Engineering Mathematics-II
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Lecture 59
Applications of Laplace Transform

So welcome back to lectures on Engineering Mathematics 2, this is lecture number 59 on Applications of Laplace transforms for solving differential equations.

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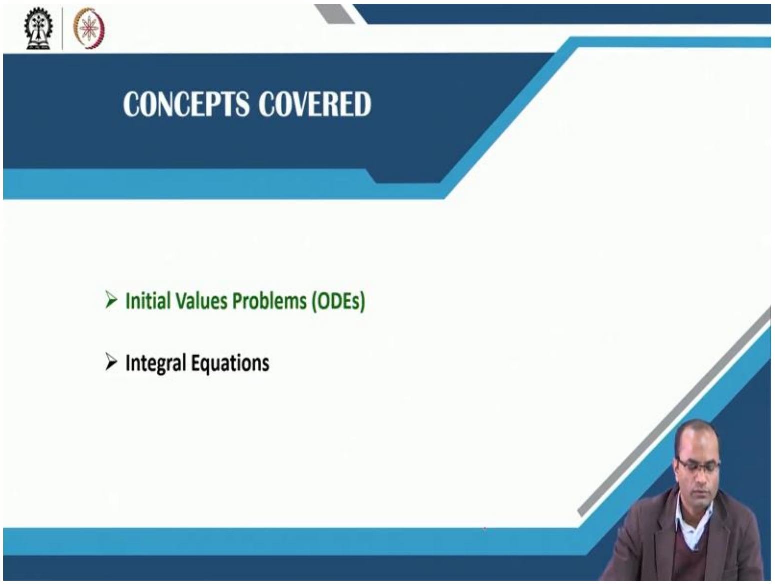

So in this lecture we will be discussing the initial value problems and the Laplace transform technique for solving such initial value problems and also, we will go through some integral equations.

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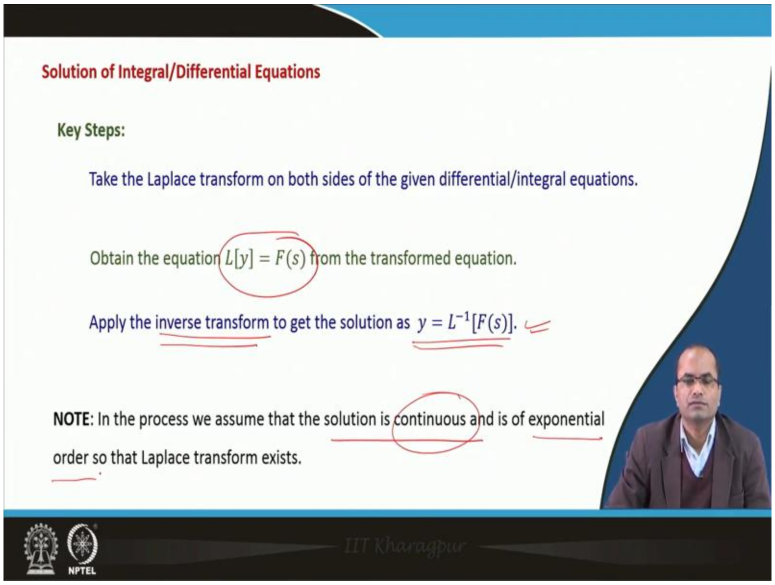

So in general when we are talking about the solution of the integral or differential equations these are the key steps we need to follow the first is we need to take the Laplace transform on both the sides of the given differential or integral equation. The second step so we will simplify this to get something like Laplace of y is equal to some function of s from the transformed equation.

So we have the given equation after taking the Laplace transform we have the transformed equation and that we will simply to have this form ly is equal to fs. And then finally we will apply the inverse transform to get the solution y is equal to l inverse fs. So these 3 are the key steps for solving any differential equation using this Laplace transform. Just a note in this process we will assume that the solution of the given differential equation is continuous and is of exponential order so that the Laplace transform exist.

Indeed we need piecewise continuity but in differential equations we most of the time we have a solution which is even continuous and of course of piecewise o of exponential order so that all the Laplace transform when we are talking about taking the Laplace transform of both the sides of a given differential equation. So naturally, we have to assume that the Laplace transform exist.

