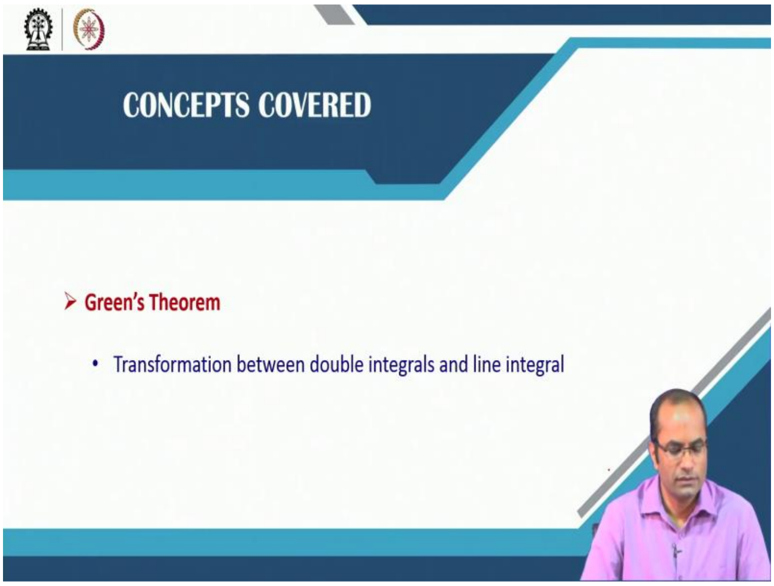
Engineering Mathematics II
Professor. Jitendra Kumar
Department of Mathematics,
Indian Institute of Technology, Kharagpur.
Lecture 06
Green's Theorem

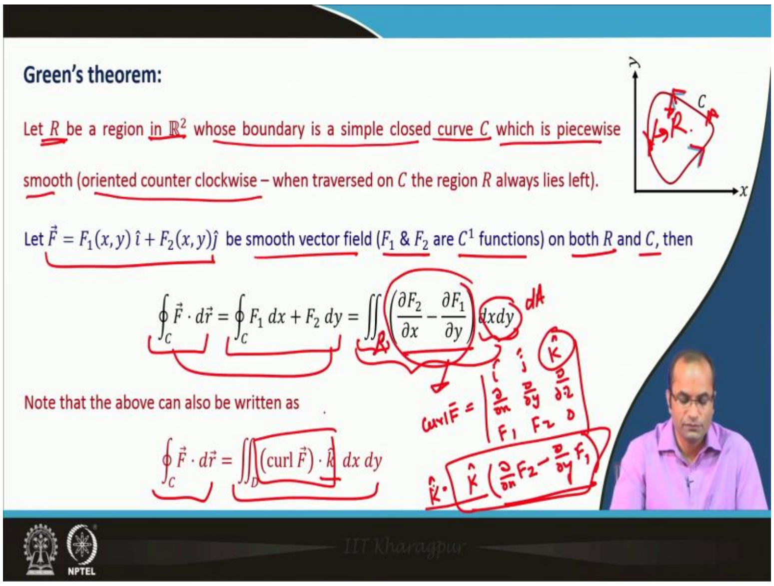
So, welcome back to lectures on Engineering mathematics II and this is module number 1 vector calculus and the lecture number 6 on Green's theorem.

(Refer Slide Time 0:24)



So, today we will discuss Green's theorem, basically this theorem tells about the transformation between double integrals and line integrals. So, in the last lecture we have already discussed what are the line integral or the curve integrals.

(Refer Slide Time 0:43)



So, coming back to the theorem let's just go through the statement, let this R be a region so, here we have R region in R 2. So, we are talking about the 2 dimensional plane now, whose boundary is a simple close curve, so we have a region R in R 2 and whose boundary is a simple closed curve that means the curve does not intersect itself, this we have already discussed in the last lecture.

So, which is piecewise is smooth that term is also discussed already. The another thing here is oriented counter clockwise that means, when we traversed on this curve. So, for instance we are traversing in this direction, then the region R always lies left to the tracing direction. So, if you are tracing in this direction, then this region is always towards the left so, this is what we call the positive orientation or the here the oriented counter clockwise.

