Engineering Mathematics-II
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Lecture 60
Applications of Laplace Transform continue

So welcome back to lecture on Engineering Mathematics 2 so this is 60 on Applications of Laplace Transform. And this is second lecture on this topic we have already gone through some applications for solving for instance initial value problem, integral equations.

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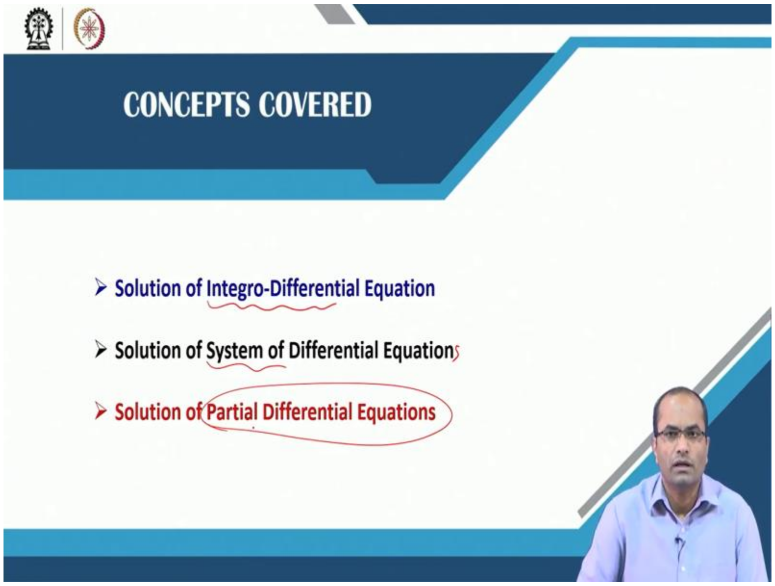

So in today's lecture we will go through how to solve using Laplace transform and integro differential equation. And as well as we will also look into the possibility of solving the system of differential equations and finally we will be talking about how to solve partial differential equations using Laplace transform.

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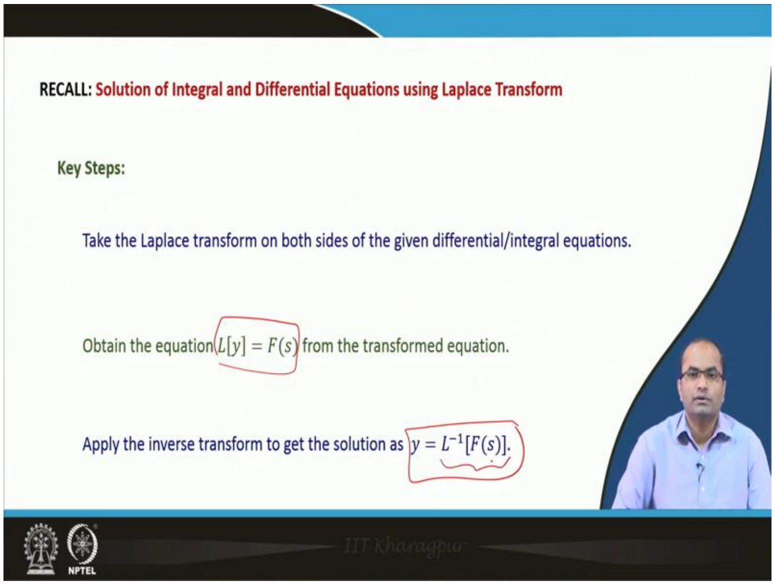

So just to recall from the previous lecture we have gone through several examples of integral and differential equations that can be solve using Laplace transform. So the key steps for solving all those integral or differential equations were that we need to take the Laplace transform on both the sides of the given differential or the integral equations. And then we will obtain by simplifying the given or the transformed equation into this form.

So that the Laplace transform of y is equal to some function of as appeared. And here during this transition the equation was or the differential equation was with constant coefficient then this was much easier to get this form. But if the equation was having variable coefficient in that case we need to solve again a simple in most of the cases linear differential equation to get this Laplace transform of I equal to fx. And once we have this form we need to take the inverse Laplace transform so that we get the desired unknown y by taking the inverse of this function fs.

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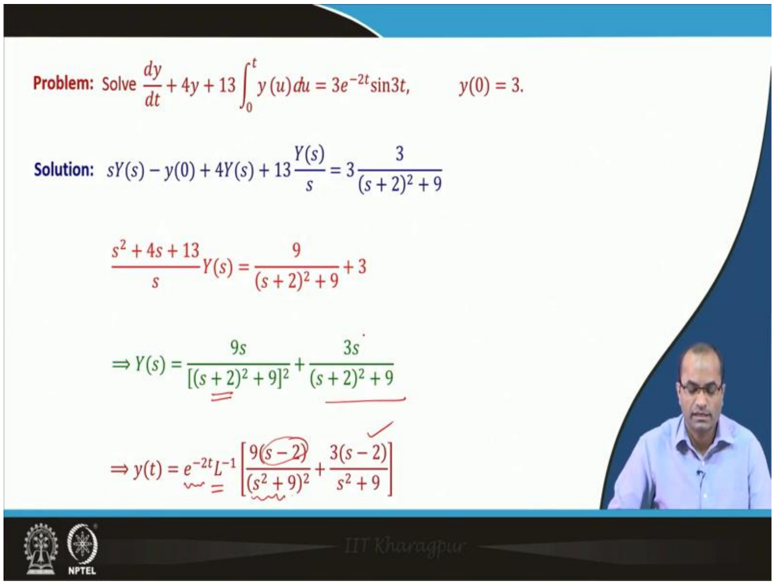

So today we will continue with integro differential equation, so in integro differential equation we will have differential term as well as integral term. Like in this example we have dy over dt plus 4y plus 13 and there is a integral where this unknown y is there as an integrant. And the right hand side is a function of t and the initial condition because this derivative is given so we must have some condition there. So we need 1condition because it is a first order in y so here the y is given at 0 and the value is 3. So the process will remain the same and we will take the Laplace transform on both the sides of this equation.

So here so instance this dy over dt we have this derivative theorem so s and the Laplace transform of y minus this y 0 then 4 times the Laplace of y plus 13 times and here it is an integral. So 2 types of these integrals we can handle one is this direct integral of this kind 0 to t then we can apply simply that property that what is the Laplace transform of the integral. So here we have to just divide by y and the Laplace transform of this integrant. So that is one possibility other possibility we have seen in previous lecture that if we have this convolution type integral that can also be handle using this Laplace transform by applying that convolution theorem.

So right hand side also we need to take the Laplace transform and we taken. So we need to use this shifting theorem because e power minus 2t is there with sin t. So the Laplace transform will be this 3 over s square plus 9 for sin square for sin 3t. And due to this e power minus 2t there will be shift here so s is replaced by s plus 2. After doing this we will use the given initial condition so y0 will be taken as 3 and this is this has gone to the right hand side.

The rest here so we this s square when we make this common denominator so we have s square and then there will be a term with 4 all with ys and here also this 13 with ys. So we have the left hand side this s square plus 4s plus 13 by s into ys and remaining we have the right hand side this is 9 over s square plus 9 and plus 3. So we can now take everything to the right hand side here expect this ys. So ys is equal to 9s over this whole square term will come because this also the same term s plus 2 whole square plus 9.

And here also will get then 3s over s plus 2 whole square plus 9. And now as a final step we need to go for this inverse transform. So we have to take here the inverse transform and we have use this shifting theorem again so e power minus 2 t will come out and the Laplace transform of 9s will be shifted as s minus 2. Here also this s is shifted by 2. So we have s square plus 9 whole square plus here also we have s square plus 9 and there 3 and s is now written as s minus 2.

