Engineering Mathematics -II

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Lecture 09

Stokes Theorem

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So, welcome to lectures on engineering mathematics -2 and this is a lecture number 9 on Stokes Theorem.

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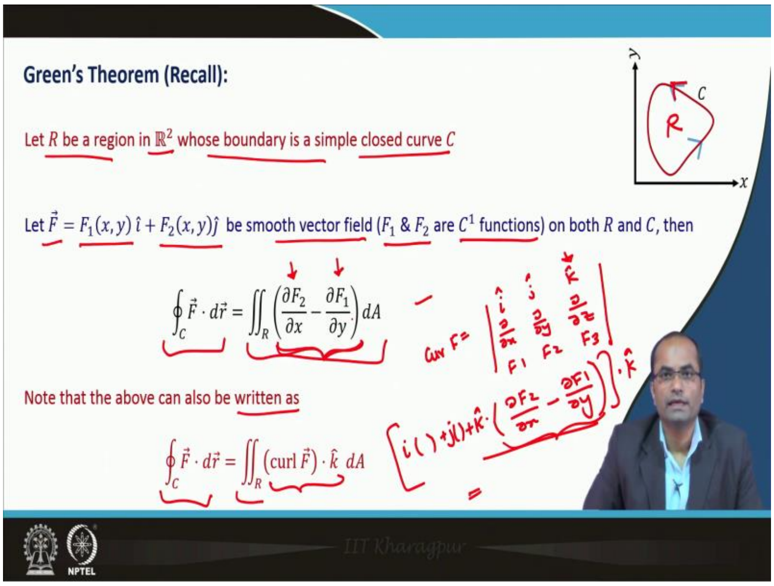

So, today we will discuss what is the Stokes theorem and basically this is the generalization of Greens theorem. So, we have already discussed the Greens theorem in previous lectures. So, the Greens theorem just to recall was to let R be a region in this R2 whose boundary is a simple closed curve C.

So, we have a region R here and its boundary is given by this simple close curve C and we take that this F, F1 i and F2 j be the smooth vector field, smooth means F1 and F2 are C1 functions on both R as well as on the curve C, then that Greens theorem says that this line integral F dot dr is equal to this area integral, where the integrant is now the difference of these 2 partial derivatives, so partial derivative of F2 with respect to X and the partial derivative of F1 with respect to y.

So, having this what we observe now, we will rewrite this Greens theorem in a slightly different way. So, this is the result here now that this curve integral can be written as this area integral. And now the difference is that instead of this difference of 2 a partial derivatives, we can also write it as curl F and it is dot product with the vector k, so, what is curl F?

So, the curl F would be i, j, k and the partial derivative with respect to x partial derivative with respect to y partial derivative with respect to z. And then we have of F that means F1, F2 and F3, so its dot product will be done with this case. So we will go for this third component only that will survive all other will vanish while this doing going for dot product.

So, this with K we are going to have this del F2 over del X and minus del F1 over del Y. So, this will be the third component and then we will have something with i and then we will have something with j and then k. So, with this curl F if you put a dot product with this unit vector k then we will get this del F2 x minus del F1, y this is exactly the integrant of this area integral in Greens theorem.

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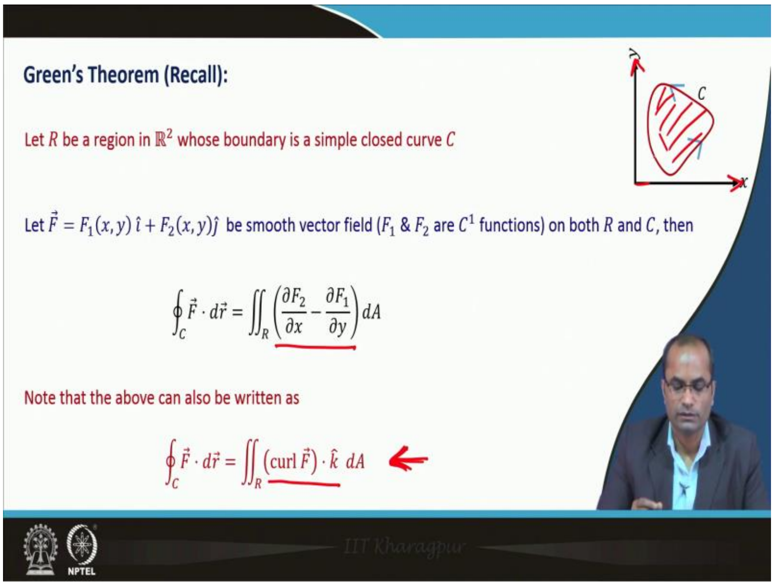

So, instead of writing this integrant we can also write curl F dot product with K. So, this is the idea now, that we can generalize this Greens theorem with this form of this equation to a higher dimensions. So here this Greens theorem was stated for R 2 for a plane. So, here we had the region in R 2 and this curl was also in R 2. So, if we generalize this in 3 dimensions, that means, we have now X Y Z and there is a curve which is given in 3 dimensions and then the bounded region by this curve will be a surface. So what will happen now to the new result that we will come up now?

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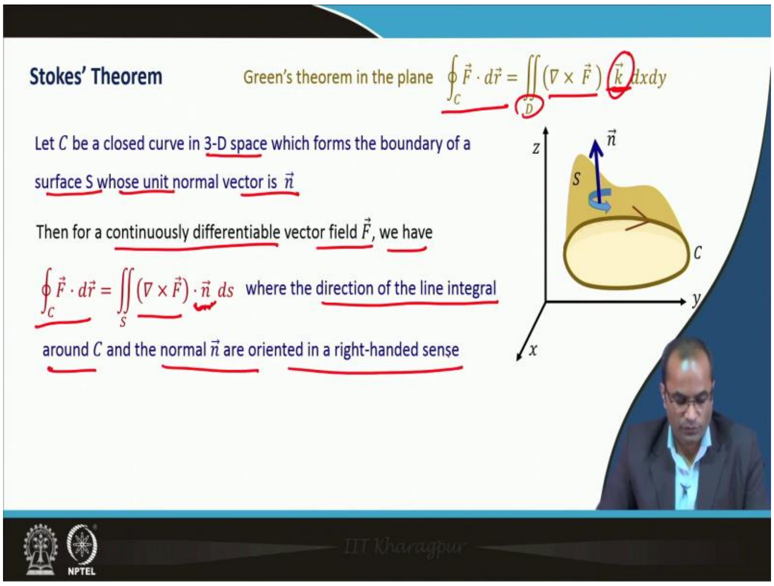

