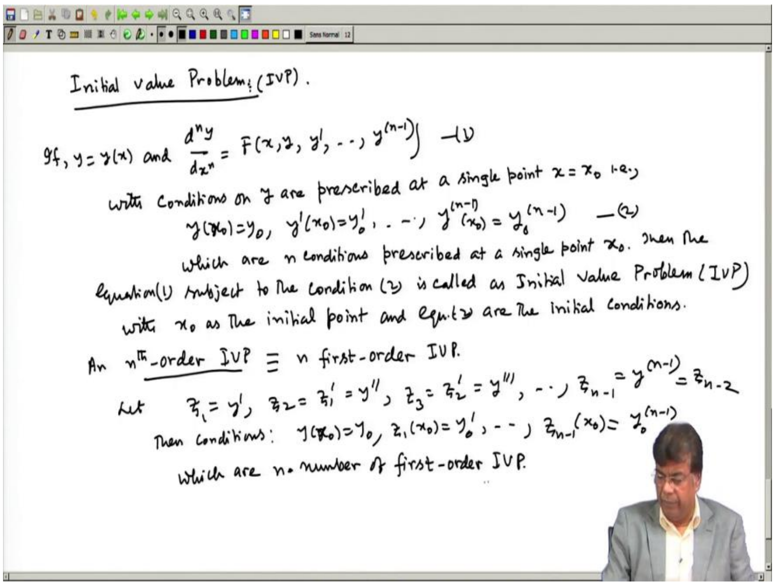

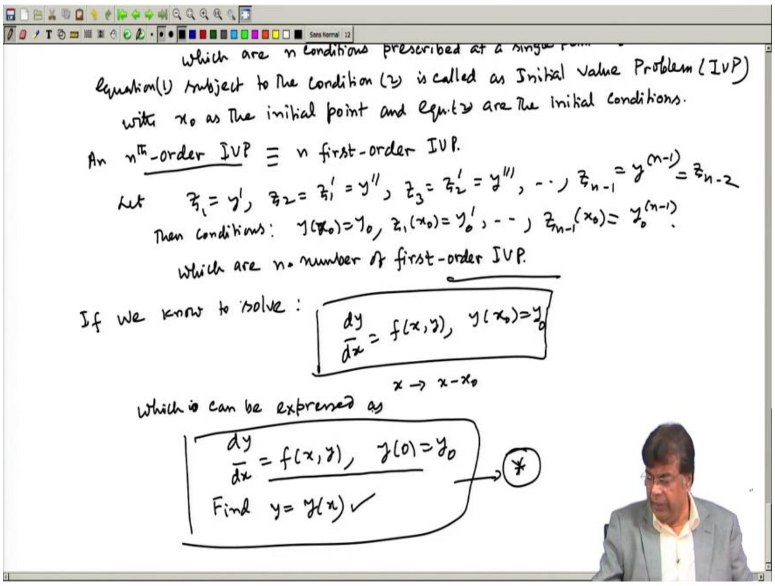

Our next topic is solving an initial value problem. So, we will describe an outline for solving the initial value problem in a couple of lectures. So, what is initial value problem? So, if we have a problem consists of differential equations given by

$$\frac{d^n y}{dx^n} = F\left(x, y, y', \ldots, y^{(n-1)}\right) \ldots (1)$$

and it satisfy this equation with subject to the boundary conditions

$$y(x_0)= y_0,\ y'(x_0) = y'_0, \dots, y^{(n-1)}(x_0) = y^{(n-1)}{}_0 \dots(2)$$

So, that means conditions are prescribed at a single point x_0 . So, then in this case equation (1) subject to condition (2) is called or is referred as initial value problem. IVP in short, with x_0 as the initial condition, initial point.

Now one simplicity for this initial value problem or IVP is that any nth order initial value problem can be converted to n first order initial value problem. So, if we substitute what we can see very straightforward that and nth order IVP is equivalent to n first order IVP, very simple to say or to prove. So, you have equation (1) and (2) which are nth order IVP. Now if I substitute, let

$$z_1 = y', z_2 = z_1' = y'', z_3 = z_2' = y''', \dots, z_{n-1} = y^{(n-1)} = z_{n-2}{}'$$

These right-side values are all prescribed. So, that means we now have n number of first order IVP.