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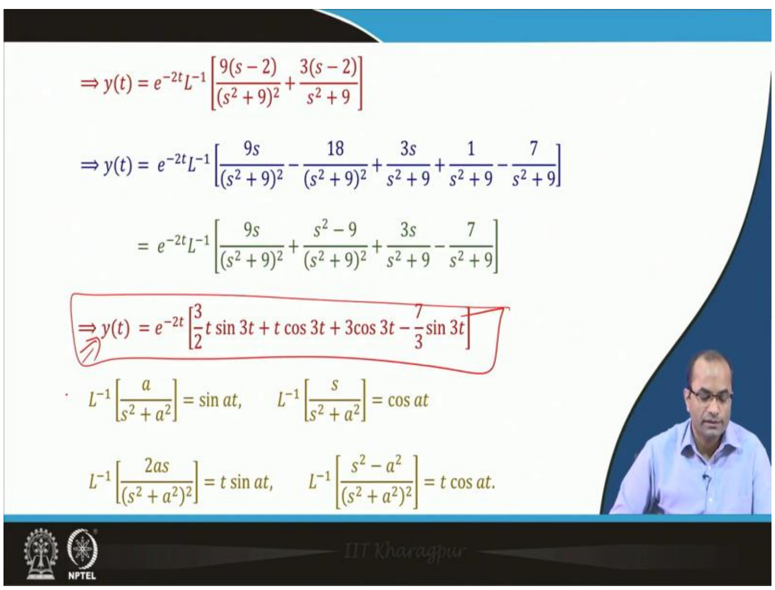

So having this now expression for yt that we need to take this inverse Laplace transform. So let us break into this fraction so we have 9s over this s square plus 9 whole square then we have 18 there and here we have this 3s over s square plus 9. And there is a term minus 6 which is written here plus 1and minus 7 so that is add up to minus 6 over this s square plus 9. This is the exactly the term there so and now what we will do.

So here 9s over s square plus 9 as it is and this term together with this term here so these 2 terms are merge now to have a special form. So here s square plus 9 whole square and then we need to have here multiplication by s square by 9 and also division by s square plus 9 to this term to have common denominator. So that s square plus 9 together with this minus 18 will give this s square minus 9.

And then we have 3s over s square plus 9 as it is and we have minus 7 over s square plus 9. And now we know these Laplace inverses so Laplace inverse of a over square plus a square that is sin at we know and when we have s over s square plus a square we have these cos at and a s over s square a square whole square that gives t sin at. And then here s square minus a square where s square plus a square that is t cos at.

So having all these in mind we can now look for these expressions so we have for instance this one 9s over s square plus 9. So that is exactly fitting into this form s square plus 9 square that is feeding into this form and these are some cos n and sin form. So taking these Laplace transform what we get yt is equal to e power minus 2t.

Then we have here 3 by 2 t sin t because of this then we have here t cos t because of this formula and then 3 cos t 3 t and then minus 7 by 3 to just compensate to have 3 there on denominator. So we have this sin t. So this is the inverse Laplace transform and the solution of the given differential equation or integro differential equation.

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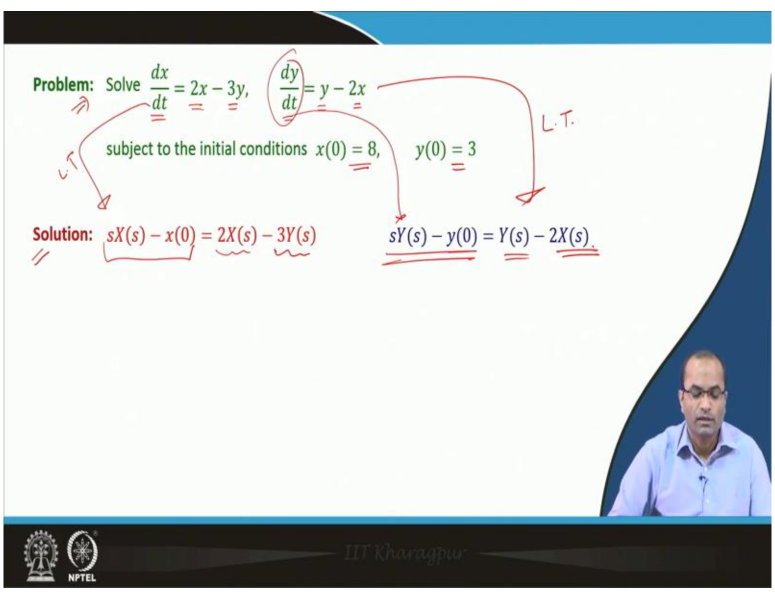

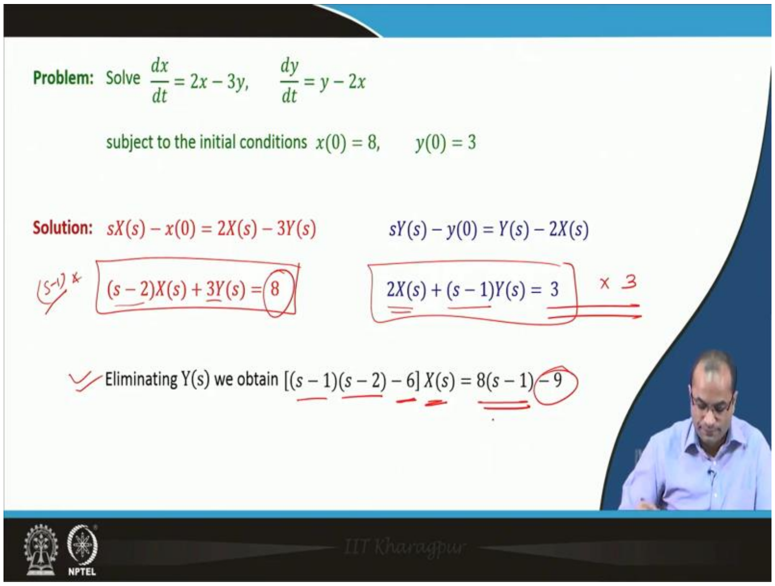

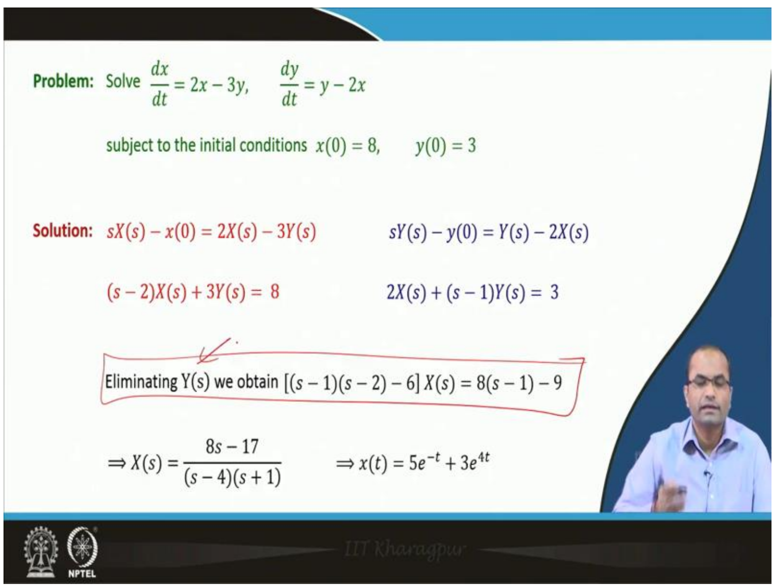

Here we will just take 1example of the system of differential equations. So in this case we have 2 equations here dx over dt written as 2x minus 3y dy over dt as y minus 2x. So we will demonstrate the idea by this simple example but one can go through many more. So subject to here the initial conditions we have for x0 is equal to 8 and the y0 is given as 3. So solution we have to again use similar steps again.

So dx over dt we apply the Laplace transform so we will get s xs minus x0 the right hand side we have 2 xs and then minus 3 ys. Similarly for the second equation we have to apply this Laplace transform so after taking this Laplace transform for the second equation we will get again we will apply here this derivative theorem. So sys minus y0 and the right hand side we have for y ys and minus 2 xs.

So these x and y they are unknown and t here is the dependent independent variable so we have applied Laplace transform to both the equations and we got these transformed equations. Now each we can use of instance here this x0 and here we can use y0. So we will get simply here x minus 2xs plus 3ys equal to 8. Similarly here after using this y0 s3 we will get 2 xs plus s minus 1y. So y sys here and then we have minus y there so that is that what s minus 1ys and the right hand side goes to 3.

So now we have these 2 equations these 2 transformed equations and we can eliminate for instance y first or x first whatever ways. So we eliminate 1variable either x or y and then we will get one expression for our xs and then in this different situation for instance if we eliminate sx then we will get function y s in terms of s. So for instance if we eliminate ys from both so to eliminate ys here we need to multiply for example this first equation by s minus 1. And the second equation we can multiply by 3.

So in that case these ys term will be similar here we will have 3 s minus 1here also we have 3 s minus 1and then we can subtract from each other we will get the equation. So here we have s minus 2 and then s minus 1with xs and here we have 6xs. So this s minus 1 s minus 2 and then when we subtract this 1we have minus 6 with xs. The right hand side this is going to be 8 into s minus 1and then we have 9 there so minus 9. We have this equation which is in xs only so we can bring this xs everything this function of s to the right hand side.