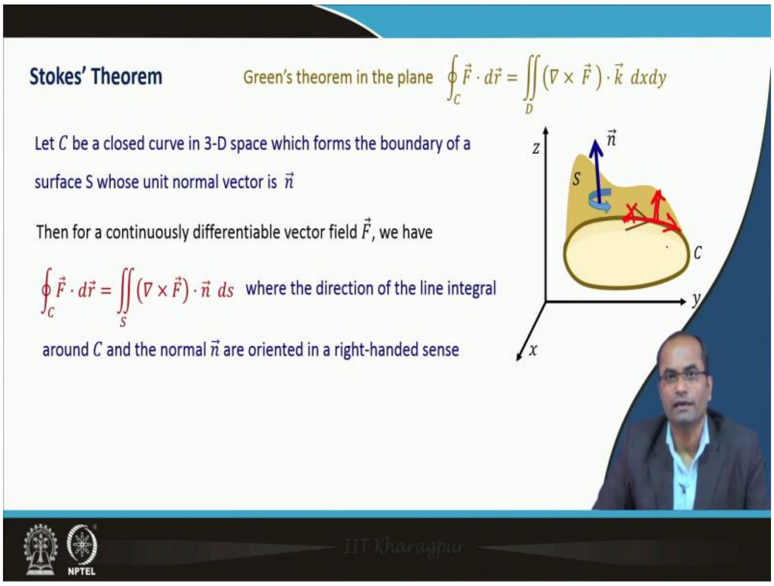

So, the Greens theorem again in the plane was given by this. So, now, we let c be a close curve in 3D space. So instead of 2D space, we are talking about 3D space, which forms a boundary of a surface. So naturally if we are talking about a curve in 3D space, its boundary will be a surface whose unit normal vector is n.

So suppose its unit normal vector is n then for a continuously differentiable vector field, so now F is also defined for 3 dimensions and the result we have in this case, the curve integral F dot dr, the same integral what we have in Greens theorem will be equal to so we have this curl F. There also we have curl F and in Greens theorem, the dot product was just with the K and K was the unit normal to the plane X Y.

So, we have the, we have the curve and we have this region D in the X Y plane. So, basically this k was the normal to the D, to the region D. Now, we have the same situation. So instead of that k now, we have here n we have a unit normal to the surface, it was earlier also unit normal to the surface but the surface was in the plane X Y. So, the unit normal was just K, but now we are talking about the surface, so it is normal, we have denoted here by n.

So now the important points the direction of the line integral around C and the normal n are oriented in a right handed sense, because now this is important. This curve integral the value will change if we take instead of anticlockwise if we go for clockwise directions, here also this unit normal. So, there are at each point they will be two normal, one in the outward normal or the inward normal whatever we call there will be 2 normal there.

So, now when this exactly equality will happen, because otherwise there will be a problem with the minus sign. So, when this equality will happen that we have to understand that what should be the orientation of the curve and what should be the orientation of the normal. So, that has to be in line with and to understand this the one way would be that you think that, suppose here you have the unit normal n on this curve and now having if we walk through.

So suppose this is a man standing at this position whose head is exactly this arrow and if he or she walks around this curve, then the surface should be the left hand side. So, that is one way of understanding this, so again just to repeat, so let us stand here and then draw a normal at this point. And then if we walk for instance, if he start walking in this direction, then there is no surface to our left hand.

So, we cannot go in this way otherwise there will be a problem we have to take the normal to the other direction. So if you want to have we have fixed the normal in this direction, then we need to walk in this direction, so that the surface will lie left side to us. So this is one way of understanding this.

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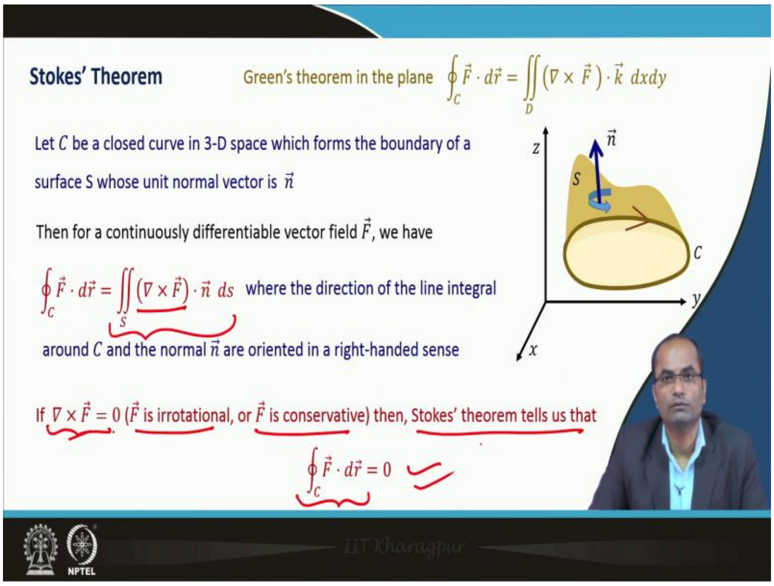

The second way we can always have this right handed system. So, suppose this is the normal direction, we have taken at any point and then we go with this right handed system. So, this normal direction suggests that this rotation will be in this direction. So, rotation will be in this direction and that finally will can suggest us what will be the rotation on the curve C. So, this is one way of understanding the direction of the (curve) orientation of C with respect to the orientation of n.

So, we will go with this. So, again we will have on the curve we will draw a normal and then we will walk through the direction and it should be the surface should lie left to us. So, if you are walking in this direction then the normal should be in other directions so that again the same thing happened that when we walk around this the surface will be left to us okay.

So, having this knowledge now we can go through. So just a corollary here that if this curl F is 0 that means if F is irrotational or F is conservative this we have already discussed before, then it Stokes theorem suggest us that this F dot dr is going to be 0 because this curl left is 0 and the Stokes theorem says that this is equal to the surface integral and if this curl F is 0, the surface integral will become 0 and we have this curve integrant equal to 0.