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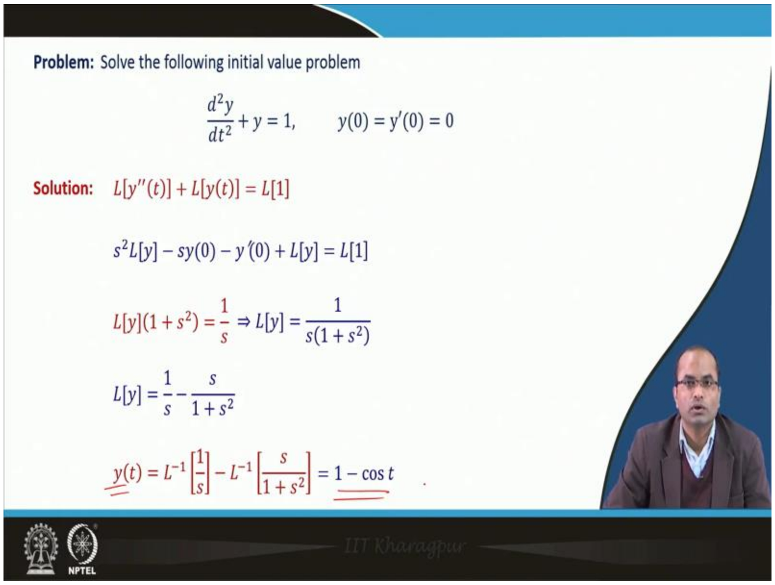

So the process will be clear with the following example and then we will be doing many other examples. So here we need to solve the differential equation or the initial value problem this d2 y over dt square plus y is equal to 1. Initial values means we have these conditions are given at a single point t equal to 0. So this t equal to 0 two conditions are given it is the second order differential equation.

So we need two conditions to solve to get unique solution of this given differential equation. So we will apply the Laplace transform so Laplace transform on this double derivative plus Laplace transform on this y and due to linearity, we can apply to each. And the right hand side it is the Laplace transform of 1. So this is the first step in each problem we will sue it. The second one now we have to use this y double prime that means the derivative theorem to remove these derivative.

So we have this derivative theorem s square ly minus sy 0 minus y0 that is exactly the Laplace transform of this ly double prime. Then we have ly and the right hand side we have l1. So if this step itself we can use these initial conditions y0 is given as 0 and also y prime is also given as 0. So we can use these given initial conditions so these conditions are incorporated just after apply the Laplace transform taking the Laplace transform on both the side of equation.

So here this is 0 and this is 0 so we have s square and then plus 1 when we take this ly common so this is ly s 1 plus s square and the right hand side this Laplace of 1 is 1 over s. So now we can divide this by 1 plus s square we get ly is equal to 1 over s 1 plus s square and then the final step is to take the inverse here. The third step is to take the inverse so that we get y from here.

So taking the inverse of this we can either apply the convolution theorem on this or we can do the partial fractions this we have already discussed before. So we will not spend time on taking this inverse Laplace transform. So for instance we can do this partial fractions 1 over s minus s over 1 plus s square.

So this is the Laplace transform of y and then we can apply this inverse technique so yt will be l inverse 1 over s minus this s over 1 plus s square. So l inverse 1 over s we know it is 1 and the l inverse s over s square plus 1 that is cos t. So we have the solution yt as 1 minus cos t.

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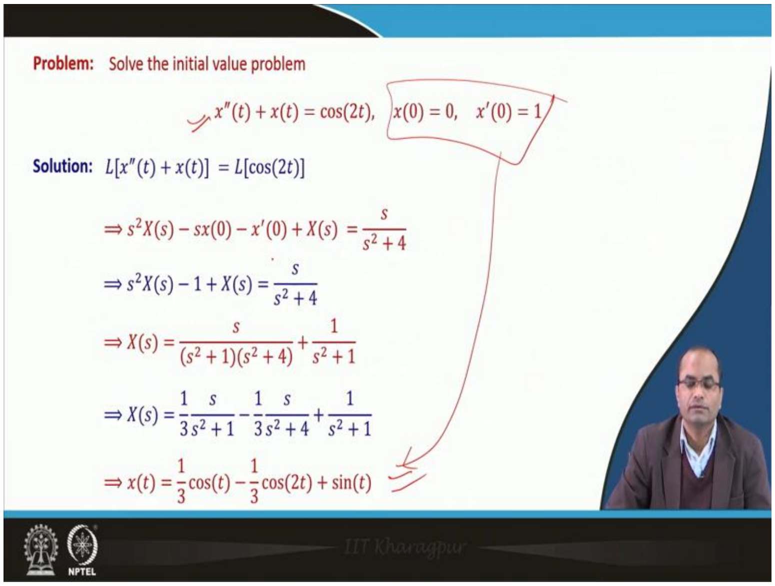

Here we have another problem x double derivative with respect to t xt and cos 2t here these are the initial conditions given. So again the process will remain the same we will take the Laplace transform simplify it incorporating these initial conditions and finally we will go for the inverse Laplace transform.