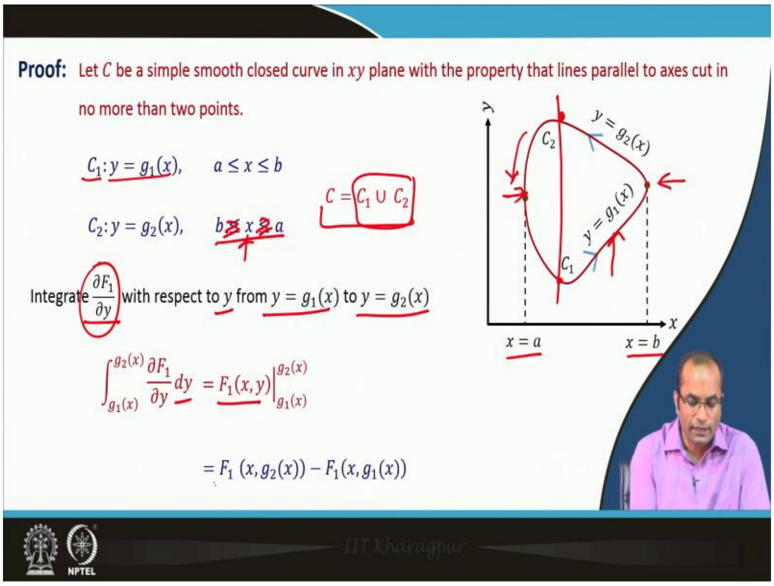
So, then if we assume that this F is a vector here F 1 x y F 2 x y on 2 components we are talking about the 2 dimensions now. And be a smooth vector field that means the F 1 and F 2 these 2 functions of 2 variables, are C 1 functions so, they are continuous and differentiable on both the region R and as well as on the boundary, then what we then we have this interesting result that this curve integral C F dot R or we can define this curve integral with this F 1 dx plus F 2 dy if we write this in component wise. So, this curve integral is equal to this double integral.

So, this double integral is taken over this domain R and we have here these 2 partial derivatives, the partial derivative of F 2 with respect to x and minus this partial derivative F 1 with respect to y and this difference and we are integrating over this region R. so this curve integral is transformed to this area integral, the integral over the domain R. And just to be noted that thus this integral can also be written as. So here we have the curve integral, the right hand side part which was written here in component wise that Del the partial derivative of F 2 with respect to x minus partial derivative F 1 with respect to y. And this integral over we can also replace by da or dx dy, it's a double integral over this R.

So, this integral we can also rewrite in curl F and the dot product with the k because if you recall that the curl F is nothing but the determinant of Del over Del x Del over Del y and Del over Del z and then we have from F let us say F 1 F 2, the third component is anyway 0 here. So, if we just get this corresponding to this Kth component, because here we are doing this dot product with the k. So, we do not need the other I j. they will be anyway 0, the component sitting with i and j will be anyway 0. So, this with the k here, what we get Del over Del x of this F 2 minus this Del over Del y of F 1. So, this is and when we have the dot product with this k, then we will get just this difference, which was mentioned here in the theorem.

So, either we write in this component form or we write in this curl F dot k form because this form will be used later on for generalization to the higher dimensions. So, at that time We will discuss this form again.

(Refer Slide Time 5:06)



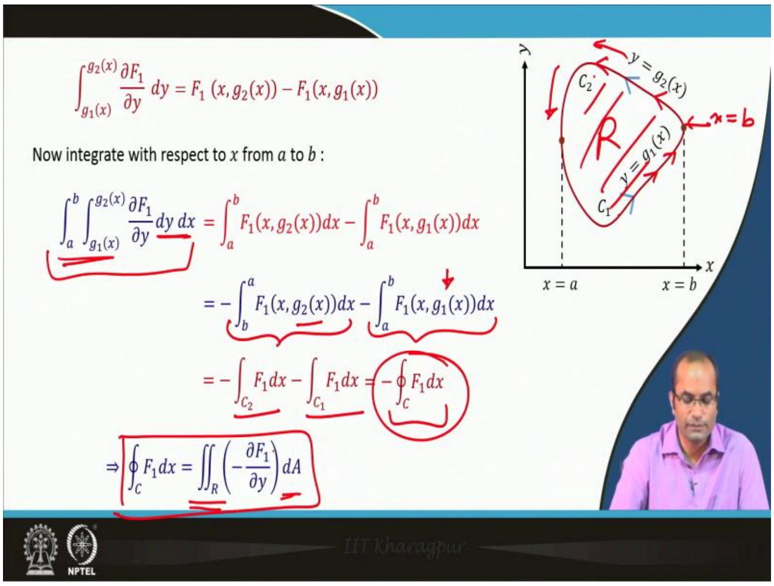
Coming back to the proof, so simple idea of the proof is. So we consider that the C is a simple, nice smooth close curve in this xy plane, and with the property that the lines parallel to the axis of parallel to the x axis and the parallel to the y axis cut no more than 2 points. So, this is what the restriction we have on the curve. So, this is the close curve, we have considered a simple smooth close curve and if we draw these lines parallel to parallel to the x axis, or we draw the line parallel to the y axis in both ways the lines will cut not more than 2 times this graph for example, if you draw this it's cutting 2 times are we drawing this direction, again it will cut not more than 2 times.

So, and then we just we break this curve into the following 2 curves. So one, so here these 2 lines, so this is the whole curve lies between these 2 lines x is equal to a and x is equal to b. And what we are doing now we assume this C 1 which is y is equal to g 1 x that is the curve here.