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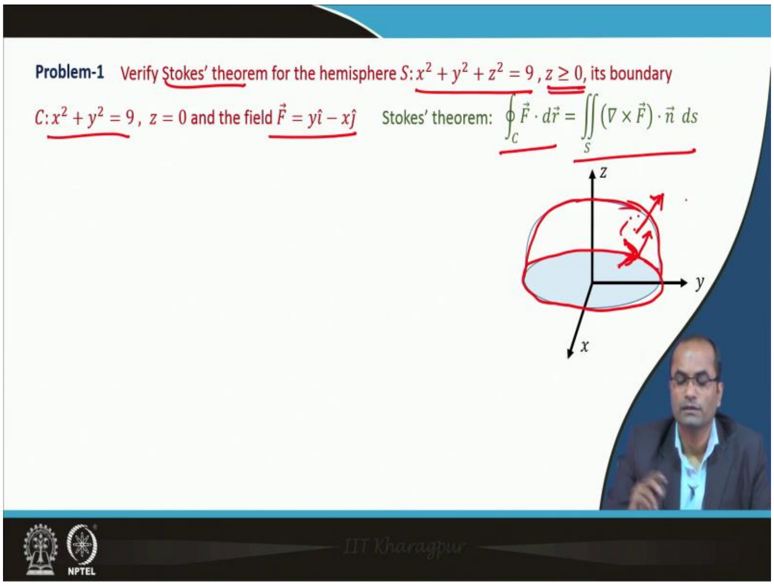

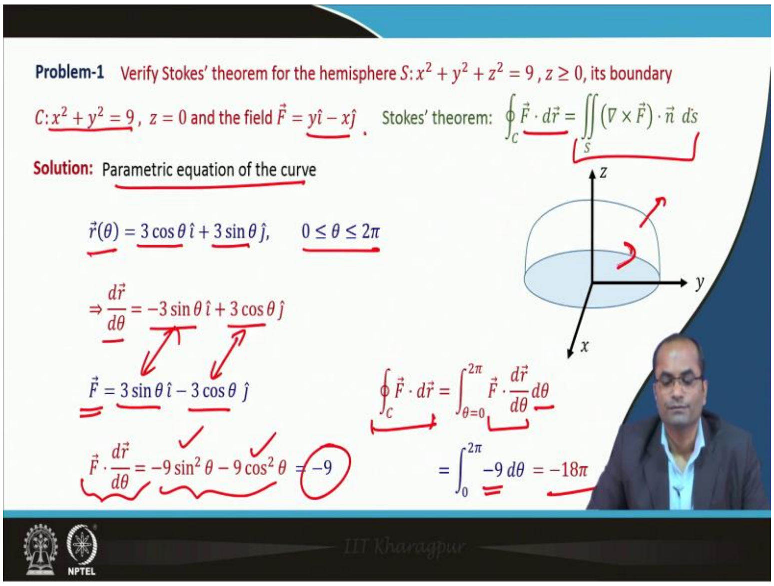

So, will verify in this problem the Stokes theorem for this hemisphere. So this is the hemisphere for just z greater than equal to 0, so the upper part there on the X Y plane and its boundary is x square plus y square equal to 9 at z equal to 0 plane. So the boundary is of course on z equal to 0 plane on X Y plane and the vector field is given by yi minus x j.

So, the Stokes theorem that we need to verify this curve integral we need to verify the right hand side the surface integral, we already know that how to compute surface integral. So, this is the situation here we have this hemisphere and its boundary is given by this circle here on the X Y plane. So, again the same idea if we take the outer normal and we walk along this curve here in this direction, then everything is in line.

The surface lies left to left to us. So, we will take the outer normal and this anti clockwise direction for the circle, the boundary. So the parametric equation of this curve of the circle we can easily write down. So it was x square plus y square is equal to 9 a circle of radius 9. So r theta is 3 Cos theta 3 Sin theta and theta is 0 to 2 Pi. So dr d theta, we can compute out of this, so minus 3 Cos theta and then 3 Cos theta.

So the F was given y i minus x j, so y is 3 Sin theta and x is 3 Cos theta, so F we have also substituted this parametric form. And then F dot dr we need for this curve integral, F dot dr So F dot dr d theta will be minus 9 Sin square theta, so the multiplication of the 2 here and then multiplication of these two, so minus 9 Sin square theta then again minus 9 Cos square theta, and we have this factor minus 9.

So, this F dot dr this integral, the curve integral, we know that this is equal to F dot dr and then d theta and theta varies from 0 to 2 Pi. So theta is 0 to 2 Pi F dot dr is 9, so we have minus 9 and then the 2 Pi is the value there. So, we have minus 18 Pi is the value of the curve integral. And now in this in the second case, you will consider this right side integral where we need to compute the surface integral over this hemisphere.

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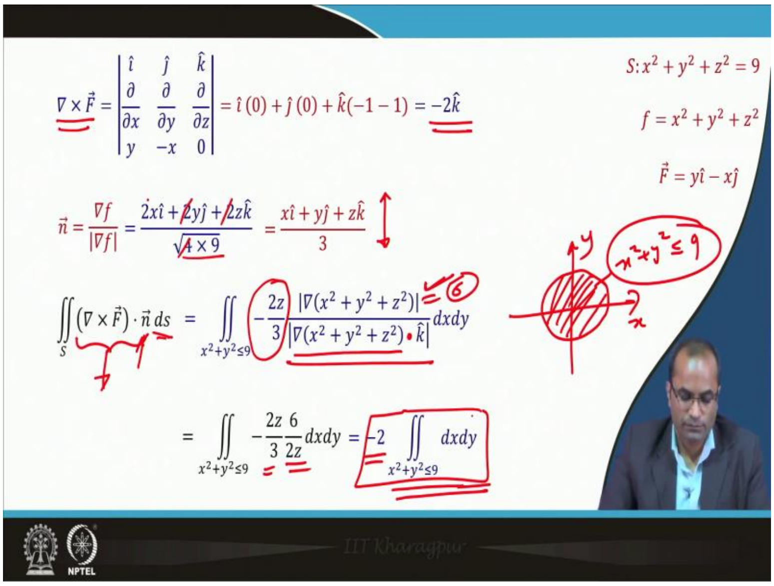

So for that, so we have now of S, the surface we have the F, the equation of the surface this F the function and the vector field F. So this curl F we need to compute that means this determinant i, j, k and then we have the partial derivatives and then we have these first x component then y component and then 0. So in this case I and then this will give the 0 that product with the j also this is going to be 0 and then for k as well, we have this for k we have the minus 1 and then again minus 1, so minus 2 k is the value there.