So taking this Laplace transform on this x double prime plus xt and the right hand side on cos 2t we will apply again this derivative theorem s square xs minus sx0 minus s prime that is the Laplace transform of this double derivative plus the Laplace transform of xt and the right hand side Laplace transform of cos 2t. So now we can substitute this x0 which is 0 there x prime is 1 so we have 1 term there.

So then we can simplify for this xs which is given here now and again the same process. So we can do partial fractions at this stage and then we have to take the inverse finally which is from it is a cos t the inverse of this will be again cos 2t and here it will be sin t.

So this is the solution of the given differential equation which also incorporates this initial conditions because they are used in the process. So the advantage of this Laplace transform is that directly after taking the Laplace transform we can use the given initial conditions and finally we will get just the solution of the given problem.

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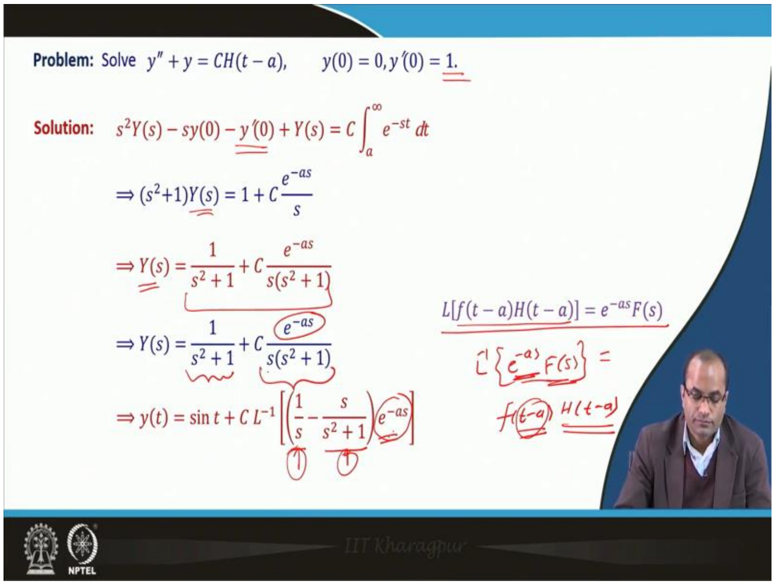

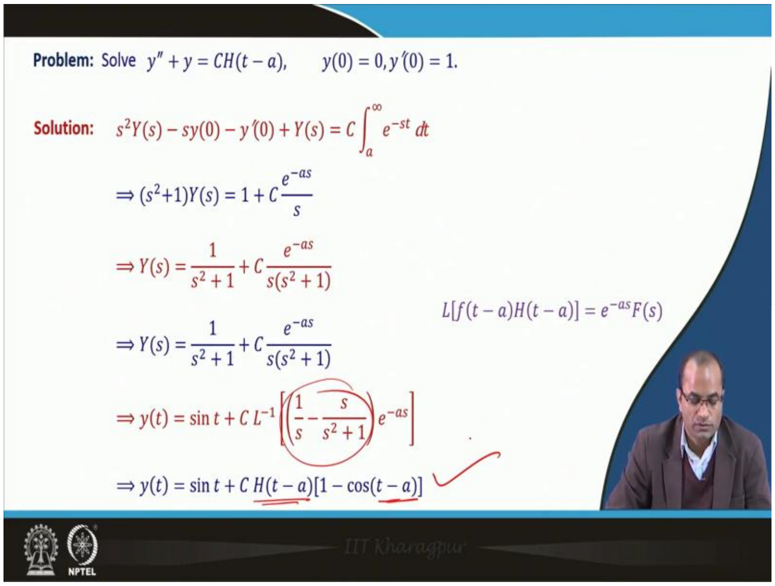

So here we have y double prime plus y is equal to c times some constant times this heavy side function ht minus a. So it takes the value 1 when t is greater than or equal to a otherwise 0. And then we have initial conditions and this is indeed a very good application of the Laplace transform because this is a discontinuous function sitting at the right hand side dealing with this directly by some other means would be difficult.

