Advanced Computational Techniques

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Lecture 01

Polynomial Interpolation

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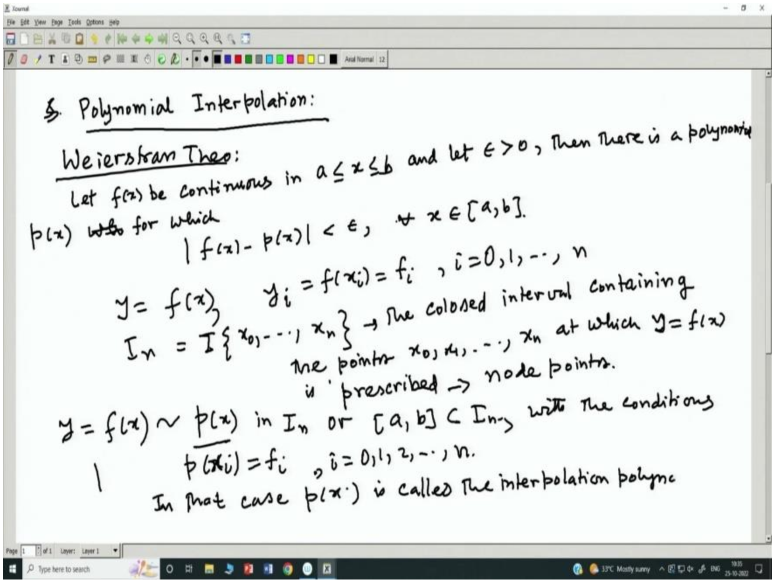

Good morning. First we will talk about the polynomial interpolation. Here what is our intention is to approximate a function by a polynomial formula. The basic for this approximation is the Weierstrass approximation theorem. So, what is Weierstrass approximation theorem? It says that if there is a function f(x), any continuous function, in an interval [a, b] then that function can be expressed in the form of a polynomial. This is the Weierstrass theorem. Let f(x) be continuous on $a \leq x \leq b$ and let $\varepsilon > 0$ then there is a polynomial p(x) for which

$$|f(x) - p(x)| < \varepsilon \text{ for all } x \in [a, b].$$

That means the function can be approximated by a polynomial, but this theorem does not say what will be the degree of the polynomial. So, this is the basic for the representation of a function. Now, many functions which we know that, say, y = f(x) is a function, a relation is there either the function is very complicated i.e. f(x) is very complicated or the function form is not known. In spite of that or in instead of that what we have a set of data points.

Let us call this f_i, i= 0,1,…,n and let I_n which we call the closed interval containing all these points $x_0, x_1, \ldots, x_n$ at which y= f(x) is prescribed. So, this also can be said as the node points at which y= f(x) is prescribed. So, our intention is that we want to approximate y= f(x) by a polynomial p(x) in I_n.

Within that interval or interval of interest, say [a, b] an interval which belongs to I_n. Polynomial p(x) we need to find out with the conditions

$p(x_i)=f_i$ (*) where, i=0,1,2,…,n.

So, that means the polynomial is passing through the prescribed values if I act these n+1 data points.

So, this condition is imposed. In that case p(x) is called the interpolation polynomial.

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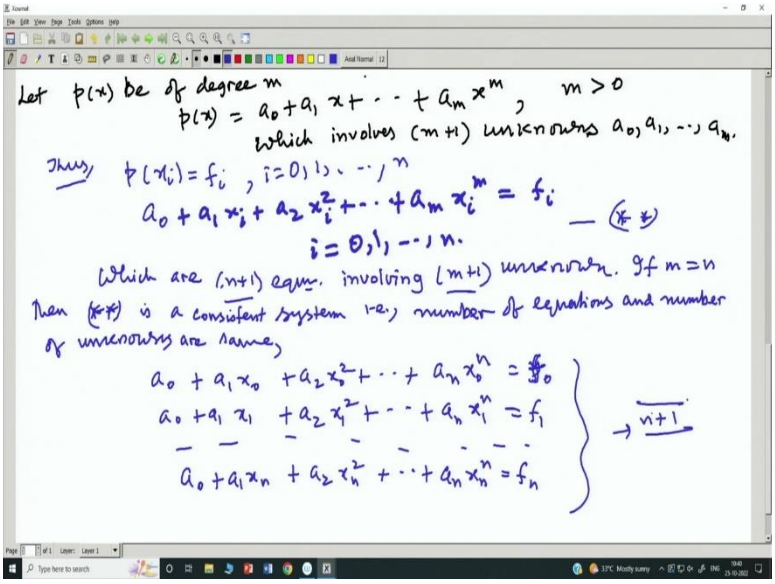