So that means, in other words, that if I know, if we know to solve a first order IVP then we can extend our knowledge to solve any nth order IVP because any nth order IVP can be equivalently reduced to n first order IVP. So, that is why we restrict our attention to know how to solve an equation of this form. If we once know how to tackle this kind of situation, so then we are equipped enough to solve any nth order initial value problem.

Now for the sake of simplicity, x can be transferred, we can make a coordinate transformation x→x-x_0, say and instead of wired initial point as x_0, we can consider initial point as 0, so without loss of ambiguity

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

our task is to find y(x). y_0 is given value.

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Well-posed Problem:

The governing equations and auxiliary conditions (initial / boundary) conditions are well-posed if the following conditions holds:

- Solution exits
- Solution is unique
- Solution depends continuously on the auxiliary data. (*)

Numerical Methods for IVP:

$$\frac{dy}{dx} = f(x,y), \quad y(0) = y_0.$$

Find $y(x)$ for $x > 0$.

For, $x_0 < x_1 < x_2 < \cdots < x_n < \cdots$

Find $y_i = y(x_i)$, $i = 1, 2, \cdots n$.

where $y_0 = y(x_0)$ is given.

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Define $x_i = ih$, $i = 0, 1, \cdots$

$h = x_i - x_{i-1}$,

$y_1 = y(x_1) = y(0 + h)$, Expand by Taylor series,

$$y_1 = y(0) + h y'(0) + \frac{h^2}{2} y''(0) + \cdots$$

$$y_1 = y_0 + h f(x,y)\big|_{x=0} + \frac{h^2}{2} y''(0) + \cdots$$

retaining only up to linear order of h,

$$y_1 = y_0 + h f(0, y_0).$$

In general, $y_i = y_{i-1} + h f(x_{i-1}, y_{i-1})$. $i = 1, 2, \cdots$

where y_0 is prescribed. Which is referred as Euler Method.

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Truncated: $y_1 = y(0) + h y'(0)$ $+ \frac{h^2}{2} y''(0) + \cdots$

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- Truncation Error: Whi

Now before solving the few conditions that will post problem. So, how we define? When we go for a numerical solution, numerical solution is little bit of conditional with an ideal situation. So, that means when you are going to solve the numerical solution that means we are assuming that solution exists and that solution is unique.

So, if you have a solution which is already proved to be exist and it is unique then through the numerical procedure, we obtain that solution. So, this is the initial assumption or initial action before we proceed for starting the numerical solution. By the numerical solution we cannot prove that solution does not exist or multiple solution exists. So, that kind of complications we are not dealing with.

So, that kind of problem we call as well processed problem. The conditions, the governing equations and auxiliary conditions maybe the initial condition in this case, can be boundary condition, also, in the same way we can also define for the boundary value problem, if the following condition holds. Following solution exists which we should not ask whether solution exists or not, no doubt about that.

Solution is unique, solution depends continuously on the auxiliary data. Auxiliary data means all the boundary conditions, initial conditions. So, if these two happens, this thing happens we are going to solve under this assumption that we have a solution which exists and unique and the solution continuously depends, so if we change the initial condition or the boundary conditions that reflects the solution of the numerical solution as well in continuous fashion.

So, under these assumptions we will talk about some numerical method to solve this. So, what we now focus on numerical methods for IVP. So, what you have given is

$\frac{dy}{dx} = f(x, y),\ y(0) = y_0$.

So, you need to find out y(x) for $x \geq 0$, from 0 onwards. So, that means something has happened at x=0. So, we know the form of the solution and then we proceed on x. So, what is the solution is, that means a train has started with an initial acceleration or initial velocity is prescribed at x=0 and then what is the distance covered by the train as time progresses?