But here since we know the Laplace transform of the heavy side function it can easily be handled. So taking the Laplace transform we use this derivative theorem here and then we have ys the right hand side we have c and then we have the Laplace of this t minus a heavy side function t minus a which is given here. We can just integrate a to 1 the function is e power minus st dt. So which is just e power minus a s over s.

So here we have s square plus 1 y is equal to 1 plus c e power minus as over s and we have used this initial condition y prime 0 as 1. So simplifying this for ys we have these expressions now for ys including the c there and then we can go for the inverse. So after taking the inverse here it is clear it will be sin t and now here we have e power minus as and then we have s s square plus 1.

So we have to first go for this partial fractions for instance this is 1 over s and s over s square plus 1 and then e power minus as. So we have to deal now how to get the inverse remember the shifting theorem where we have the l inverse of this e power minus as and fs will be just ft minus a ft minus a and ht minus a. So this is the inverse so e power minus as is sitting there 2 and we have fs so sf is 1 over s and then s over s square minus 1 so there are 2 functions.

The inverse of this is 1 and the inverse of s is s over s square plus 1 is cos t. So then there will be a shift in cos t with the t will be replace by t minus a and there will be a factor ht minus a. And then we need to that is all this we can handle easily now so we have the heavy side function t minus a and this the 1 minus cos t inverse of this but because of this theorem there will be shift here also. So we have constant times the heavy side function t minus a and 1 minus cos t minus a.

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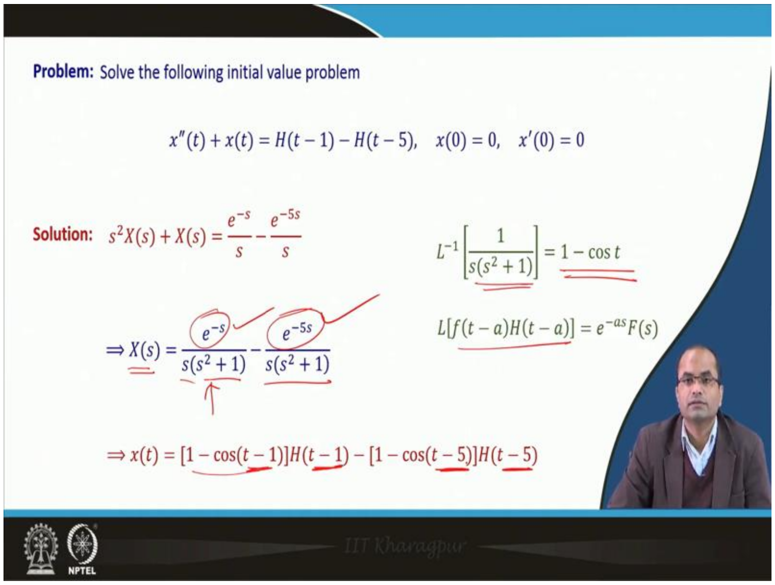

Solve the following problem we have the double derivative plus this x right hand side there are heavy side functions but they can be handle similarly as in the previous problem. So after taking the Laplace transform we have s square xs plus this sx coming from here because these 2 are 0 x0 and x prime so there will be no other term other than this s0 xs once we apply here the derivative theorem. Here the Laplace of this heavy side t minus 1 is e power minus a i minus s over s and in this case it is e power minus 5s over s. And then this xs will be e power minus as over s square plus 1 e power minus 5s and again s into s square plus 1.

So again we have to realize that the l inverse over 1 over s s square plus 1 is 1 minus cos t and because of this e power minus 5s will have the shifting theorem which was used in previous example as well. So we have 1 minus cos t minus 1 this is the shift t minus 1 because of this first e power minus s in the second case we have shift by this 5 so t minus 5 and again same thing 1 minus cos t minus 5 the heavy side function t minus 5.