So we have 8s minus 17 s minus 4 and s minus 1and that will be the last step now we have to take the inverse and we will get xt. So taking the inverse from here we will get the variable xt in terms of t. So 5 e power minus t3 e power 4t. So having this xt there either we can get from 1of the equations so getting this derivative there we can also get y directly from this equation or we can proceed in a similar fashion that as we have eliminated here yt now we can eliminate xt.

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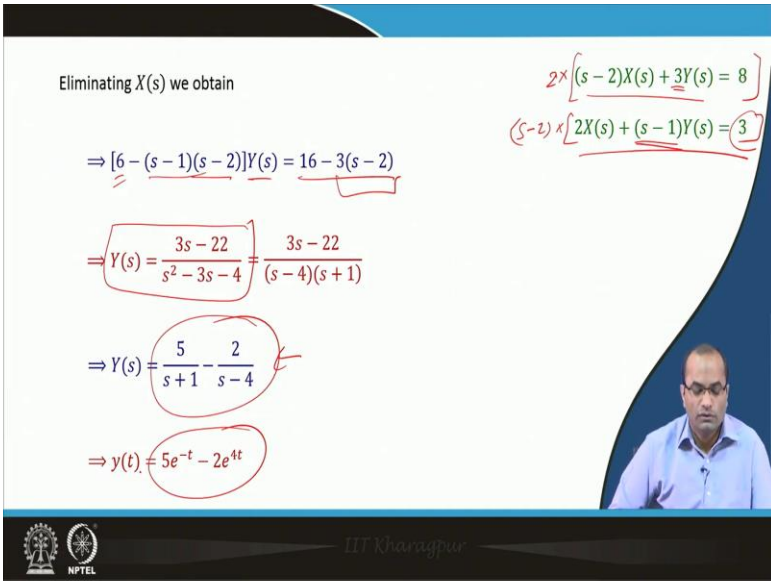

So having these equation and if you want to eliminate xs for instance so here we need to multiply the whole equation by 2 here we have to multiply the whole equation by s minus 2. And the we can subtract so this what we will get there we will have the 6 there and minus s minus 1and s minus 2ys. Right hand side will be 16 there and minus then 3 times this s minus 2. So having this again we can rewrite this as ys equal to 3s minus 12 over s square minus 3s minus 4 and again as a final step we have to go for the inversion of this.

So we will take the inverse transform after doing this partial fractions. So we will get 5 e power minus t and 2 e power 4t. So in this way we can also handle a system of linear equations we can follow the similar steps we have to take the Laplace transform and then this transformed equations will be solved again to get xs and ys and finally we have to just get this inverse transform.

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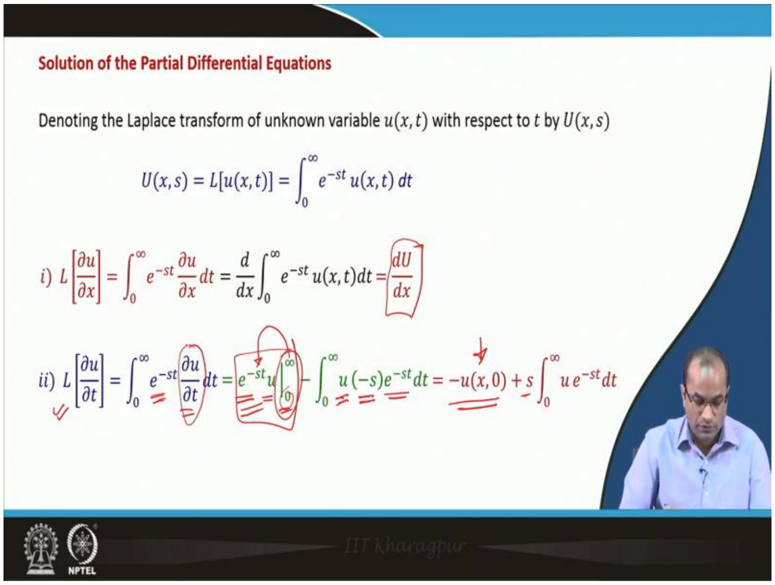

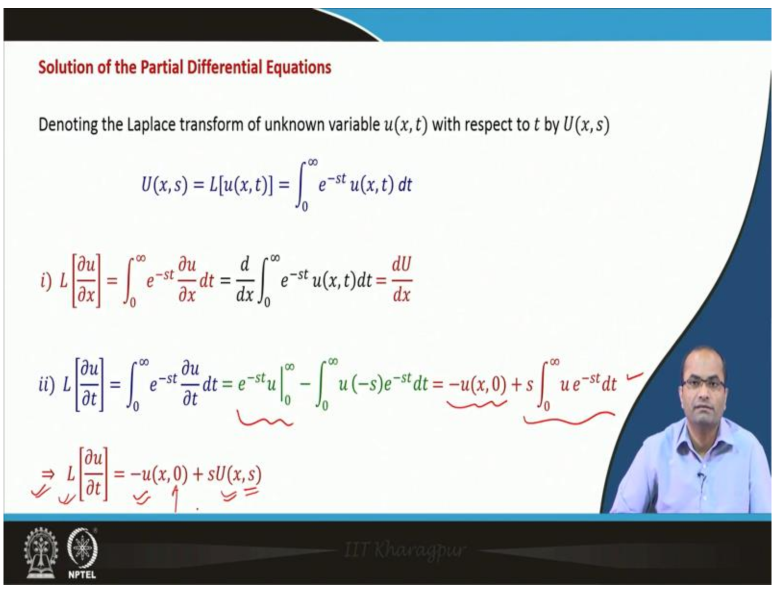

Now we will come the solution of partial differential equations the major difference here for partial differential equations would be more than one independent variable so we have to think basically with respect to which variable we need to take Laplace transform. And this case it is very obvious that we will just go with the time variable whenever there is a t because usually starts from 0. So from 0 to infinity as the range of the Laplace transform we do have.

So here we denote the Laplace transform of the unknown variable u x t let us say we will work with the u x t or y x t whatever. So there are two these independent variables x and t. So we will take this Laplace transform by with respect to t and this we will denote by this big U, x will remain as it is, and this t will be transform to this parameter s. So this U x s the Laplace transform of u x t is defined as 0 to infinity e power minus st u x t dt.

So here this x will be treated as constant and we are integrating over t and finally we are introducing this s into this transformed known unknown so U. So U x s is equal to this 0 to infinity e power minus st u x t dt. So we have then all those possibilities that if we want to apply this Laplace transform to this partial derivative with respect to x. So since we are not taking this derivative, this Laplace transform with respect to x, so this s will be treated as constant.

So for instance if we apply the definition we have e power minus st del u over del x dt. And this del u over del x we will write outside as d over dx because this will be integrated over this t. So we will have actually the only one variable. So s is a parameter for the transformed case. So we have 0 to infinity e power minus st and u x t dt and as per our notation this we are denoting U big U that is the Laplace transform of u x t with respect to t.

So here we have dU over dx, so this derivative term dU over dx will remain as it is, because we are not doing or we are not taking the Laplace transform with respect to x. But when we have this del u over del t, so now the derivative is with respect to t and having this derivative with respect to t when we take the Laplace transform that derivative will be removed so it is like a derivative theorem will be applicable now.

So if we have this e power minus st del u over del t dt, so if we take the partial derivative so here e power minus st in del u t over del u over del t will give u. And then this limit 0 to infinity and again the same thing so e power minus st with minus s and then del u over del t will be u while taking the integral of this del u over del t. So this first term when u when this t goes to infinity because of e power minus st this term will vanish and when t goes to 0 then we will have u 0 term there.

So we have minus u so we have this e power minus st u and this integral term, so when u t goes to infinity that will vanish. And we have this minus u x 0 because of this 0. For the second term we have to differentiate this e power minus st so minus s term will come. And then we have u because of the integration of this term del u over del t. So as a result what we will get? We will get because of this minus u x 0 and the second term we will have s integral u e power minus st dt.

So this will be treated as when we are taking the Laplace transform with respect to t here del u over del t, it will be minus u x 0 plus s and u x 0. So this is exactly or precisely the derivative theorem we have. Because this is the derivative with respect to t and we are taking he Laplace transform also with respect to t, so this derivative theorem will work now. So we have minus u x 0 with replacing this t by 0. And s U x s, so x will remain as it is and with respect to t we have actually the derivative theorem.