This curve from x is equal to a to this x is equal to b, this is denoted by y is equal to g 1 x and C 1, the another curve which starts from this endpoint here and goes along this up to this b point, that is what we are calling the C 2. And let us assume that the equation of the C 2 is given by y is equal to g 2 x. So here for g to x, we have this x which is which lies between this b and a. So, here the here we have this x which varies from B to A. So that is exactly given here. Now, this curve C the given close simple close curve C is the union of the 2 because this the entire smooth close curve was divided into 2 portions, one was C 1, the other one was C 2.

So now what we do here we integrate Del F 1 over Del y, because that was a part in this result when we have this curve integral to the double integral. So we integrate this Del F 1 over Del y with respect to y from this g 1 x to g 2 x. So we are doing this in the direction of y and the limits will be from this curve to this curve. So from this g 1 x to the g 2 x, we are integrating this Dell F 1 over Del y with respect to y. So since we have this derivative and then we are also integrating with respect to y, so this is simply F 1 and then we have to put the limits from g 1 to g 2. And then we have this F 1 the y is replaced with the g 2 x and here the y is replaced with g 1 x.

(Refer Slide Time 8:31)



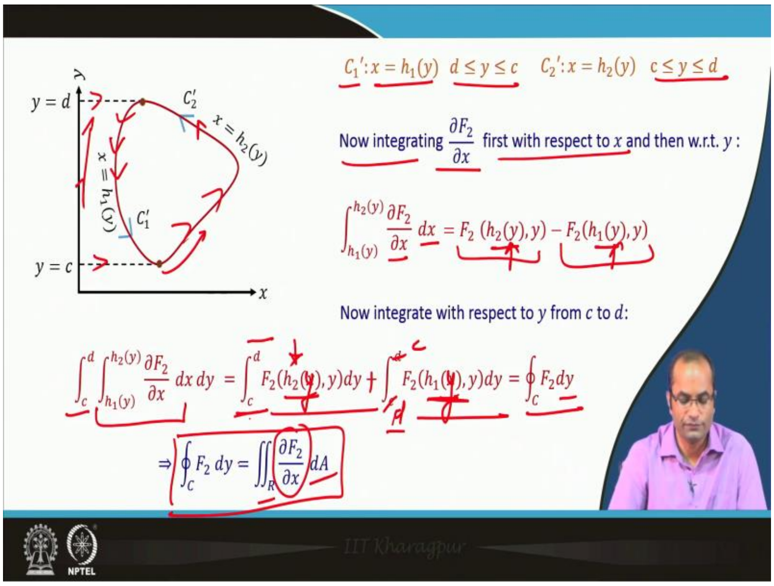
So this is the result we have just by integrating Del F 1 over Del y with respect to y. Now, we integrate with respect to x from A to B. So in the direction of y we have integrated already from this boundary to the upper boundary and now in the direction of x we have the maximum range here from x a to b. So, basically we are integrating over the whole given region R so that is a result. Now here a to b over x and then this was the already we have computed. So, this was F 1 x g 2 x and F 1 x g 1 x. So, this is integrated over a to b and here also we have the integral over a to b for x.

So, we have just changed the limit here so, with the minus sign, so then B to A and we have F 1 and just note that is y component is g to x. So, which was if we look at the figure now. So, this g 2 x, so, here we are in g 2 x which varies from b, so, this was x is equal to b and then we are going in this direction up to x is equal to a, and the y is replaced already with g 2 x that means we are integrating over this curve C 2. So, this is the integral over the curve C 2. And similarly here if you note this y here is replaced with this g 1 x. So, here we are integrating over the C 1 curve.

So, this is the integral over C 2 and this is the integral lower C 1 curve. So, then we can combine the two. So, that means this C 2 F 1 dx and this is C 1 F 1 dx. So, with minus sign if we take common that means this plus this so we are integrating over the curve this F 1 with respect to x.

So, this is the result of this area Integral, this is the integral over this whole region R and the result is this minus C F 1 dx. And there was another term Del F 2 Del x so now we will repeat this so the first we got here, this result that the curve integral F 1 dx this minus sign also we have taken to the left hand side. So, the area integral this over R of Del F 1 over Del y. Now, the similar steps we will repeat or for Del F 2 over Del x. And first we will integrate with respect to x and then with respect to y.