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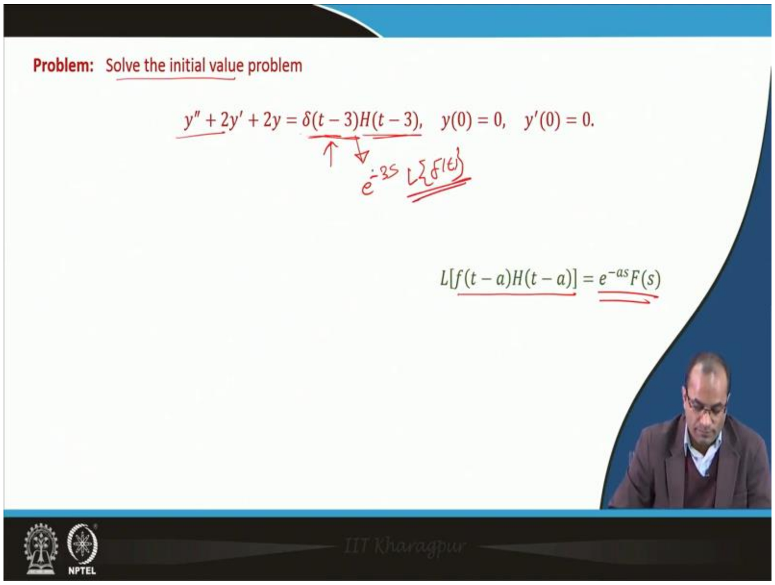

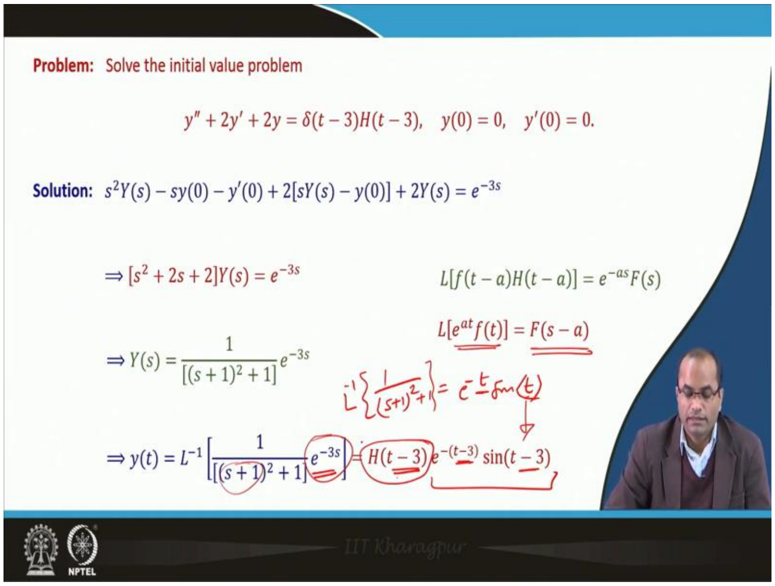

Solve the initial value problem so we have here again with the delta term delta t minus t ht minus 3. So here also we can use this while taking the Laplace transform there we can use the shifting theorem ft minus a ht minus a is equal to e power minus as fs. So the right hand side here after taking the Laplace transform we will have e power minus 3 s and the Laplace transform of this delta t which is 1. So here the right hand side will be having only e power minus 3s once we take the Laplace transform.

Here we have for the double prime these 3 terms using this derivative theorem then we have 2 times again the derivative theorem there and then 2 times the ys. So here now we can combine so with ys we have s square 2s plus 1 the right hand side we have e power minus 3 s and we have used these initial conditions. Then we have ys in terms of completely the function of s so 1 over s plus 1 whole square plus 1 and e power minus this 3 s. So the yt when we take the inverse there so we have to now get this inverse and again this shifting theorem will help for the inverse.

So whenever e power minus as is sitting and we are taking the l inverse this will be ft minus a so there will be a shift in the function. So here also we have the shift t minus t will be replaced by t minus 3. But we need to get first the Laplace inverse of this 1 over s plus 1 square plus 1. So it is of kind 1 over s square plus 1 and that first shifting theorem we have to apply there to because e power at ft is s minus a so that we can apply here to remove this shift there.

So we will be e power minus t and then this is sin t. So there will be e power minus t sin t and because of this e power minus 3s there will be a shift in t by t minus 3. Because the Laplace inverse of this 1 over s plus 1 square plus 1 will be e power minus t and sin t. And then because of this e power minus 3 s this 3 will be replaced by t minus 3 this will be also replaced by t minus 3 and there will be factor of this heavy side function t minus 3.