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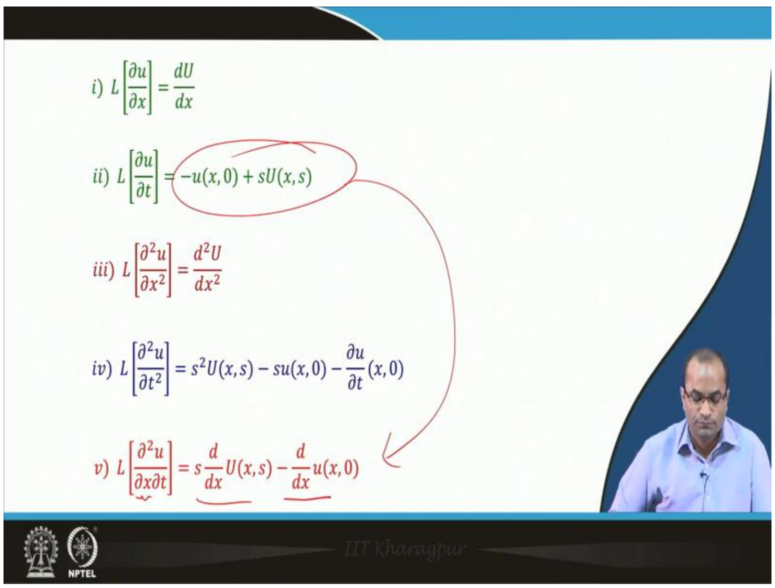

So we have these two rules whenever the Laplace transform we apply to del u over del x we will have dU over dx there. And when we apply Laplace transform to del u over del t in that case we have minus u x 0, so this is the derivative theorem. And for instance, we have the double derivative with respect to x, so that is not going to be affected we will remain with the derivative and this u will be transformed to this big U as its Laplace transform.

So next when we have the derivative with respect to t so two times derivative with respect to t. So we have the derivative theorem for the second order case, we have s square U x s minus su x 0 minus del u over del t x 0. So keeping this x as constant and we have applied this derivative theorem. So u the last result we may be using in some of the partial differential equations that is a mix derivative.

So whenever we have mix derivative the derivative with respect to x will remain as it is, and the derivative with respect to t will be treated as this derivative theorem. So we have this d over dx as it is, and this first derivative will have this minus x u x 0 term there. And this s U x here we have the derivative with respect to x. So s derivative with respect to x. So it is the derivative theorem as we have here del u over del t, so that will be used and d over dx will remain as it is.

So we can just see from there. So this only the derivative term will come extra with respect to x because of this second derivative.

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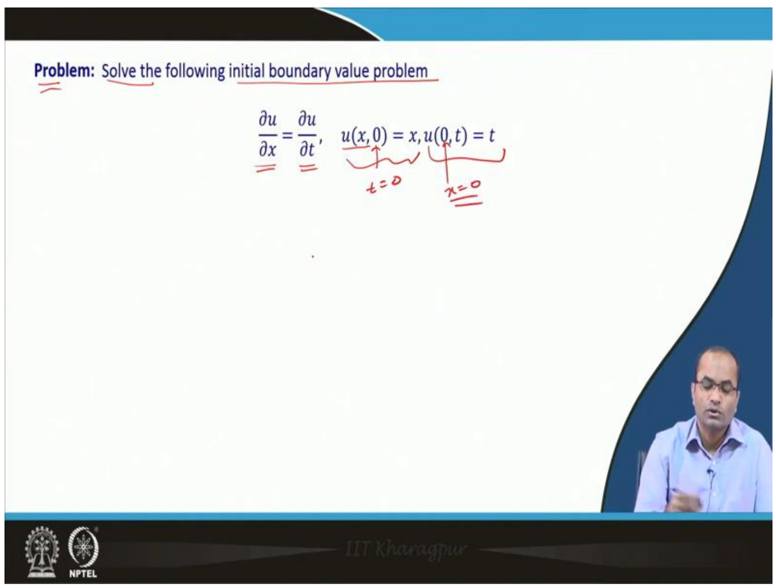

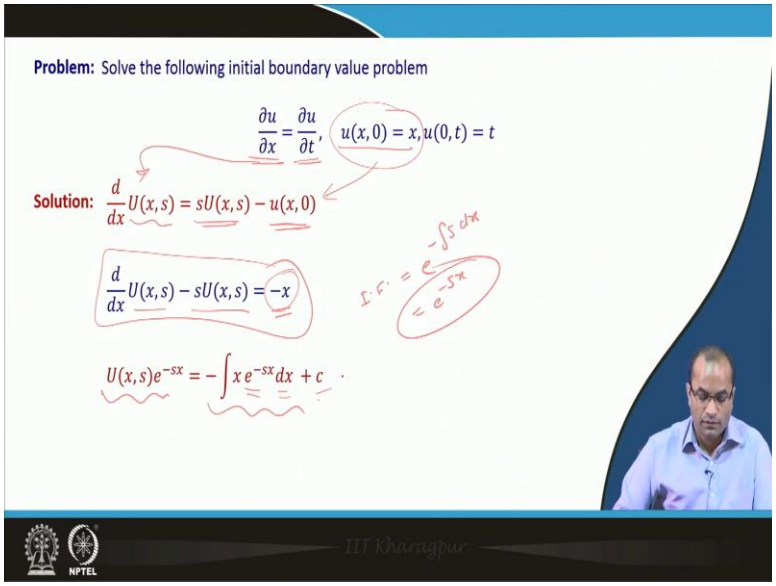

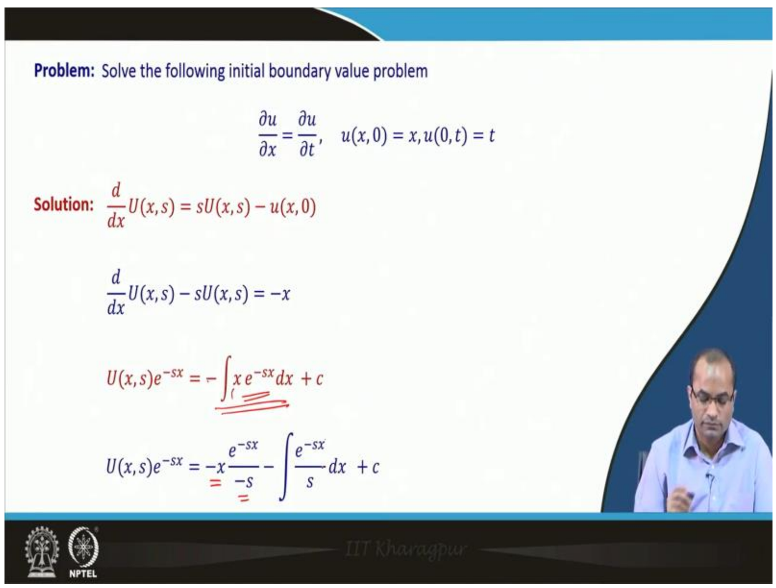

So we have now the example where we want to solve this following initial boundary value problem. So we have del u over del x is equal to del u over del t this is a first order equation. And these are the conditions here, so normally this u x 0 when t equal to 0, so this is called the initial condition.

And then we have boundary condition here when we have said this x equal to 0. So with these two conditions we can now apply the Laplace transform and these initial condition will be incorporated at the beginning itself. So while taking this Laplace transform this we will get here d over dx and U x s the right hand side because of t that will be derivative theorem. So we have sU x s minus this u x 0.

So u x 0 we can use from there, that means we have d over dx U x s minus sU x s is equal to minus x. And then we can solve this linear equation, so again we are getting a simple linear ordinary differential equation. So here e power minus this sx is going to be integrating factor. So here the integrating factor is exponential minus this s dx that is e power minus sx that is the integrating factor. And then we can write down the solution as U x s integrating factor is equal to minus this x multiplied by the integrating factor dx and plus a constant c.

That means what we have now e power minus sx after this integration, so x remain as it is and the e power minus x is integrated. So e power minus sx over minus s, then minus sign and then we have e power minus sx over minus s and this minus and that minus will make this plus. But here that sign will remain and then the derivative of x will be 1.