(Refer Slide Time 11:16)

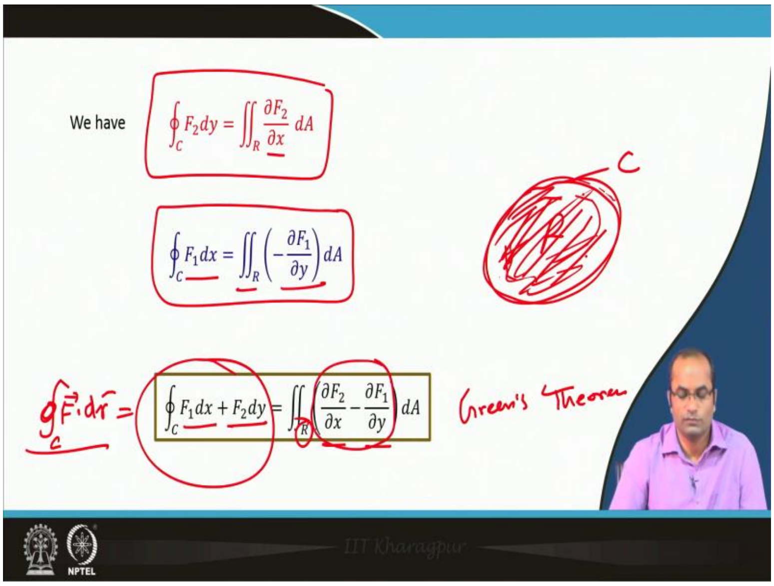


So now, this time we will consider the same figure, but now we will fix in the direction of why these limits C to D and then we have these 2 curves h 1 y and this h 2 y. So, here we have this h 1 y and then here we have this h 2 y, x is equal to h 2 y. So, the C 1 prime curve is the x is equal to h 1 and this is our C 1 curve. So, here y varies between C and D so y is bigger than C and the less than D. And similarly here y again is between C and D for the C 2 curve as well but here it goes from C to end then D, the other was D to C well. So, now we integrate this partial derivative of F 2 with respect to x with respect to x and then with respect to y.

So, here we have Del F 2 over Del x and we are integrating over with respect to x. So, here we have the F 2 now, and the upper limit is h 2 and then the lower limit for x is h 1 . So again the similar situation so, we will integrate now with respect to y from C to D. So, if we integrate from C to D, the this expression from C to D, so we will get the F 2 where we have h 2 x and here we have x one x. So if we consider this one that this F 2 and we have already this substitute this x 1 is replaced by this h 2 x. So just consider this one the x component is replaced by h 2 y. So, this is this is y and this is also y because this was y here, this was also y.

So, this is replaced by h 2 y that means we are on this curve moving from C to D and then along this curve, C 2 prime and here we have the other situation that this first component is x is replaced with this h 1 y, and then we are integrating over this y from C to D. So, we can also now change here plus and then D to C. So, we have C to D and then for the C 2 curve for the C 1 curve we have D to C that means we have a closed curve now from C to D and then D to C on this Curve, Curve C 1 and C 2. So, these 2 integrals will again make the whole closed curve where the F 2 is integrated with respect to y. So here we have this conclusion that the curve integral of this F 2 dy is equal to this area integral of this Del F 2 over Del x. And now we combine this result and what we obtained in the previous slide.

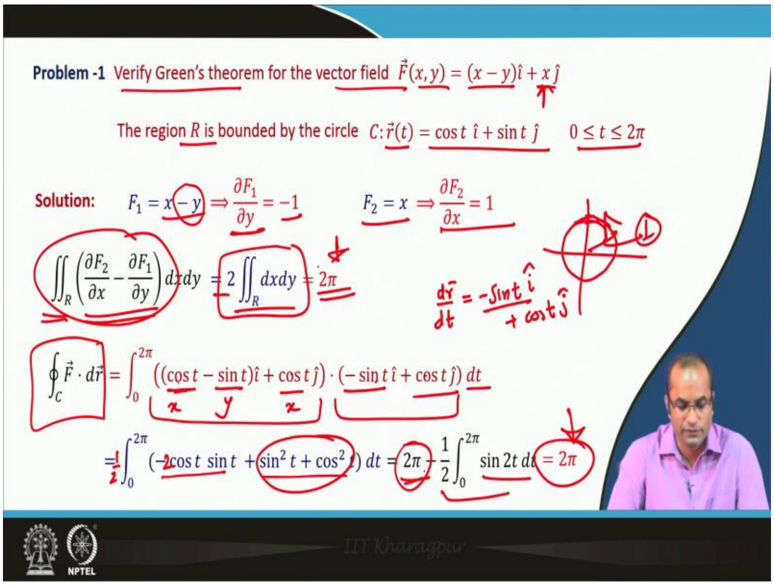
(Refer Slide Time 14:27)



So, we have these two results, the F 2 with respect to y, then we have this area integral the partial derivative of F 2 with respect to x, and here it was the other way around. So, the F 1 dx this curve integral is equal to this area integral of this F 1 with respect to y dy. And now we can add the 2 so we have this F 1 dx F 2 dx that is the curve integral this F dot dr. And here the right hand side is there Del F 2 Del x minus this Del F 1 over Del y integrated over the given region R. So this is the Green's theorem, the statement of this Green's theorem the conclusion that this curve integral which was also can be written as F dot dr. So this close integral, the curve integral F dot dr is equal to the area integral and this area is nothing but the enclosed by this curve C, so this is the area R.