So this is the curl F and then we need to know the unit normal vector to this surface del F over its modulus and that can be computed. Now del F so here we have this F so thus del F will be 2 x i and then 2 y j and then we have 2 z k so that is del F. And to divide with this absolute with this modulus with this will get 4 x square plus 4 y square and then 4 z square and then the square root.

So this 2 will come out and then we have again here x square plus y square plus z square in the denominator and then we have the numerator there. So, this x square plus y square z square on the surface that value is given 9. So therefore, we have this 4 into 9. So, this 4 will get canceled with this 2 there and then we have x i y j plus z k by 3, this is the unit normal vector to any point at any point of the surface of this or this hemisphere, then we need to get this surface.

So we know this already del cross F. We know the n, so we can get this dot product. So there is only the k factor there. So this will be just z by 3 and with the minus 2, so minus 2 z by three, this is the result of this integrant.

And then for ds, we need to have this factor which we have discussing the surface integral that means the del F we need to get and del F the cross product, the dot product with this in this case k because we have projected that surface on the x y plane and once you projected on the x y plane, this is a disk with radius 9 whose boundaries just a circle of radius 9.

So we have projected on this X Y plane, therefore we have to also multiply this dot product with the vector p which is k here. And now we can simplify this. So this is minus 2 z by 3 and then this we need to compute this will come as 4 into 9, so 36 and then as a result, this will be just 6 there and here with this dot product, so this will be coming as 2 z, the absolute value because z positive.

So we have written just 2 z there. So it is minus 2 and then we are integrating over this disc, dx dy. So this was the projection on the X Y plane, this was X and then Y. So, this is the projection this disc x square plus y square less than 9, this is the disc there, where we have to this perform this integration and this integration is nothing but the area because this is the area of that disc.

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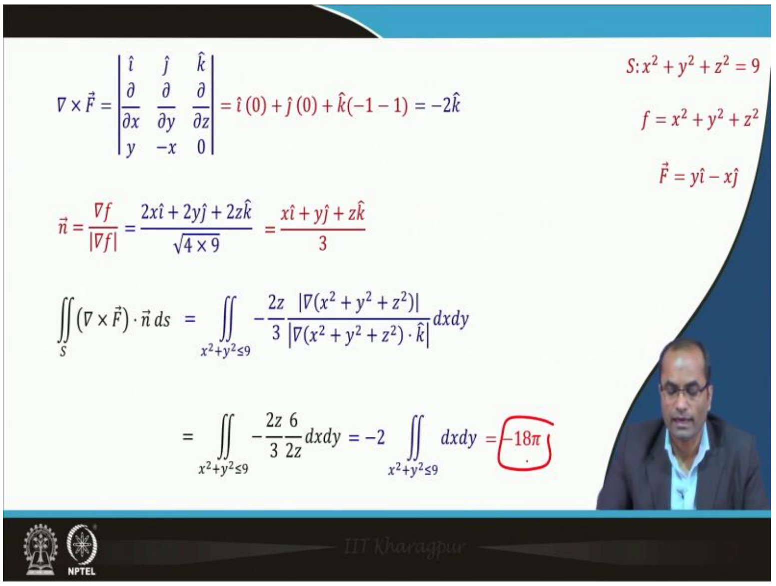

So it is Pi r square, so minus 2 and then we have So, minus 2 into this Pi and r is 3, so 3 square 9, so 9, so minus 2, so minus 18 and then Pi. So this is the integral which we got through the line integral as well. So, the line integral and the surface integral, the value is the same. So, it verifies the Stokes theorem.

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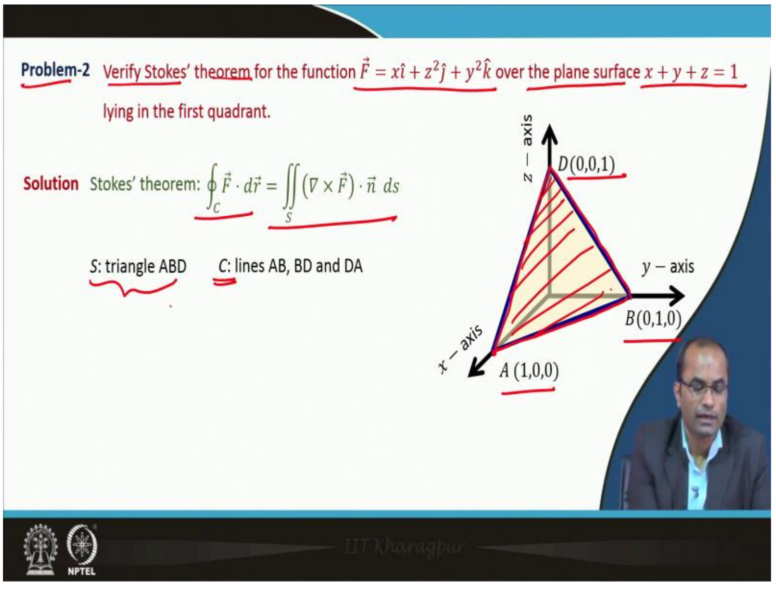