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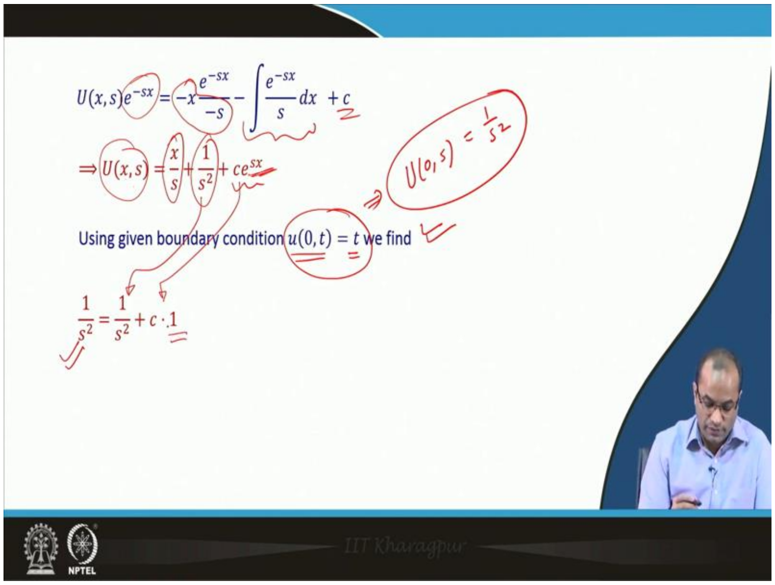

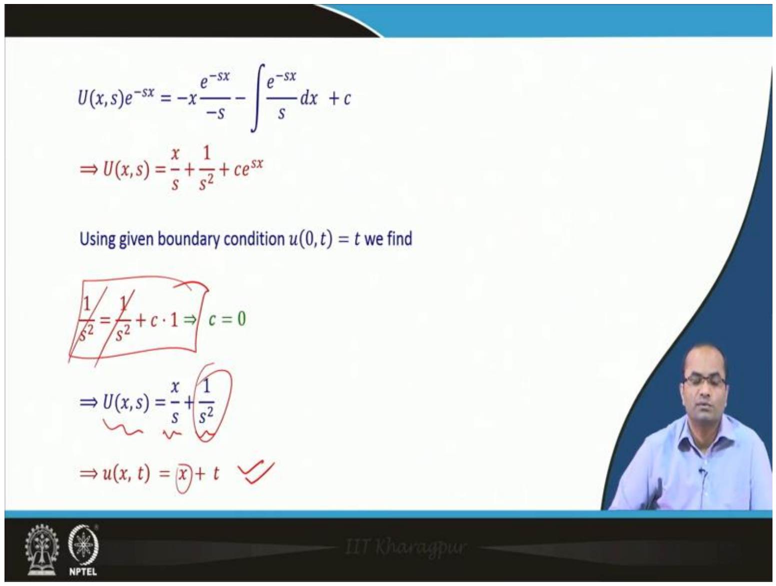

So we have this equation which again can be integrated this, so we have e power minus sx over this s square with the plus sign and when we bring this e power minus sx so that e power minus sx will cancel out. So in this first term we have x over s, in the second term we have this 1 over s square and then the third c and from the left hand side we got this e power sx. Now we can use the boundary condition which is u 0 t is equal to t.

So using this boundary condition we have to now take the Laplace transform of this boundary condition as well. So taking the Laplace transform we will get u 0 and s is equal to the Laplace transform of t that is 1 over s square. So having this information now we can use in this equation. So u x 0 u 0 s that is 1 over s square x we have taken 0.

So that will disappear, now because of this 1 over s square is there. And then c into e power sx and putting this x to 0 we have 1. So we have this equation now 1 over x s square is equal to 1 over s square plus this c, and this will get cancel. And we will get the c equal to 0. So having this c equal to 0, we got this U x s as x over s and plus this 1 over s square.

And after again taking the inverse Laplace transform, so x as it is and 1 over s so that is the Laplace inverse of 1 over s that is 1. And the Laplace inverse of this 1 over s square that is t there. So the solution of the given partial differential equation is simply x plus t.

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Problem: Solve the following partial differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x, \quad x > 0, t > 0$$

with the following initial and boundary condition $\underbrace{u(x,0) = 0}_{I.C}, x > 0,$ $\quad \overbrace{u(0,t) = 0, t > 0}^{B.C.}$

$t=0$ $\qquad x=0$

Solution: Taking Laplace transform with respect to t we have

$$sU(x,s) - u(x,0) + x\frac{d}{dx}U(x,s) = \frac{x}{s}, \quad s > 0$$

$$\frac{d}{dx}U(x,s) + \frac{s}{x}U(x,s) = \frac{1}{s}$$

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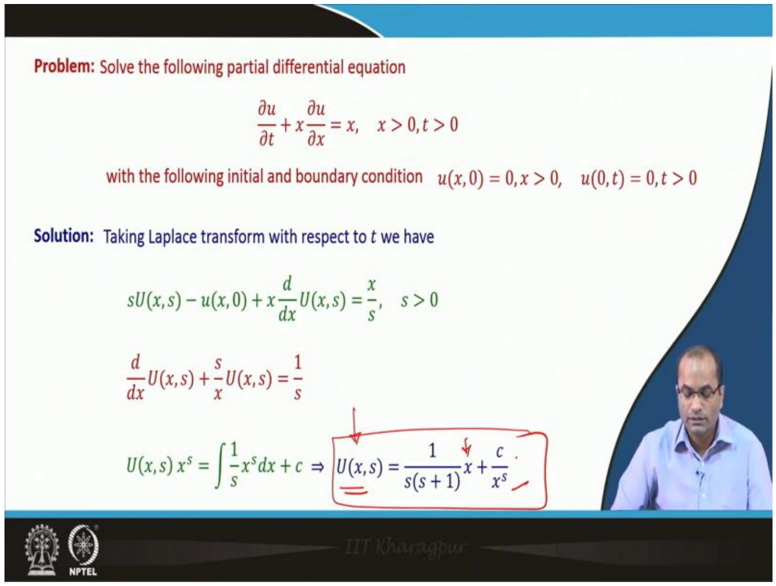
$$\frac{d}{dx}U(x,s) + \frac{s}{x}U(x,s) = \frac{1}{s}$$

$$I.F. = e^{\int \frac{s}{x}dx} = e^{s\ln x} = e^{\ln x^s} = x^s$$

$$U(x,s)x^s = \int \frac{1}{s}x^s dx + c$$



This is another differential equation we have del u over del t plus this x and del u over del x the right hand side is x for x positive t positive this equation is given. With following initial and boundary conditions so we have this u x 0 so this is our initial condition because at t equal to 0 this is prescribed and in this case this is x equal to 0 so this is another condition which we call as the boundary condition.

So with these initial and boundary conditions depending on this order we have the number of the supplementary conditions. So these are both first order terms there and therefore we are having only one initial condition and one boundary condition. Taking Laplace transform now with respect to t again what will happen? We have here the derivate theorem sU s x minus u x 0, the second term x, then del u over del x that will not be (())(26:42) because we are taking Laplace with respect to t.

And this U will be this transformed variable big U x s the right hand side we have x and then the Laplace transform of 1 that is 1 over s. So this s greater than 0 this is valid. Now we can apply the initial condition that is u x 0 is 0 so this term will vanish. And we have this here d over dx U x s plus this x over s U x s and then we have the right hand side 1 over s.

So U x s into x power s that is the again the linear differential equation and we can think about this integrating factor. So the integrating factor of this will be e power this s over x and dx, so this is e and then we have s Ln x there, so this is e and then ln x power s, so that is x power s. So this is the integrating factor x power s which is used here. So U x s into this integrating factor xs then we have right side 1 over s multiplied by this integrating factor integrated over dx and plus this c.

Now this x say power s we can also bring to the right hand side and we have U x s is equal to this x, s will be integrated so we have x power s plus 1 over s plus 1. And this x power s so will cancel the x power s plus 1 we will have only one x there, and c over x s. So we have this now the transformed variable everything is aggregated. So left hand side only U x s right hand side the function of the s and the x which we can now think for the inverse.