So, we can instead of integrating over the curve this theorem says that you can also it is equivalent to integrate over this region R, but then the integrant will change which is Dell F to Dell x minus Del F 1 over Del y.

(Refer Slide Time 15:53)



So, we will go through some problems here; one, verify the Green's theorem for this vector field where the vector field is given by x minus y and the second component is x. So the region R here is given which is bounded by this cos t sin t so which is a circle with t varies from 0 to 2 pi, it is a whole closed circle, which is given by this parametric equation.

So we want to verify the Green's theorem that means, first we will compute the area integral or the line integral and then the other one and see the both the values are same. So here the F 1, the first component of this F is x minus y, that means the Del F 1 over Del y because in the Green's theorem we need to compute Del F 1 over Del y and Del F 2 over Del x.

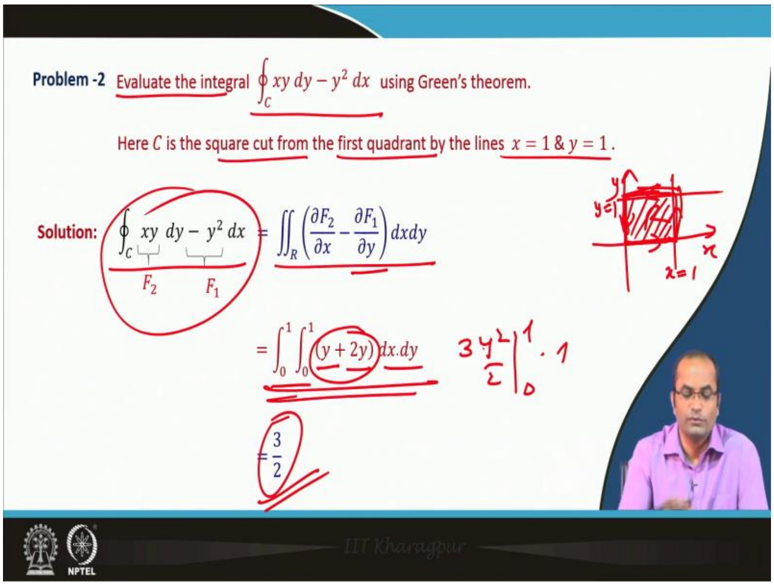
So Del F 1 over Del y since minus y is there we have minus 1 and F 2 here is x, this is F 2. So Del F 2 over Del x is 1. So if we go first with the area integral, where the integrant was this partial derivative with respect to x minus this partial derivative of F 1 with respect to y. So, what we have here 1 and then minus minus 1. So we have 2 times and this dx dy over this region R. So we have the circle the unit radius centred at 0. So, this was the circle and the area of the circle we know because the radius is just the one unit here. So Pi R square that means the Pi, so here this is 2 Pi, the value of this area integral the double integral is 2 pi.

And now we can evaluate this F dot dr which we have already learned in the previous lecture. So here the F the this was x minus y, so x is cos Theta, this is y sine Theta and then here again we have this x which is not Theta sorry it is a t, so the cos t and then dr over dt. So we have this R there and we can get this dr or dt s. So cos will be minus sin t and then plus this cos t and j. So, this is minus sin t and this cos t j. So this is Dr over dt so we have F when we substituted already these parametric form dot dr dt and then integrating over dt. So this dot product will be minus sine t cos t and then here the plus sign is square t and this will be cos squared.

So, minus cos t sine t, sine square t cos square t and this is 1. So, the one when we integrate this is 2 Pi and then minus half so two sign t cos t, here we can divide by half and multiply by two. So, this 2 sine t cos t will be sine 2 t and this 1 is integrated and we got this 2 Pi and then we have minus sign there. So, sine 2 t which again will give cos 2 t and then upper limit lower limit that will become 0, so we have just the answer here 2 Pi. So this was the curve integral, which gives us this value 2 Pi here it was the area integral which gives us the value 2 Pi.

Sometimes this theorem is very useful, because very often we will observe now that this curve integral is easier for some problems whereas, the area integral is easier for the other problems. So accordingly, we can either perform this curve integral or the double integral to get the area of the, to get this such integral.