Now, so first of all in the numerical solution we have to specify that at what points we are going to obtain y. So, that means the solutions, at what values of x, we are going to determine. So, that has to be specified. So, $x_0 < x_1 < x_2 < \cdots < x_n < \cdots$ the points where we need to find out the solution. So, find

$y_i = y(x_i), i = 1,2, \dots, n$

For i= 0 $y_0 = y(x_0)$.

So, now these points the discrete points where we have to find out the solution say x_i, so these discrete points define

$x_i = ih$, i=0,1,...

$h = x_i - x_{i-1}$

So, that means this x_i's are all equi-spaced points.

So, first of all if we want to find out y_1 so first thing first. So,

$$y_1 = y(x_1) = y(0 + h) .$$

If I expand by Taylor series, I get

$$y_1 = y(0) + hy'(0) + \frac{h^2}{2} y''(0) + \cdots$$

$$y_1 = y(0) + h\mathrm{f}(\mathrm{x}, \mathrm{y})|_{x=0} + \frac{h^2}{2} y''(0) + \cdots$$

So, now if we retain only up to linear order of h, we get

$$y_1 = y(0) + h\mathrm{f}(\mathrm{x}, \mathrm{y})|_{x=0}$$

So, this is the procedure.

Now I obtain the solution at x_1 by this method by this manner. So, once I get that then with this x_1 I can go to x_2 and so on. So, in general I can write this

$$y_i = y_{i-1} + hf\left(x_{i-1}, y_{i-1}\right), \qquad i = 1,2, ..$$

So, this is the general method for i=1, 2 up to whatever the points where y_0 is prescribed. So, this simple procedure is called the Euler method which is very well known which is referred as Euler method.

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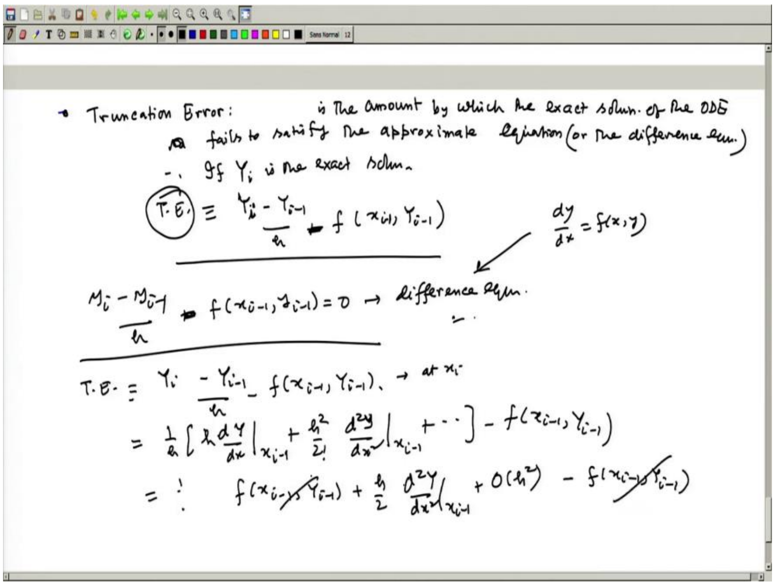