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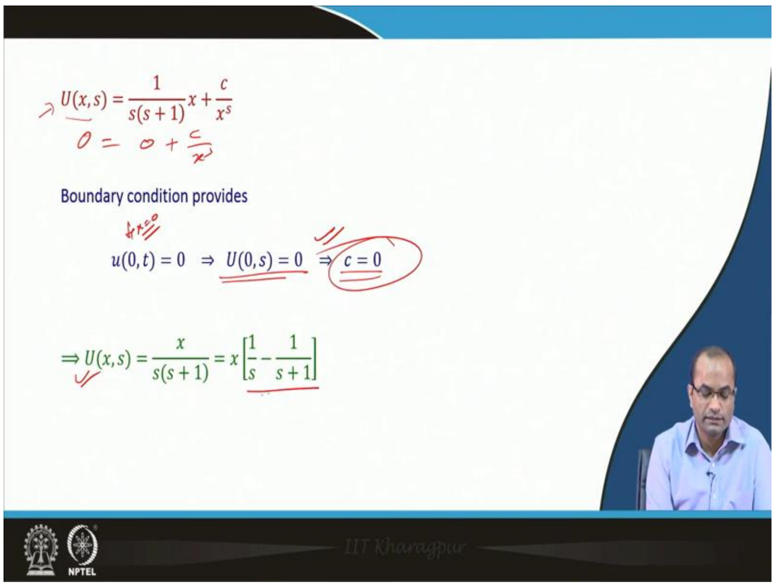

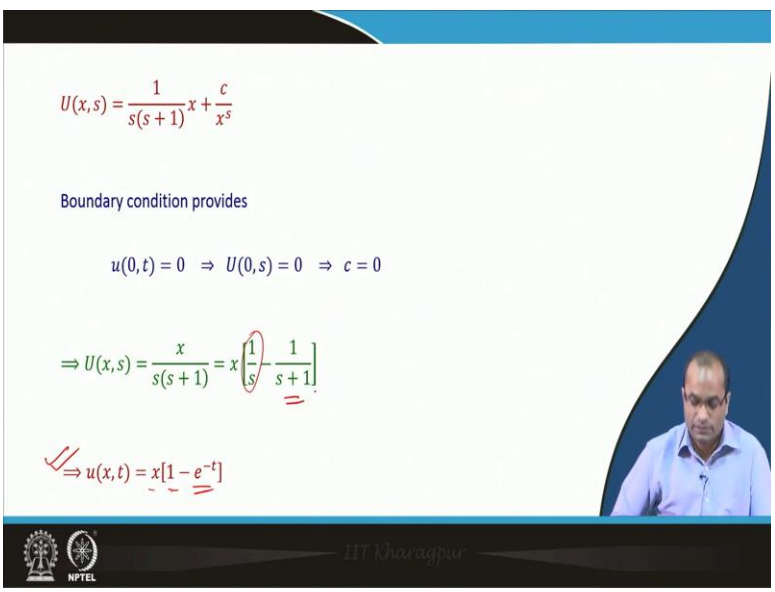

And so we have U x s is equal to 1 over s s plus 1 x plus this c over x s. And now we have the boundary condition, so boundary condition was given as u 0 t so at x equal to 0 this was prescribed. So u 0 t was 0 when we take the inverse when we take the Laplace transform we will get U 0 s equal to 0.

And if we apply this there so if we put x equal to 0 and so this will be 0 here also 0. Then we have c over x power s, so that will give actually c equal to 0. So U x s will be x over s s plus 1 because c is 0. And then taking the partial fractions we have 1 over s minus 1 over s plus 1. And then we can go for the inverse Laplace transform having this x there and 1 over s will be just 1, and then we have here e power minus t, so that is a solution of the given differential equation.

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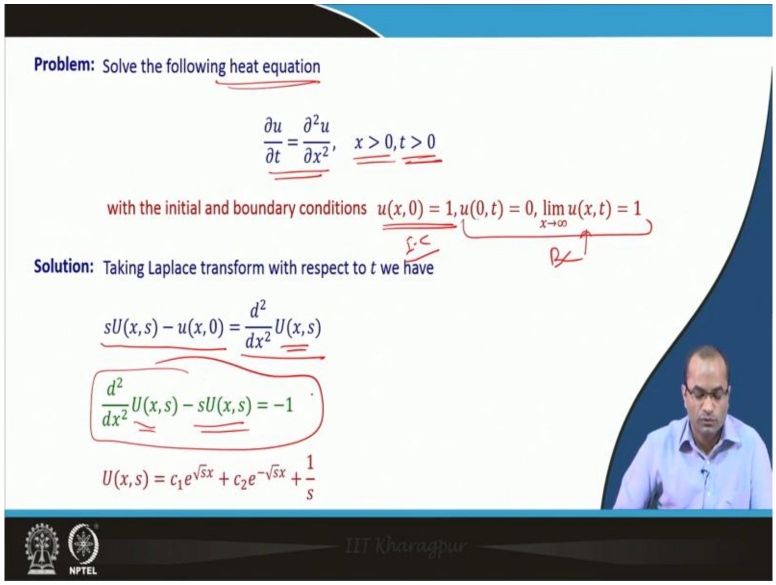

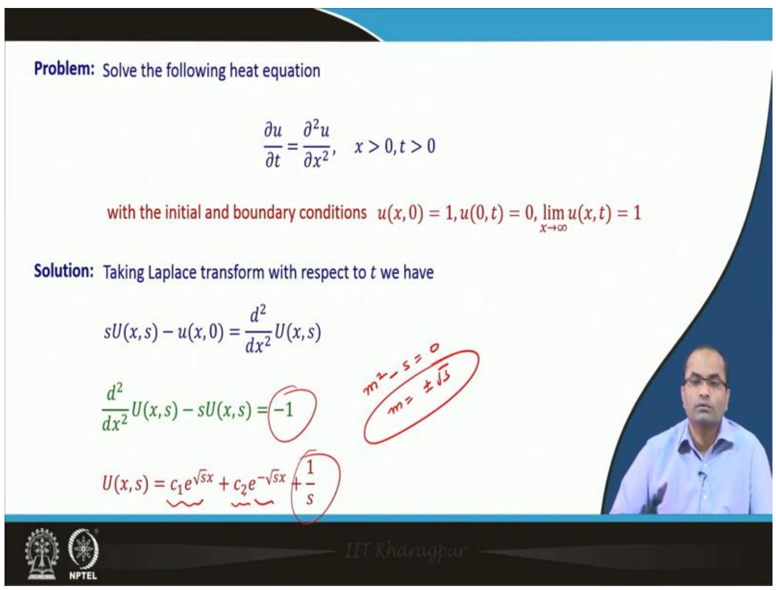

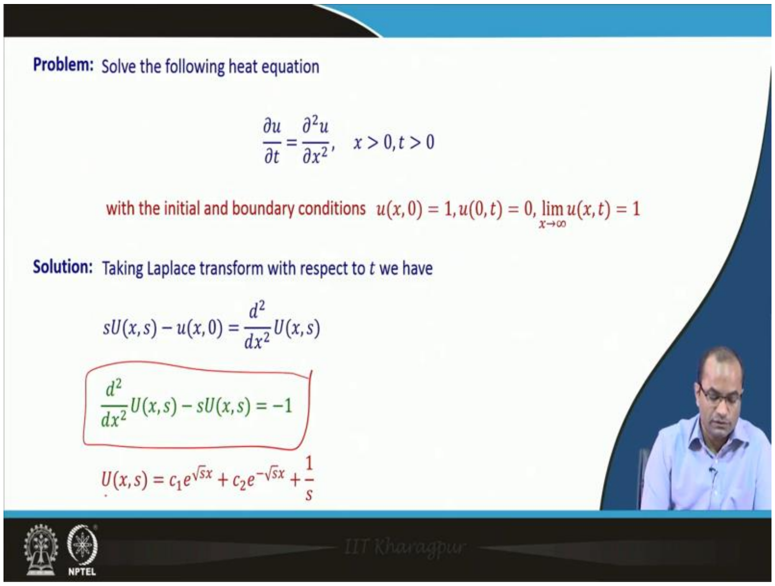

Here we have this heat equation which we can apply here which we can solve using Laplace transform we have solved this using Fourier transform earlier, now we will see the application of Laplace transform for solving such equations where x is positive t is positive. These are the initial conditions given u x 0 1 and these are the boundary conditions at x equal to 0 and here at when x goes to infinity.

So these are the boundary conditions and this is the initial condition. And now we apply the Laplace transform as usual, so we will get s this derivative theorem with respect to t the right side will be having double derivative with respect to x of this U x s. Then taking to one side we have d 2 over dx square U x s minus this sU x s and equal to minus 1 which can be solved again it is a linear differential equation and having this constant coefficients.

So we can get this auxiliary equation m square minus s equal to 0 so m will be here plus minus this square root s. So these are the two distinct roots, so we will get c1 e power the first root square root sx c2 e power minus square root sx and then so that will be the complimentary function and then because of this we will also get this particular integral as 1 over s. So having the solution of this differential equation which is written here.