Now go for interpolation polynomial. To know the form of this polynomial p(x) you can write p(x) be of degree m as:

$p(x) = a_0 + a_1 x + \cdots + a_m x^m$; m $>$ 0

which involves m+1 unknowns $a_0, a_1, \ldots a_m$. Now, we know that this p(x) is the interpolating polynomial.

so it must satisfy

$$p = f_i$$

.

Let us call this as the condition 1.

By virtue of this what you can write is that

$p(x_i) = f_i \ for \ i = 0,1,2,3, \dots, n$

(or) $f_i = a_0 + a_1 x_i + \cdots + a_m {x_i}^m$ (**)

$$for \ i = 0,1,2,3, \dots, n$$

which consists of n+1 equations involving m+1 unknowns.

Since we have not made any restriction on the degree of the polynomial as of now what we can say is that if I take the degree of the polynomial is same as the data point, that means the unknown then this forms a system of m +1 equations in n+1 unknown.

Now, if I choose m=n then (**) becomes consistent system i.e. number of equations and number of unknowns are same which can be determined by this way. We can write

$$f_0 = a_0 + a_1 x_0 + \cdots + a_n {x_0}^n$$

$$f_1 = a_0 + a_1 x_1 + \cdots + a_n {x_1}^n$$

$$f_2 = a_0 + a_1 x_2 + \cdots + a_n {x_2}^n$$

……………………………...

$$f_n = a_0 + a_1 x_n + \cdots + a_n {x_n}^n$$

This forms a system of equation with n+1 number of variables.

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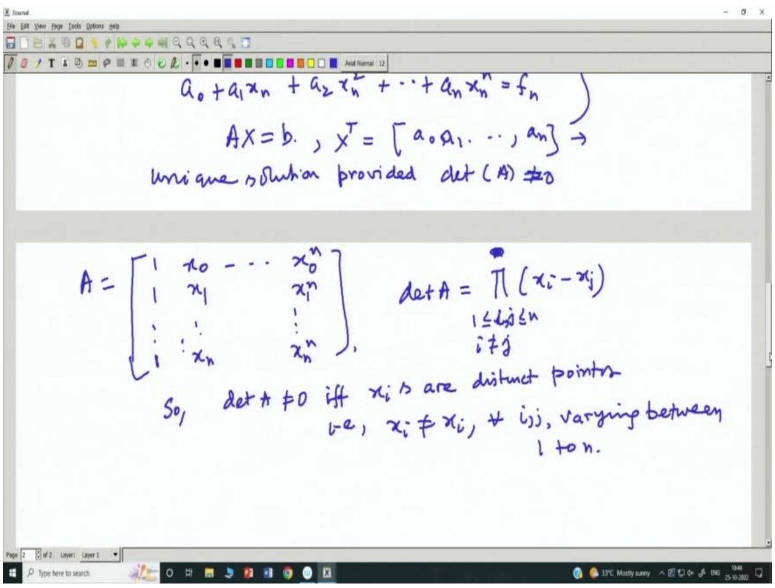

Now, this will have a unique solution if determinant of A is not equal to 0. So, this form a system

$AX = b\ ;\ X^T = [a_0 \quad a_1 \quad \dots a_n\ ]$

It is consistent so it will have a unique solution provided determinant of A the coefficient matrix is not equal to 0. Now what will be the determinant.