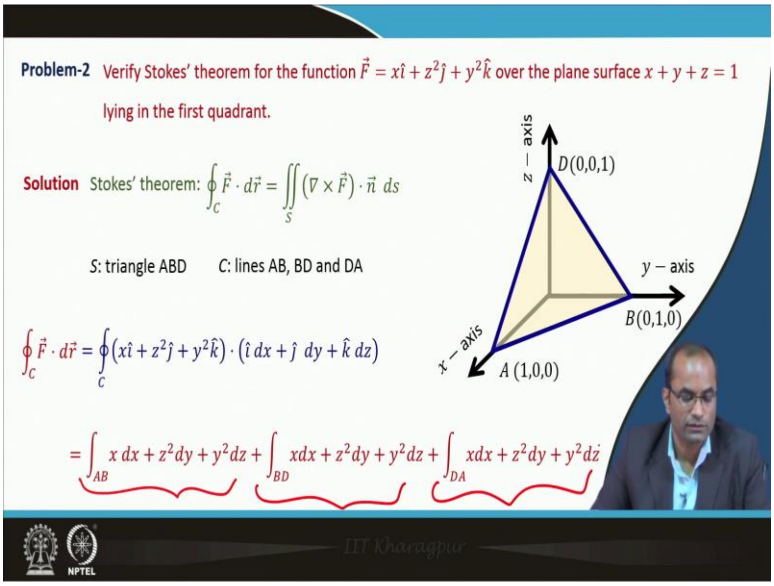

Another problem here again we will verify the Stokes theorem, but now the function is bit more complicated the vector field xi z square j and y square k and we will take over the plane surface x plus y plus z equal to 1 lying in the first quadrant. So, this is the situation this is the plane we are talking about lying in this first quadrant A, B and D.

So it is cuts here on A on X axis with 100 then here 010 and 001 on the Z axis. So, the again the Stokes theorem, we need to apply here the S, the surface is this triangle, the triangular surface and the curve C these are the lines which are the boundary which form the boundary of this surface. So here the curve the bounding curve, are just these 3 lines and the triangle here is the surface.

Well, so this F dot dr, let us compute first F dot dr and so F is given and then dr we have i d plus j d y and z k. So, whenever we have line, just direct a component wise setting works. So, in this case now A B and then we have B D and then we have D A we have the 3 integrals over the 3 lines, so we will consider each of them separately.

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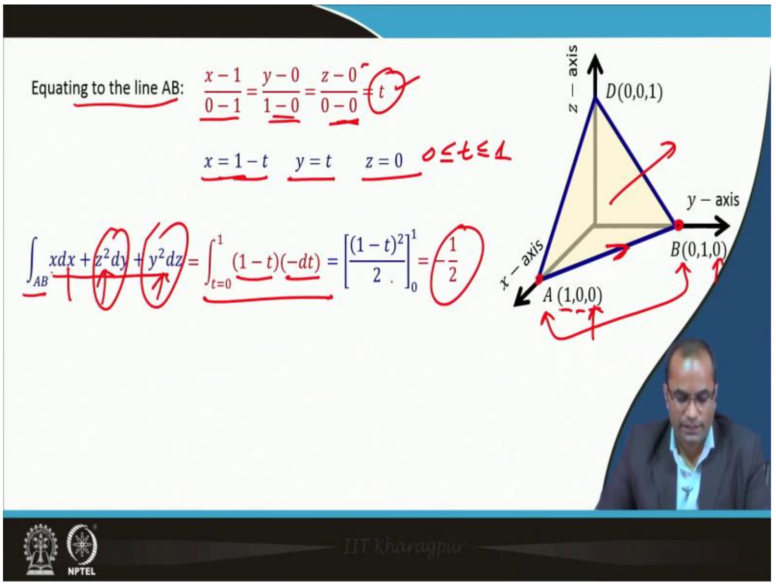

So, now on the line AB, so if we consider this line AB here, that will be the equation so x minus 1 and then y minus 0, z minus 0 and then here the difference. So 0 minus 1 and 1 minus 0 and then 0 minus 0, that has to be equal to t for linearity just we introduced a parameter t there, that means x 1 minus t, y is t and z we can take 0 because the z component is not varying along this line.

So x is 1 minus t and y is t that means and for t concerning t, t is 0 then we are at this point A because 1 0 0 and then t is 1. We are at point this B. So if we say that t goes from 0 to 1 we are going in this direction. So, we will take anti-clockwise direction and then we have to take the outer normal on the surface to have that equality of the conclusion. So, along this A B we will integrate x dx z square dy and y square dz we have these parametric equations, we will put it there.

So, x is 1 minus t, the dx minus dt regarding the second one since z is 0, so this will become 0 and the third one also because of dz this will become 0. So, these two will not contribute now, only the x dx part will contribute and that we can make, we can integrate it and put this limit 0 to 1, the value is coming as minus half.

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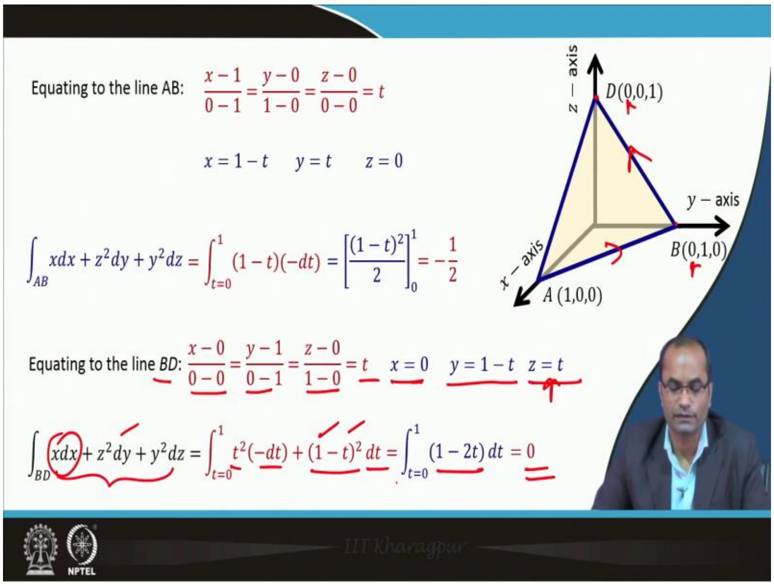

Now, for the second line this BD, so we can write down again this equation for BD. And this these ratio equated to this t to have this parametric form. So, in this case the x is not varying you have the 0, there also it is 0, so x is 0 and y is 1 minus t and z is t. So, again the t varies from 0 to 1 and we are moving in that direction anti-clockwise.