(Refer Slide Time 19:45)



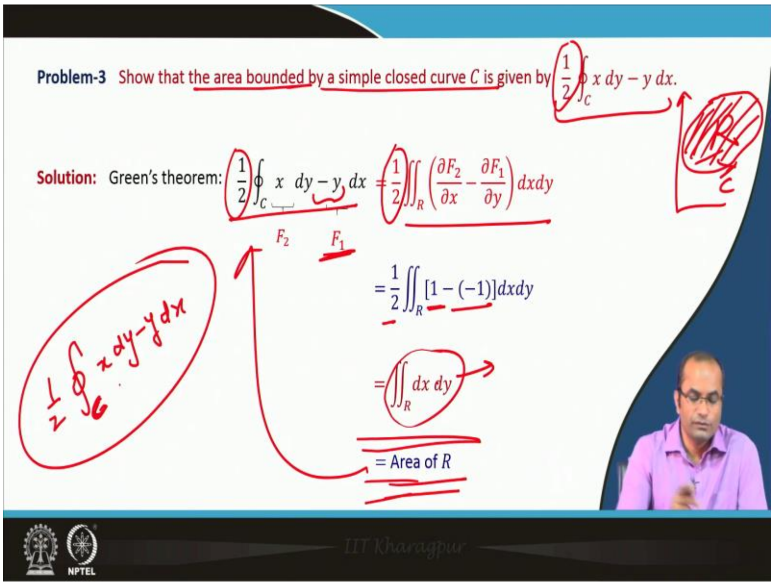
Now another problem so we will evaluate this integral. This is the line integral using Green's theorem. And the C is a square cut from the first quadrant by the line x is equal to 1 and y is equal to 1, so we have x axis we have y axis, this is line x is equal to 1 and y is equal to 1, let us say.

So this over this we are this region and then we have this curve, the boundary of this square so, we want to get this curve integral, if you perform over the over these curves, so we have the four lines there and we have to perform this integral. So naturally that will have these integrals four times which may not be very convenient. So, what we will do will use this Green's theorem and apply on this region R.

So, the Green's theorem says that this will be equal. So here we have F 2 because F 1 dx plus F 2 d y, so this is F 1 with the minus sign. And this Green's theorem says that this will be equal to this area integral which we can easily evaluate these partial derivatives. So F 2 with respect to x, that means y we will get and F 1 with respect to y we will get 2 y and then with the minus sign there also we have minus and that will become plus.

So, we have y plus 2 y integrated over this area, this region R where the limits are 0 to 1 and 0 to 1 square, this is very easy now to compute. We have here 3 y that means 3 y square by 2 and then upper limit is 1 the lower is 0, so we will get just 1 from there. So, we have 3 y 2, the 3 this is 3 y square by 2 and then we have limit 0 to 1, the other one will be also just 1 so, here we have 3 by 2 the answer. So, if you do this curve integral for instance, over these 4 curves 4 lines, then naturally it will not be as simple as we have computed this double integral.

(Refer Slide Time 22:03)

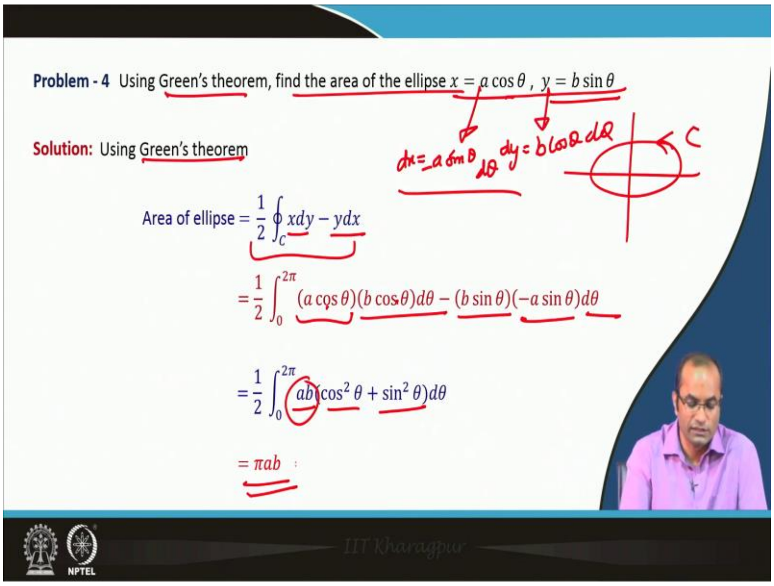


Well, so, another problem where we will evaluate or will show that the area of this bounded by a simple close curve. So, if we have a simple close curve and we will show that this area of any simple close curve C can be computed by this integral which is integral x dy minus y dx and this factor half.

So, the Green's theorem says that this line integral where we will take this F 2 and this will be F 1 with the minus sign that is half is already there. So, the half we have taken and then this is the Green's theorem that F to the respect to x F 1 with respect to y dx dy so half and then F 2 with respect to x that is 1 and then F 1 with respect to y that will be minus 1 but minus was there so minus minus 1 will have plus there.

So, this is just the dx dy over this region R and this is nothing but the area, area of this region covered or enclosed by the simple close curve C, so, this is the area of R. So, this is what we want to prove that this area can be computed with this integral, which is half the close integral over C x dy minus y dx. So, the area can be computed by this line integral which we will see in the next example just to demonstrate this.