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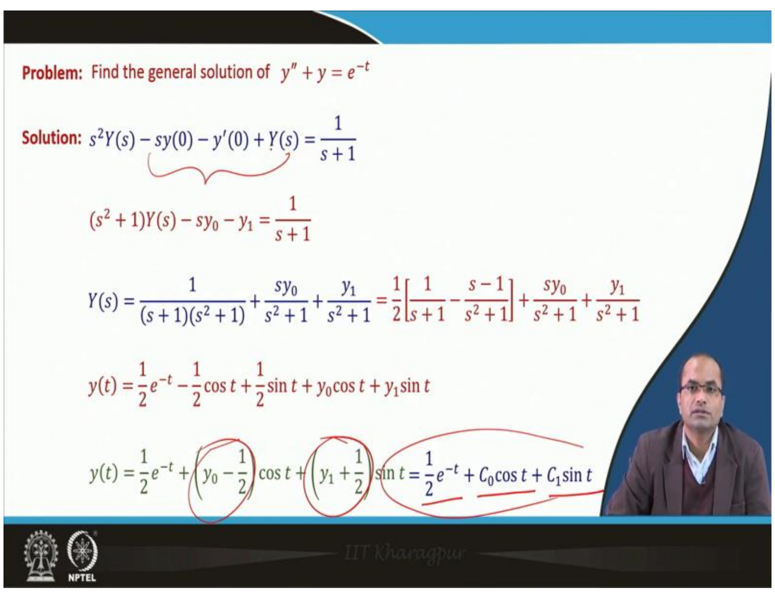

So now we will find the general solution in this case meaning that no initial conditions are given but we can still deal with this Laplace transform. So we will just keep those unknowns as it is and then proceed with the same process. So here we take the Laplace transform so this the derivative theorem for y double prime then we have ys and then the right hand side will give us 1 s plus 1 that is the Laplace transform of e power minus t.

So here this y0 and y prime 0 are not given , so we will just take them as some constant, so let us take this y0 and y1 and then we have s square plus 1 Ys is equal to this. So we will again bring into this form that Ys is equal to this function of s and then here we can do this partial fractions. That means we have 1 over s plus 1 s minus 1 over s square plus 1 and this y not s over s square with y1 we have 1 over s square plus 1.

And then so taking the Laplace inverse we will get 1 by 2 e power minus t here also we will get 1 by 2 with minus sign s over s square plus 1 will give cos t. Then we have again here the plus half then we have sin t we have y not cos t and we have y1 sin t again. So finally, we have yt is equal to half e power minus t and then this cos t will give y not minus half and with sin t y1 plus half which can be treated as a different another constant here C not and the C1.

So our general solution of the given differential equation is half e power minus t some constant C not cos t some another constant general constant C1 sin t. So given the conditions we can compute at the end also once we have the general solution. But in general when we apply the Laplace transform with a conditions are given we can fit them in after just first step and then calculations will be much easier.

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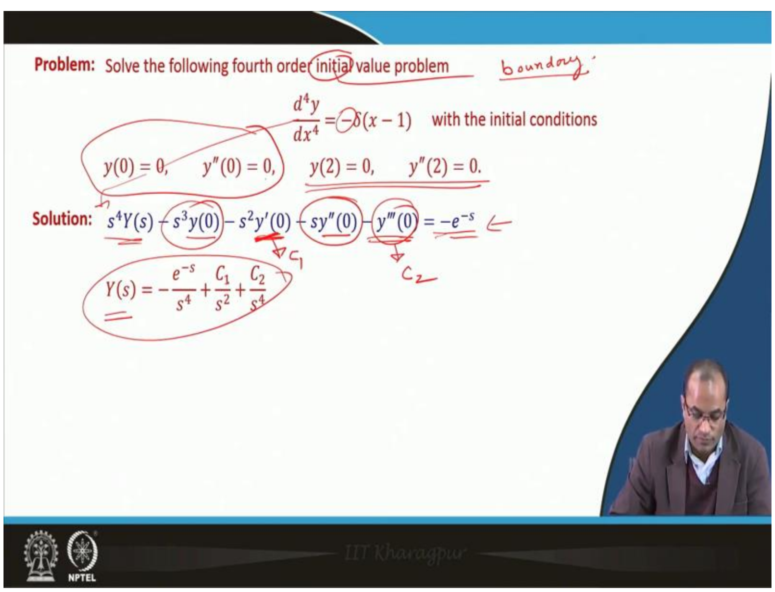