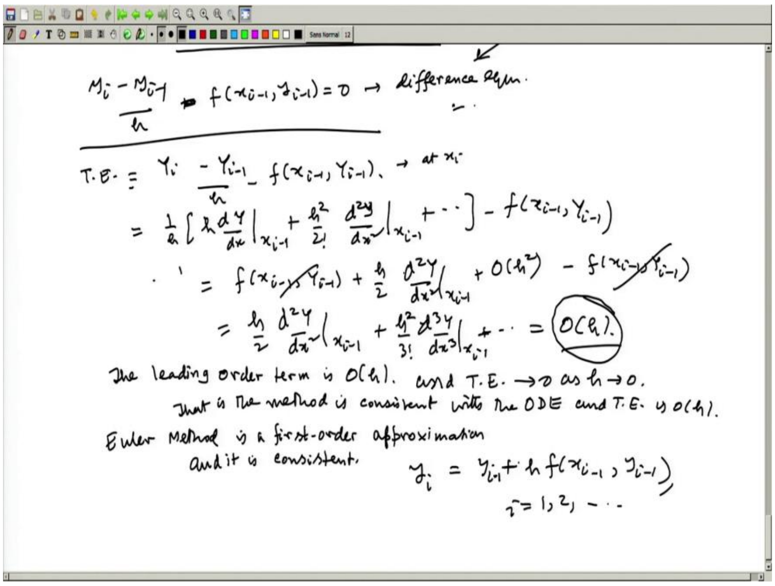

Now, so obviously this is a very simple procedure that you start with the initial solution. Then add these term you get the next one, y_1 once you get y_1 then you go to the y_2 and so on. So, this simple procedure, now what we need to know the truncation error or because what we did here that instead of solving the whole set of infinite series, we have truncated here.

So, there is an error committed. So, that error is referred as the truncation error which is the amount by which the exact solution of the ODE fails to satisfy the approximate solution.

So, this is the truncation error we are talking about, if Y_i is the exact solution, then the truncation error T.E. can be written as

$T.E. = \frac{Y_i - Y_{i-1}}{h} - f(x_{i-1}, Y_{i-1})$

So, all these things we are replaced. So, if I replace by the solution of this difference equation $\frac{y_i - y_{i-1}}{h}$ - $f(x_{i-1}, y_{i-1}) = 0$. This is the difference equation.

Now when I am replacing this approximate solution by the exact solution, so, this side may not be 0. So, whatever the residue that residue is called the truncation error. So, this is the difference equation. So, you are asked to solve the

$$\frac{dy}{dx} = f(x, y)$$

So, instead what we are doing is we are solving this difference equation and getting this y_i. And demanding that this y_i is the approximate solution of the differential equation that is approximate solution of Y_i.

Now we are defining a parameter what you would refer as the truncation error, truncation error is the one when the difference equation is replaced by the exact solution of the differential equation. So, the residue, I think this will be the amount by which the exact solution of the ODE fails to satisfy the difference equation. So, that is the one we call as the truncation error.

So, we need to have an expression for truncation error. So,

$$T.E. = \frac{Y_i - Y_{i-1}}{h} - f(x_{i-1}, Y_{i-1})$$

Now if I expand by Taylor series, so what I get is

$$T.E = \frac{1}{h}\left[h \frac{dy}{dx}|_{x_{i-1}} + \frac{h^2}{2}\frac{d^2y}{dx^2}|_{x_{i-1}} + \cdots\right] - f(x_{i-1}, Y_{i-1})$$

$$= f(x_{i-1}, Y_{i-1}) + \frac{h}{2}\frac{d^2y}{dx^2}|_{x_{i-1}} + 0(h^2) - f(x_{i-1}, Y_{i-1})$$

$$= \frac{h}{2}\frac{d^2y}{dx^2}|_{x_{i-1}} + \frac{h^2}{3!}\frac{d^3y}{dx^3}|_{x_{i-1}} + \cdots 0(h)$$

So, we are finding the solution at x_{i-1}.

So, what I find the leading order term, order h and one important thing is that these truncation error tends to 0 as h → 0. So, that means the method is consistent with the given ODE. The ODE and the TE is order h.

Now this is very important to check that what is the leading order term. So, leading order term is given by this way so we cannot exactly measure the truncation error because that involves the infinite degree differential equation. So, we cannot go on measuring that, what is important to measure is the order, order of h. How accurate is the method is that is depends on the choice of h. So, what should be the choice of h to measure the accuracy.

So, what I find that this method is of first order in h. So, that means Euler method is a first order approximation and up here of course it is consistent. So, obviously this is the very basic method that is

$$y_i = y_{i-1} + hf(x_{i-1}, y_{i-1}), \;\; i = 1,2, \dots$$

So, this is a very basic or the very simple method. So, this is a first order method but consistent. We stop here and then we talk about some higher order methods on the initial value problem and characterize the procedure. Thank you.