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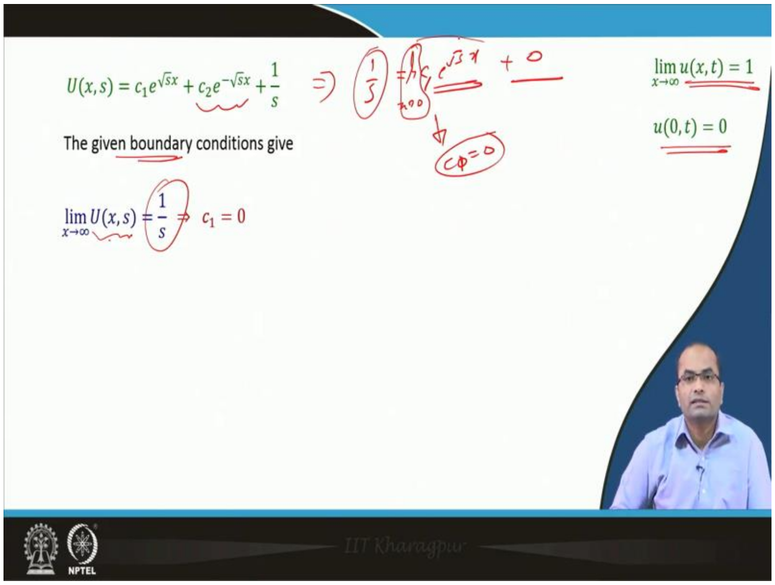

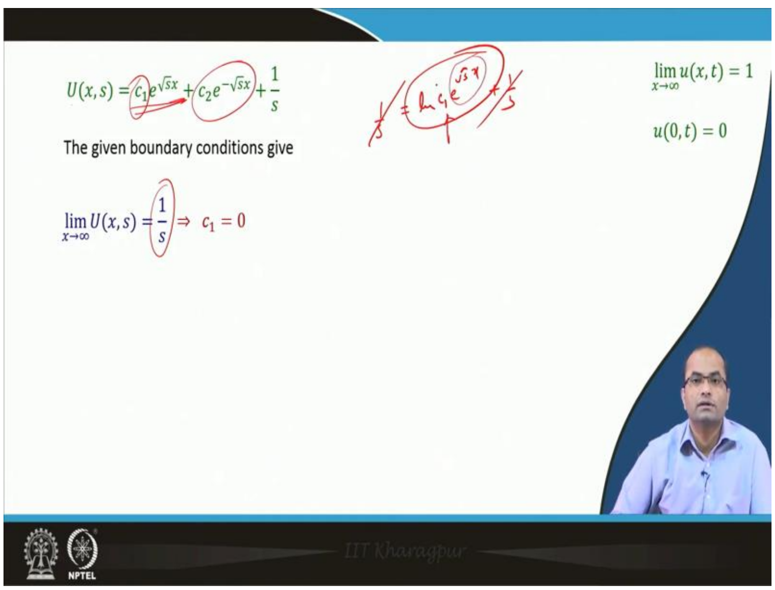

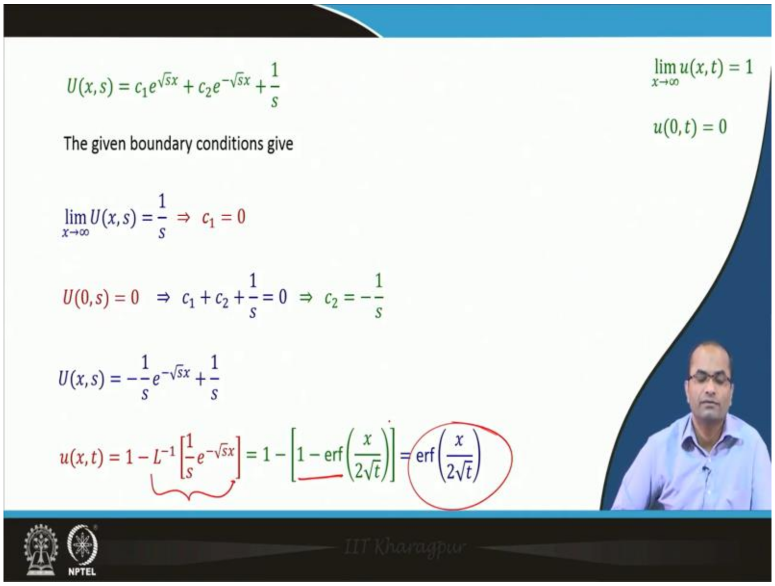

We can now consider the boundary conditions. So the boundary conditions were given one was u x t as x goes to infinity is 1, and the second one is u 0 t is equal to 0. So the first condition says that x goes to 0 when we take the Laplace transform the right hand side is 1 Laplace transform 1 is 1 over s. So that condition is given which we can apply there in our equation.

So if we take this x goes to 0 this is clear that to have this 1 over s so the when we apply this there, so this is 1 over s and then we have c1 e power square root sx and then the limit we are taking here x goes to infinity. The second term will vanish anyway, because it is a minus term. So the second term will vanish. And here to have the existence of this s equal to 1 over s, and here we have c1 e powered square root sx.

So this is possible when c1 is 0, when c1 is 0 otherwise this positive power of the exponential will take this term to infinity. But to have this equal to this minus to 1 over s then this has to be the c1 has to be 0. This is naturally 0 there and the left hand side is 1 over s. So we have the right hand side was also this 1 over s. And the only term left here with c1 e power square root sx, so this gets cancel.

So having this equal to 0 the c1 has to be 0, because this is not a 0 term. So c1 has to be 0. And we can use the second boundary condition so U 0 s equal to 0, which will give a relation between this c1 and c2 out of this equation. But c1 is 0, so c2 is minus 1 over s. So finally, what we got? We got the solution U x s equal to minus 1 over s e power minus square root sx and plus this 1 over s term which we can invert it.

So taking the inverse Laplace transform we will get 1 minus L inverse and then we have 1 over s e power minus square root sx, which if we recall from the previous lecture the Laplace inverse of this was 1 minus this error function x by 2 square root t. So then 1 and 1 get cancel and we have the error function x over 2 square root t. so this is the application of that error function for solving this heat equation.

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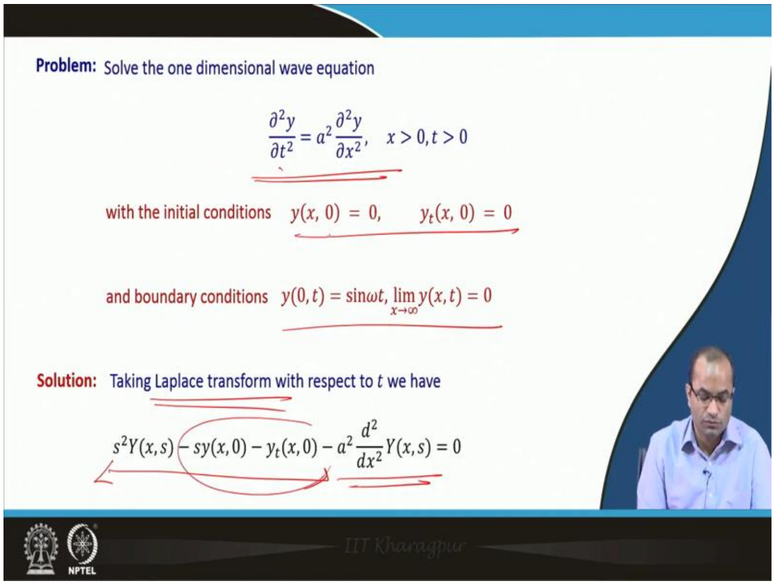

Well, so the similar steps we can follow for example this wave equation where we have the initial condition we have the boundary conditions given and absolutely we have to just take the Laplace transform with respect to t both the sides. So here this derivative theorem is applicable then we have minus a square and d 2 over dx square Y x s which after incorporating these initial conditions these two term will vanish.

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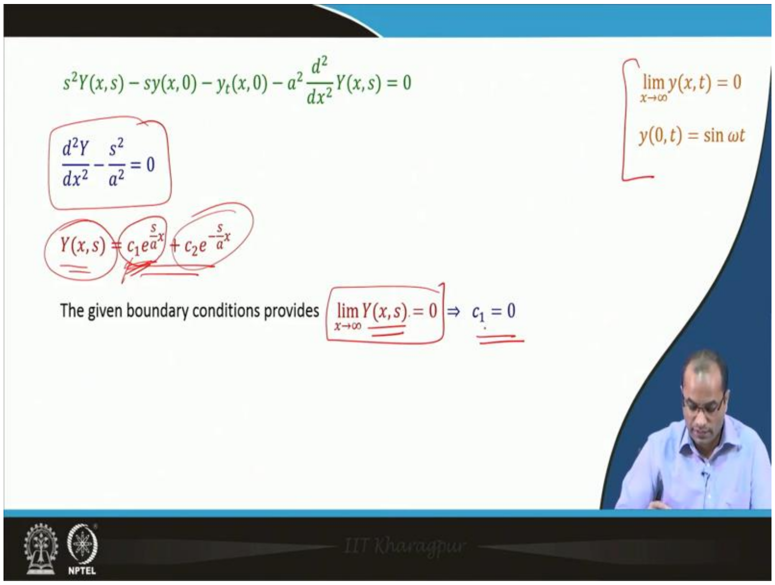