Now, to get the determinant of A is not equal to 0 if I eliminate $a_0's$ and all.

$$A = \begin{pmatrix} 1 & \cdots & {x_0}^n \\ \vdots & \ddots & \vdots \\ 1 & \cdots & {x_n}^n \end{pmatrix}; \qquad \det(A) = \prod_{1\le i,j\le n;\ i\ne j}(x_i - x_j)$$

So, determinate A is not equal to 0 if and only if x_i's are distinct points. That means

$x_i \neq x_j$ for all i, j varying between 1 to n.

So, this is the case when we say that this determinant is non-zero, that means, if the node points are distinct then there exists a unique polynomial of degree n which interpolates $y = f(x)$ at n+1 distinct points.

By some means if we can find out the polynomial of degree n then that will be the unique polynomial. So, this is the thing stated by this theorem. If you have n+1 data points are given them based on that n+1 data points you can construct a polynomial. If you can construct a polynomial of degree n then that will be the only order that is the unique.

So, there is no other polynomial or any other polynomial will be the identical with that polynomial itself. But how to obtain that polynomial. So, that means f(x) can be represented as $f(x) \sim p_n(x)$ in [a,b]

where we have

$$p_n(x_i) = f_i\ ;\ \ i = 0,1,2,\dots,n$$

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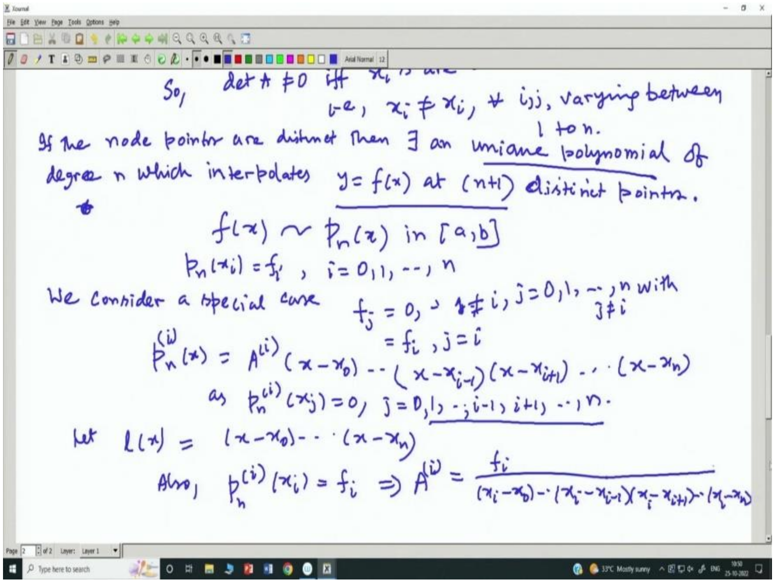

Now we consider a special case in which

$$f_j = 0 \; \forall \, j \neq i \text{ and } \; f_j = f_i \;\; for \; j = i$$

This $p_n(x)$ has zero at all these x_j's. So,

$$p_n{}^i(x) = A^i (x - x_0) \ldots (x - \; x_{i-1})(x - x_{i+1}\,) \ldots (x - x_n)$$

as $$p_n{}^i(x_j) = 0, j = 0,1, \ldots, i-1, i+1, \ldots, n$$

So, n zeroes are there that means this is forming a polynomial of degree n.

Now, let us denote $l(x)$ as

$$l(x) = (x - x_0\,) \ldots (x - x_n)$$

Also $p_n{}^i(x_i) = f_i \Rightarrow A^i = \frac{f_i}{(x_i - x_0) \ldots (x_i - \; x_{i-1})(x_i - x_{i+1}\,) \ldots (x_i - x_n)}$