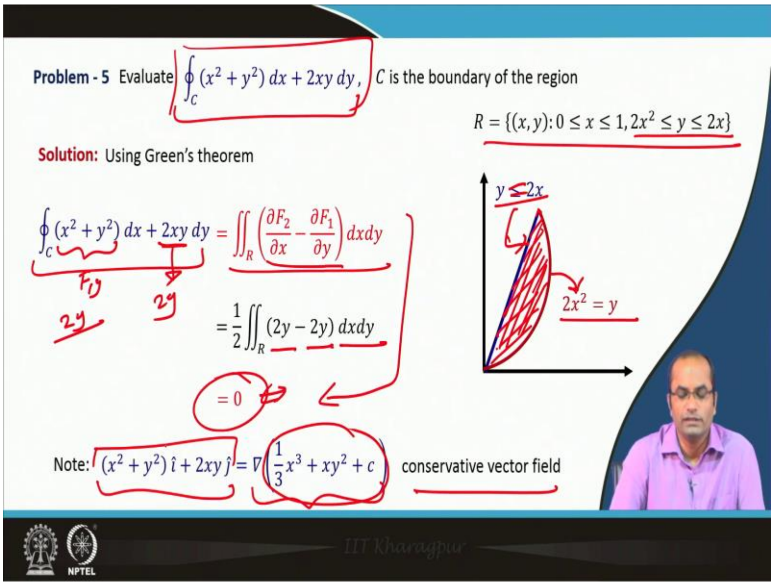
(Refer Slide Time 23:44)



That using Green's theorem find the area of the ellipse, x is equal to a cos Theta y is equal to b sine Theta. So here the solution using this Green's theorem what we have the area of the ellipse that the previous example we have seen that the area can we computed with this integral half the close integral over this C over in this case we have this ellipse. So, that is our C here, so do this integral x dy and then we have minus y dx. So let us just do this line integral directly, we have already this x is equal to a cos Theta y is equal to b sine Theta so x is a cos Theta. And then dy we can have from here that dy is B cos Theta and d Theta, b cos Theta d Theta then we have minus y, y is b sine Theta and dx so dx from this, we will get a sine Theta with the minus sign and then d Theta so minus a sine Theta d Theta.

So this integral we can perform, here the ab we can take common, we have cos square Theta and then we have plus sine square Theta. So ab and value here then ab and the 2 Pi, but half will cancel out that two. So we will get Pi ab is which is a standard result that the area of this ellipse, x is equal to a cos Theta and y is equal to b sine Theta is nothing but the Pi ab.

(Refer Slide Time 25:33)



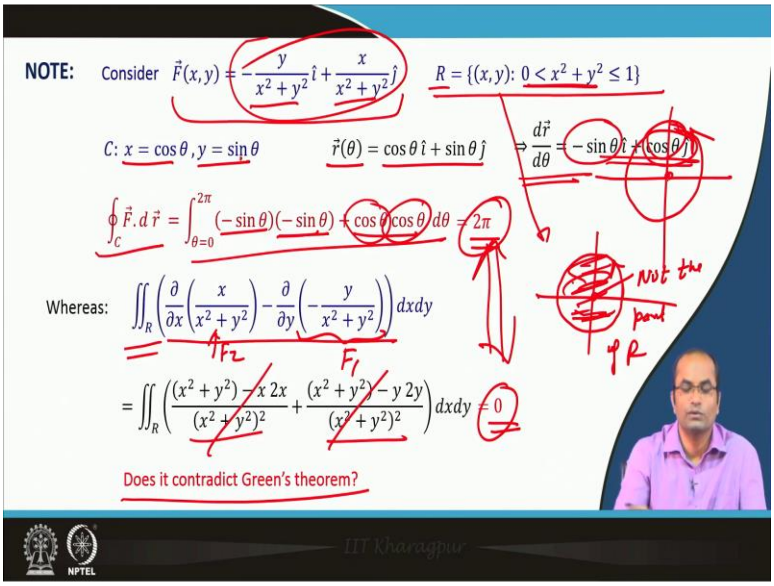
So now, we will evaluate this curve integral where C is the boundary given by this x 0 to 1. And then this is y between this 2 x the square minus and 2 x. So, we have the situation here, we are talking about this area enclosed by this 2 x square is equal to y and then y is equal to x this is the line y is equal to x and the area between these 2 what we want to covered by this boundary, where this curve integral is taking place.

So, using the Green's theorem this curve integral we can convert into the area and integral with these partial the difference of this derivative and then we can easily see the Del F 2 so, this is your F 2 is 2 x y with respect to x so we will get 2 y here and from here we will get this F 1 y, which is coming as 2 y.