So, writing this integral this time the x is 0, so the first will not contribute, but the rest two will contribute because y and z they are changing. So, here we have the z as t, so t square minus dt y square, so we have 1 minus t whole square and then we have dt for this dz out of this. Well, so just so we can simplify this. So, here minus t square there we will get the t square from here that will cancel out so we will get 1 and then minus 2 t that is the conclusion here and that value will be 0 when we integrate this one.

So, along this BD the value of the integral is 0 along this AB the value was minus half. And then the third line we will consider as the DA line.

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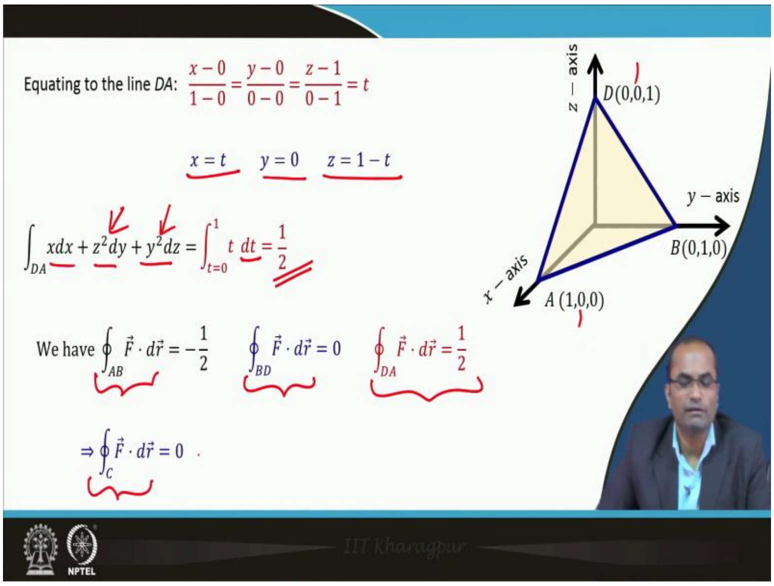

So, along this DA line, so, this y component is fixed and this x is varying from 0 to 1, so we can have like t and z is 1 minus t that is the parameterization of this line. So, again this x dx z square dy and y square dz , so in this case y is 0, so this is gone and because of d y and this is also will not contribute because of this y so we have again x dx, so x t and then dx will be also dt. So if we integrate this we will get the value half.

So, we have (contribute) we have come computed all 3 integrals, and so along AB the value was half, along BD it was 0, along DA it is half. So, if we add all 3 to get the complete this curve integral, the value is coming as 0.

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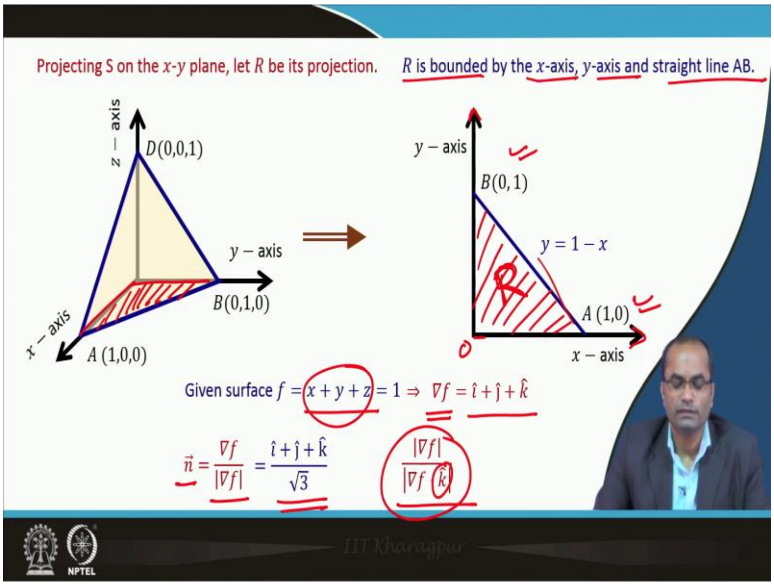

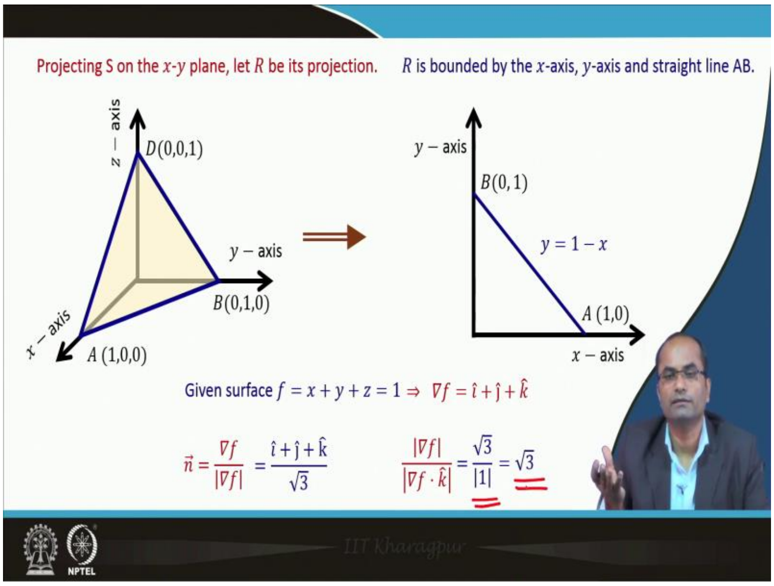

And now to verify, we have to project the surface, the triangle into one of the planes. So if we do in this X Y plane then we are getting this triangle there. So in the X Y plane if you project the whole surface, so it will just be a triangle. So just for demonstration we have taken now here that triangle. So this triangle has this point 1 0 and then 0 1 there.

So this is the region now where we will integrate that surface integral. So R is bounded by the X axis, Y axis and the straight line this AB So this is our R, the projection R and the perpendicular to this R is going to be again the k. So given surface we have x plus y plus z equal to 1. So, we will take this F and we can compute the unit normal vector. So for that we need to compute the grade F that is i plus j plus k and the unit normal.

So, we have to divide by its magnitude and we will get i plus j plus k divided by its magnitude which is the square root 3. So grad F over grad F dot k we need to compute this factor which will convert the surface to this area integral and this factor we have taken k because of the projection on X Y plane.