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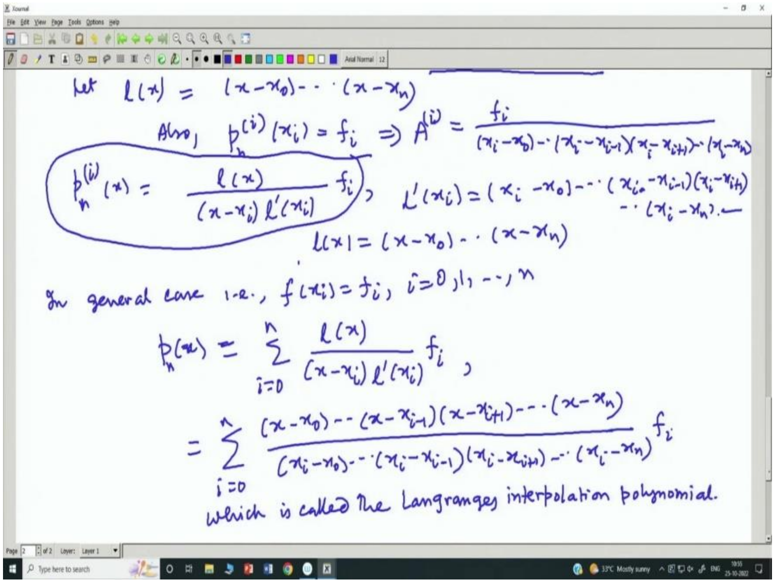

So,

$$p_n{}^i(x) = \frac{l(x)}{(x - x_i)l'(x_i)} f_i$$

why I am writing this because we have

$$l'(x_i) = (x_i - x_0) \dots (x_i - \; x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)$$

if you differentiate $l(x)$ by parts where

$$l(x) = (x - x_0) \dots (x - x_n)$$

So, that is the reason you get this. Now, if we generalize this. In general case

$$f(x_i) = f_i, i = 0,1, \dots, n$$

$$p_n(x) = \sum_{i=0}^{n} \frac{l(x)}{(x - x_i)l'(x_i)} f_i$$

So, it satisfies the condition that it is a interpolation polynomial interpolating the function at all these n+1 distinct points and also what you have is that degree is n, so, that will be the unique interpolation polynomial. So, the form is given by

$$p_n(x) = \sum_{i=0}^{n} \frac{(x-x_0)\dots(x- x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)\dots(x_i- x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} f_i$$

which is called the Lagrange interpolation polynomial.

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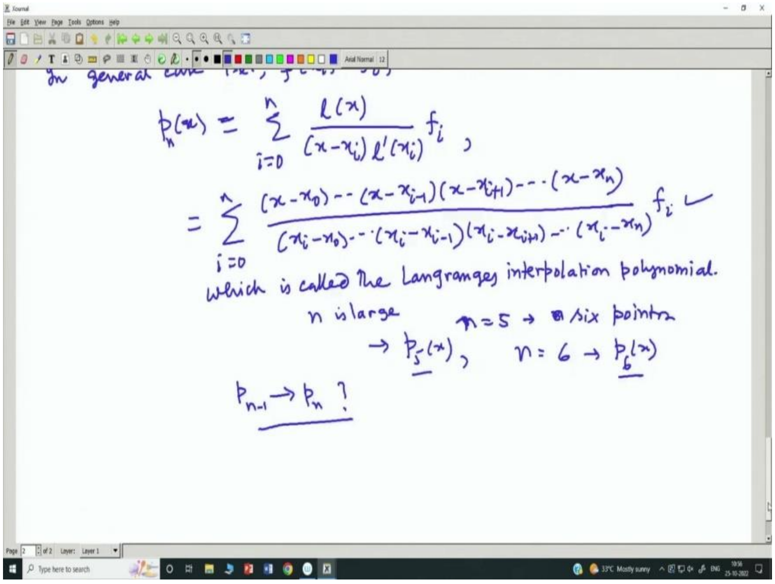

So what you find that it is very difficult to handle if n is large. So, there are huge competitions involved. That means, if I choose, for example, n=5 then you have 6 points because we are starting i from 0 to n, so 6 points you will get for a polynomial of degree 5.

This will be unique polynomial but what if I choose another interpolation polynomial and let n=6 and then I get a polynomial of degree 6, so there is no relation between these $p_5(x)$ and $p_6(x)$ So, you have to do it all over again. That means all these product 5 terms product and all these things.

So, there is no straightforward relation from $p_{n-1} \rightarrow p_n$. That is how Newton's interpolation polynomial comes into picture that we will discuss in the next class.