So, we have 2 y and minus 2 y dx dy and this cancel out so we have the 0. So this curve integral this closed line integral is 0. But one thing just to recall that this vector field here x square plus y square i into 2 x z can be written as the gradient of this scalar function and that means, which we have already studied this is conservative vector field.

So one should also note that if the vector field is conservative then along any closed path, this was one of the close path we have considered here, but we can take any other closed path also and that value will be 0 because that standard result which we have studied before. So, here this is what exactly happening, because this is the conservative vector field and we are integrating over this close curve so, naturally this will be 0.

(Refer Slide Time 27:30)



Just a short note before we end this lecture, so let us consider this vector field here which is given by this expression and the area is given by this inner part of this disk here x square plus y square bounded by this one.

So we can parameterize this circle there by x is equal to Cos Theta y is equal to sine Theta and we can write down this curve in terms of the Theta cos Theta I plus sine Theta j, vector function of single variable, and then we can compute dr over d Theta because in the curve integral, we need to compute this so we will have minus sine Theta and there we will have cos Theta.

So F dot dr we can compute, so here we will substitute the x cos Theta y sine Theta so y is sine Theta with the minus sign and x square plus y square will become 1 and then we have the dot product so this first component minus sine Theta will be there then here we have second component when we substitute this sine Theta cos Theta, we will get x is equal to cos Theta and then this cos Theta will be from the dr over d Theta.

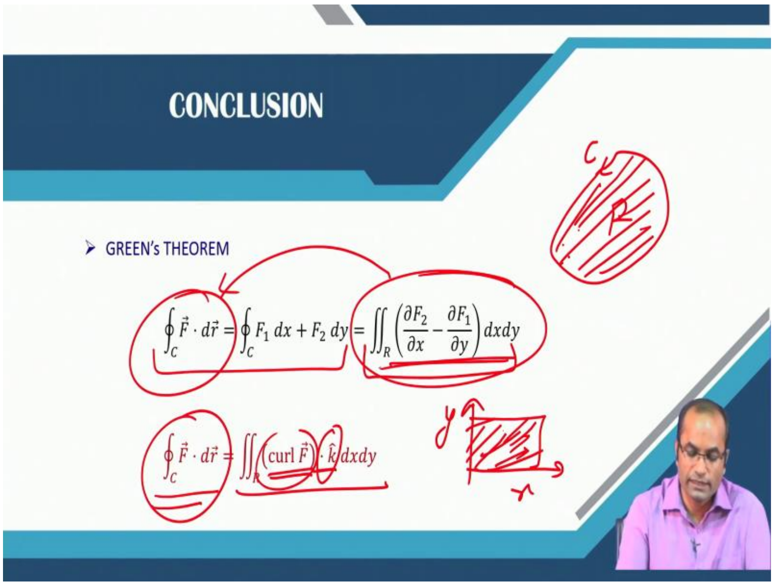
So this is the integral which we can perform the value will be 2 Pi. On the other hand, where we if we compute this area integral using this Green's theorem, then we have the partial derivative of this F 2 and then partial derivative of this F 1 which are given in the problem.

And if you do so, so, we can do this partial differentiation and then we will observe that this is coming to be 0, because here we will get minus x square plus y square whereas, here you will get x square and then minus y square. So, this will cancel out and we will get this value 0. So, here we got this value 2 Pi and for this area integral we got the value 0.

So, what is the problem why this Green's theorem is not applicable? Why these two value are not same in this case. The reason is obvious because if you consider again this R which is bounded by this circle, and if we take a close look here it is written that x square plus y square is greater than 0.

So, 0 here is not the part of not the part of the part of the region R, this is the region, but this except 0. So, we are not covering the whole region covered by this simple close curve, which is defined by this cos Theta Sine Theta the circle. So, the Green's theorem says that the we have to integrate over the region R which is enclosed by the close curve so here this one point is not in the domain because this was indeed not defined at the 00. So, therefore, this was not a part of the domain and therefore, this is not applicable. So, it does not contradict the Green's theorem, but simply that Green's theorem is not applicable because the 0 is not a part of the domain.

(Refer Slide Time 31:02)



So, these are the references we have used for preparing this lecture and just to conclude, we have learned this Green's theorem, which says that the curve integral can be written as the area integral. And the most important here is that you have the curve here, let us say C, then this is the whole region R without leaving any point there, then this is applicable, this curve integral can be computed with the help of this double integral or the other way around. If curve integral is easier, we can find such area integral with the help of the curve integral.

Another important point which will be used in next lectures that this curve integral can be also written as in form of the curl F and the dot product with the k and the scale was the perpendicular to our xy plane. So, we had the xy plane where we are working now here, 2 dimensional plane, and the the direction of this k was perpendicular to this domain given here in the xy plane. So if we do this dot product with the perpendicular to this plane, then we get exactly the integrant of this area integral. So that is all for this lecture and thank you very much for your attention.