Accordingly we can project on X Z plane or Y Z plane then this k this normal will change. So, here we will get this grad F absolute value that is square root 3 and then the product with this del F and when we do the dot product with this K, so it will be just 1 there. So, we have the 1 here. So, that means this is square root 3. So that factor from surface to this area will be now a square root 3.

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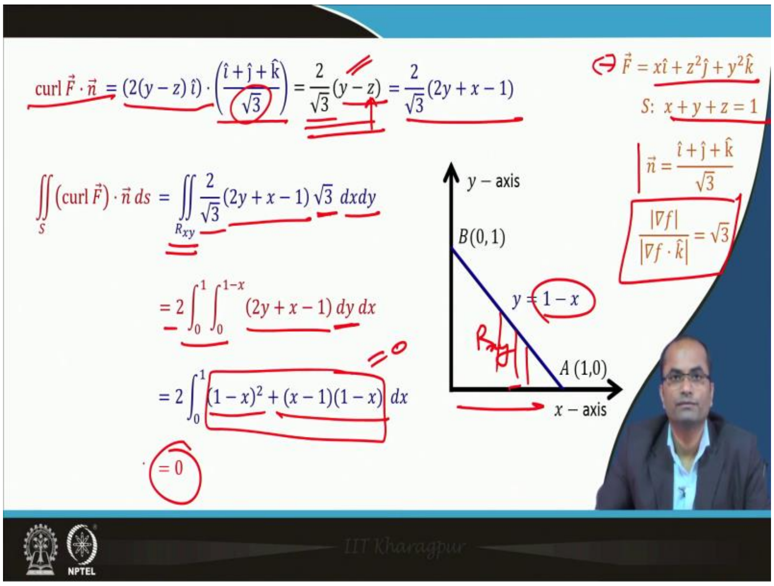

Okay, so we have all these ready for the surface integral we need this, we need the normal vector and we have the F there, so curl F dot n need to be computed, so we have the curl F now dot with this n, the curl F we can compute, so we are not going to compute now here but that will be 2 times y minus z i dot product with this unit normal vector and that is coming only because of this i components.

So 2 over square root 3 y minus z so this is curl F dot n. We are replacing this Z because we will be integrating over the X Y plane. So the Z we can replace from here as 1 minus x minus y. So that will be to 2 y plus x minus 1. So, we have this, the region R and then we can do this surface integral. So, the surface integral will convert to this area integral this is R x y we are denoting here, so 2 over square root 3, 2 y plus x minus 1, square root 3 and dx dy, this area integral.

So two times and then we need to now find the limit of x and y So, limit of x 0 to 1 and then for y we can restrict now, the y will be from 0 to this 1 minus x which is the equation of that line x plus y equal to 1 after the projection. Well, so this can be simplified now, so 2 y plus x minus 1 when can first integrate this dy with respect to y as a result we are getting this and then we will see that this is actually getting 0 the integrant. So we do not have to indeed do any interval for x.

So, the value is 0. So, again we have verified the Stokes theorem by computing the surface integral also the area integral.

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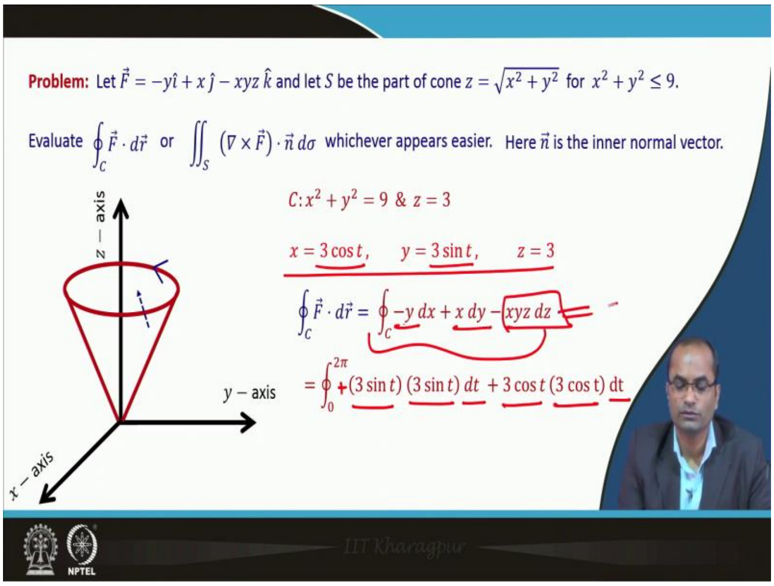

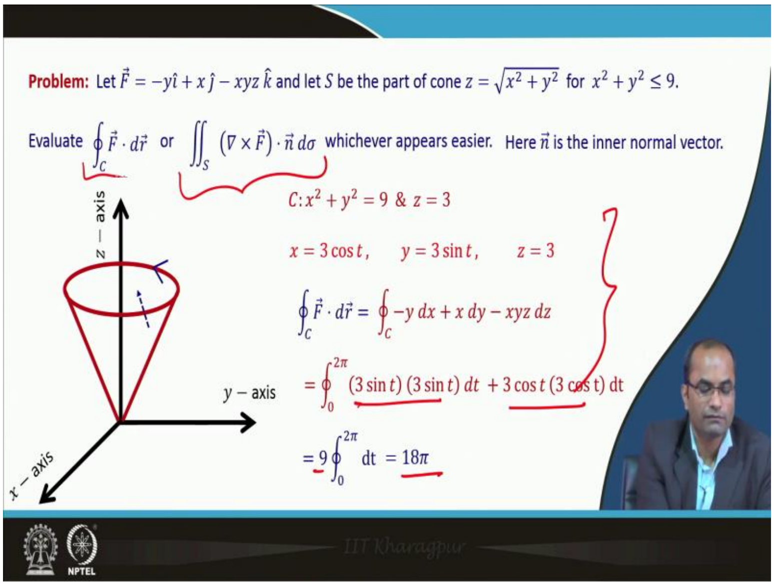

So, this is the last problem you want to do using the Stokes theorem. So, the F is given and let S be the part of the cone. Here the cone is given z is equal to square root x square plus y square and we are going up to x square plus y squared less than equal to 1. So the boundary there on this cone will be this x square plus y square is equal to 1.