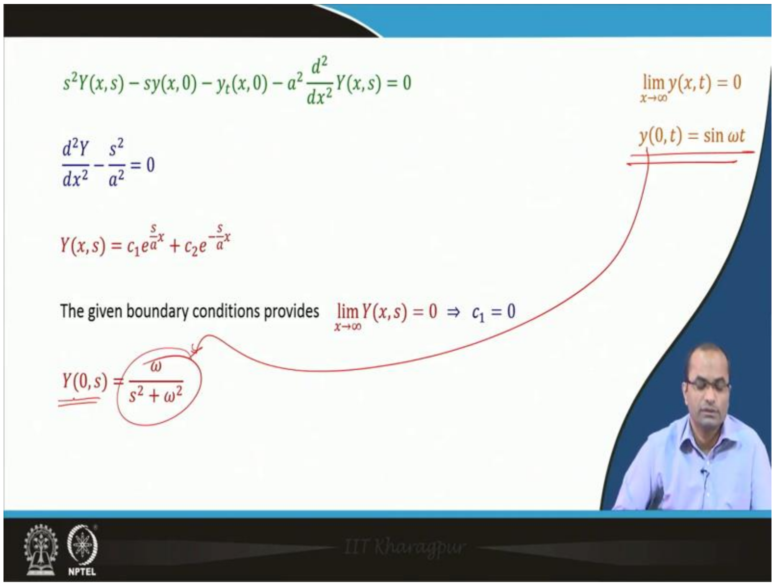

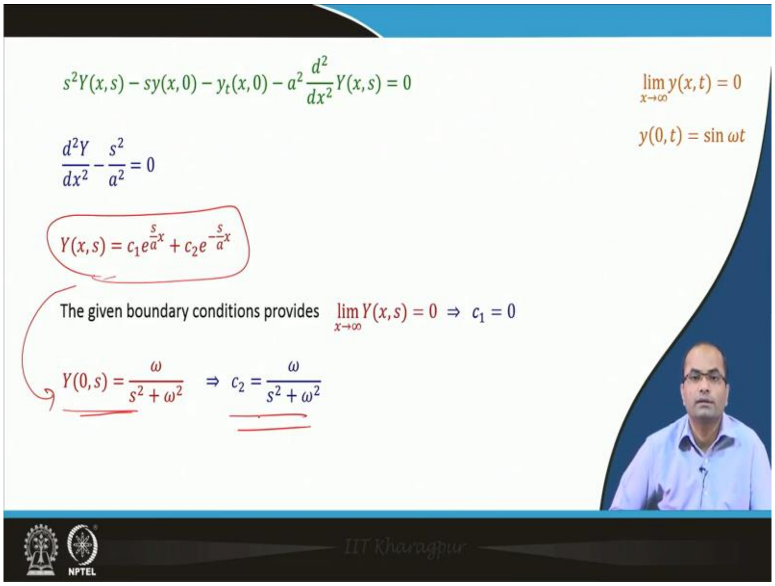

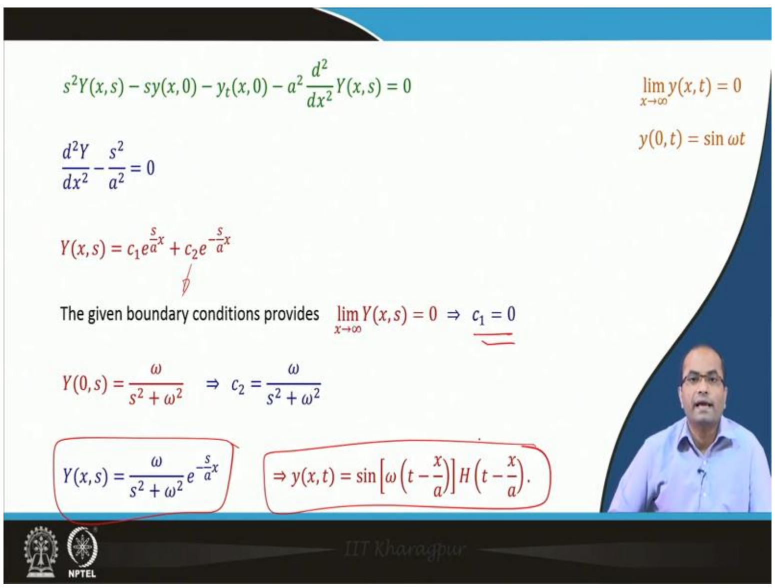

And what we have? We have a very simple differential equations, which we can easily solve. So this is Y x s is given as c1 e power s over ax and then second term there. And now we can use the boundary conditions that two boundary condition. So the first one says that when x goes to infinity Y x s is 0, applying this boundary condition there we will release that the c1 has to be 0 because if the right hand left hand side this is 0 and this is going to 0.

So this to have this 0 so c1 has to be 0. So we have the first relation, the second one is coming from the second condition when taking the Laplace transform there we have Y 0 s as omega over s square plus omega square. And from there we can get when we consider this equation here.

So we will get c2 as w or omega over this s square plus omega square. So this is c1 and c2 can be substituted there in this equation, c1 is 0 so only the c2 will survive. And then we can go for the inverse Laplace transform to get this y x t is equal to sin omega t minus x over a so that is a shifting theorem we have to apply here to get this solution.

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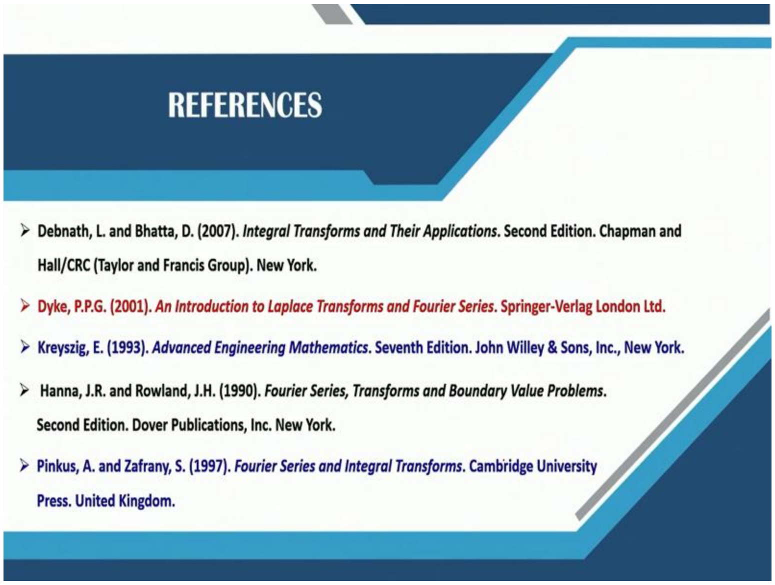

Well, so these are the references used for preparing this lecture.

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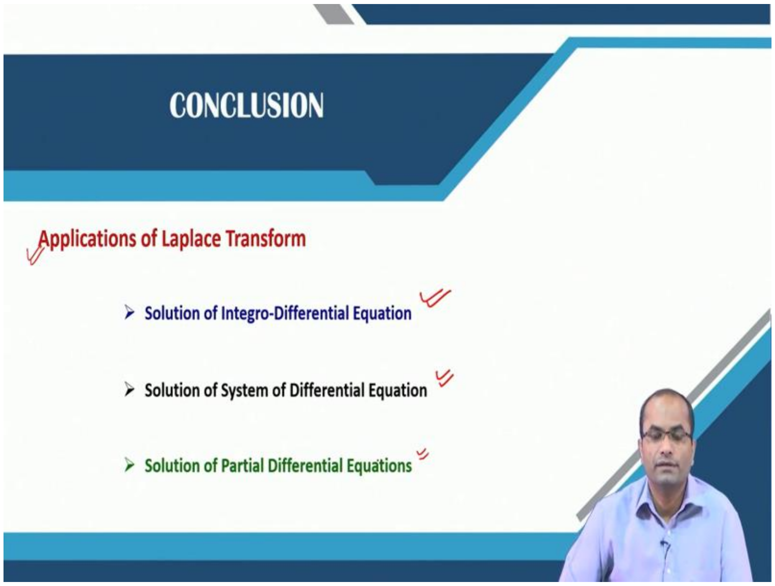

And just to conclude, so we have seen the applications of this Laplace transform for the solution of Integro-Differential equation, also the solution of system of differential equations, and the solution of partial differential equations. And we have seen the example of those standard PDs as well, so one was the heat equation other one was the wave equation which we have also solved using the Fourier transform. So that is all for this lecture, and I thank you for your attention.