So, this is the situation and we want to evaluate this curve integral or the surface integral whichever appears easier and n is the inner normal vector. So the n is also fixed here. So in a normal vector drawn here, so it is an inner one not the outer one, but the inner one. Well, so this is the cone here and now we have to see the which direction we go with the with the curve C whether anti-clockwise or the clockwise direction.

And the trick is the same that if we stand along this arrow here, with having head exactly on the step, then we have to see that in which direction we move, so that the surface lies left to us. So naturally, we have to go in this direction to have surface on our left hand side and therefore the integral has to be done in anti-clockwise on this curve, the boundary. So the curve here is x square plus y square is equal to 9 and in the plains z is equal to 3 that is the equation of the curve.

So that can be parameterized x is equal to 3 Cos t and y is equal to 3 Sin t and z is 3, why we are parameterizing now? Because definitely the surface integral will be more involved because we have to compute n then that factor which converted d sigma element to the area element and so on the projection we have to think, so definitely this is going to be much more complicated as compared to this curve integral. In the curve integral we have the circle, we know these parametric equation x is 3 Cos t, y is 3 Sin t and z is 3.

So we can easily compute this curve integral and the surface integral seems to be difficult. So coming to the curve integral F dot dr we have this F dot dr, so F is given already here y minus x j minus x y, z k and then dr is i this dx plus this dy and then j k dz. So, this dot product we can make with this F, so what was F? Minus y i plus this x j then minus x y z and it was k the dot product with I, so dx then here dy will be there and dz will be there with a component k.

So, this minus y dx will be there x dy will be there and minus x y z dz is there, so this is fine we have this integral and the parametric equations are already given. So, we can substitute y is 3 sin t, y is 3 sin t and then dx 3 Cos dx here, so dx will be 3 with minus signs so this minus-minus will be adjusted so we have plus there. So, 3 sin t dt and similarly here x is 3 Cos t and the y sorry dy will be 3 Cos t and then dt. The third one, where we have this dz this is going to be 0 because that is constant there is no variation in z.

So, having this we can now simplify this we have (3) 9 sin square and 9 Cos square, so basically 9 and 0 to 2 Pi dt, so that is the thus the 2 Pi there, so 9 into 2 Pi, so we have 18 Pi is the answer. But if you want to compute over this surface integral, then this will not be that simple as we have seen this curve integral. So, sometimes this Stokes theorem can be used for this transformation of the surface integral to curve integral. If we do not want to compute here the surface integral, the integral value we can get using this curve Integral.

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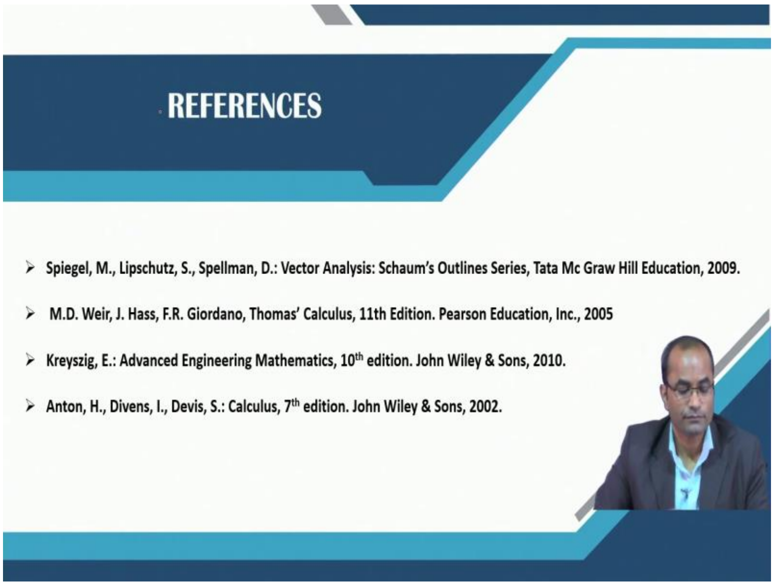

So, these are the references used for preparing this lecture.

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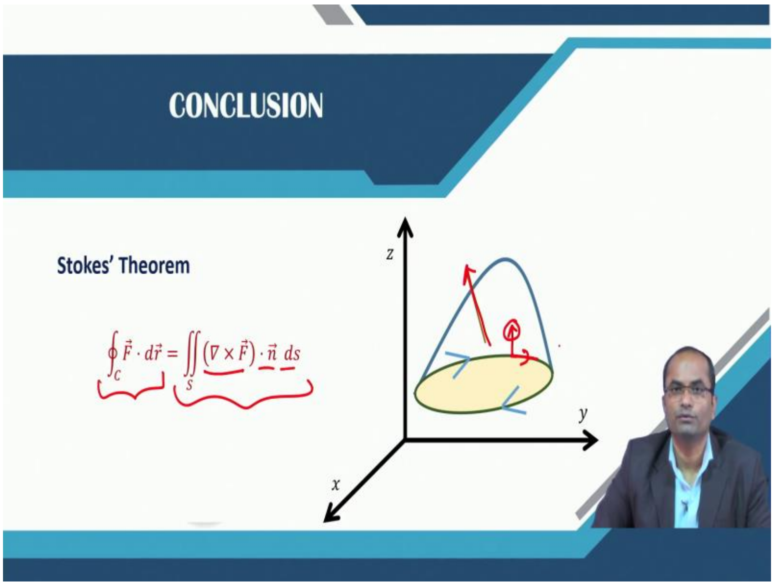

And just to conclude, we have just discussed the Stokes theorem in this lecture and which says that this curve integral of this F dot dr is equal to the surface integral of this curl F dot n and the surface. The most important point here in this equality was the direction of the normal and the direction of this curve, the orientation of the curve.

The idea we have discussed that, if we stand here on the curve along this arrow along this normal having this head on this top there and if we walk through along this direction of the curve, the surface should lie left to us and then we will have this equality there. There will be no problem with the sign, otherwise one sign will be will be different in one of the integrals there. So that is all for this lecture and I thank you for your attention.