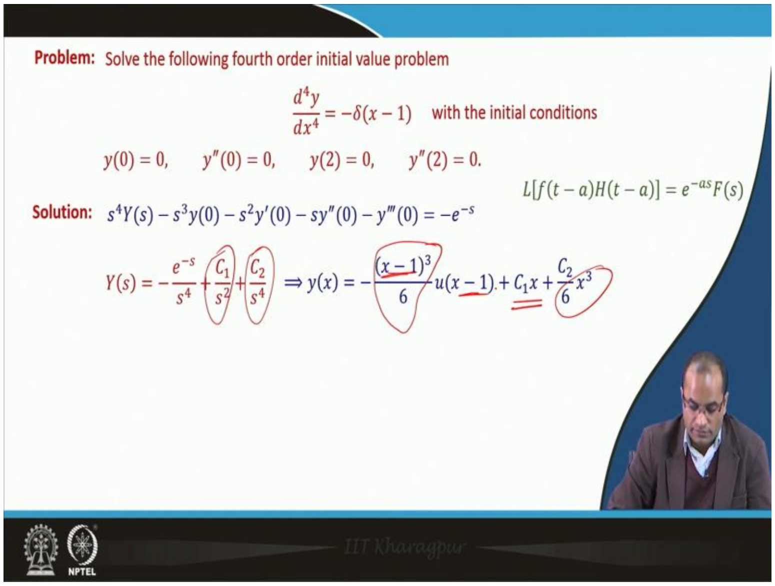

So here we have for instance this fourth order differential equation with right hand side having delta x minus 1 the initial conditions are given all these 0s. So we can apply again the Laplace transform and all these will become 0 because of this initial conditions. Sorry there are not initial conditions these are this is the boundary value problem actually, so boundary value problem because at two points this conditions are given one is at 0 and the other one other two conditions are given at 2.

So this is not the initial value problem it is boundary value problem. Boundary value problem where the conditions are given at two different points, so one the two conditions are given at 0 other two are given at 2. So here these conditions can be incorporated directly at this point after taking the Laplace transform but these two conditions are we cannot apply at this moment. First we will get the solution general solution in that sense.

So we have now the derivative theorem as 4 and then we have y0 y prime y double prime and then also the third derivative will appear in the derivative theorem. The right hand side we have the derived delta this x minus 1 and its transform will be e power minus s and there was a minus sign which remain as it is here. So after simplifying because we have to now consider for instance this y y not is 0 and y double prime 0.

So this term will vanish and also this term will vanish, but this will remain. So we can take it as like C1 and this coefficient we can take as C2, so we get this expression for ys. And then, we know this shifting theorem which can be applies here for e power minus s. And there is a s 4 term, here also the same thing but there is a no shifting theorem no e power minus s. So finally we have due to this e power minus s they will be shift in 1.

So we have x minus 1 cube by 6 and the heavy side function so sometimes we are using the notation u for the heavy side function somewhere we have used like h. So here we have the C1 x because of this and then again the same thing but without the shift here. So we have x cube y 6 here also it was x cube by 6 but because of this e power minus s there was a shift n multiplication by the heavy side function.

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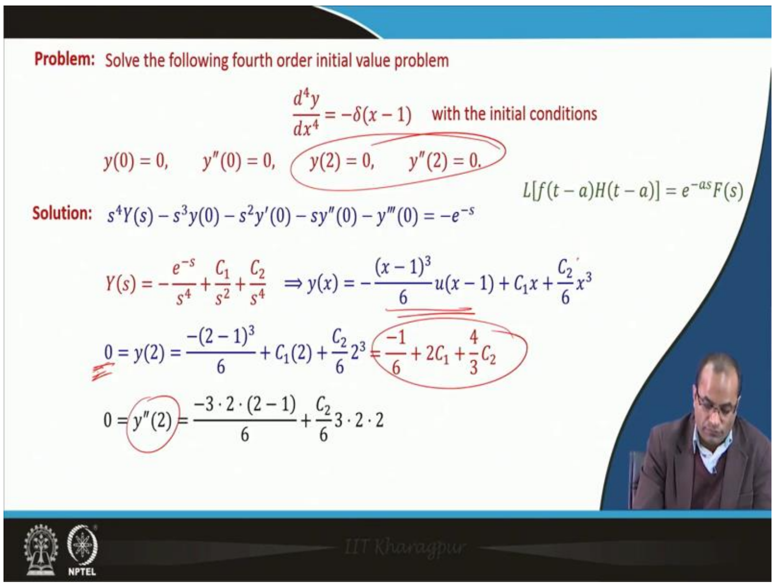

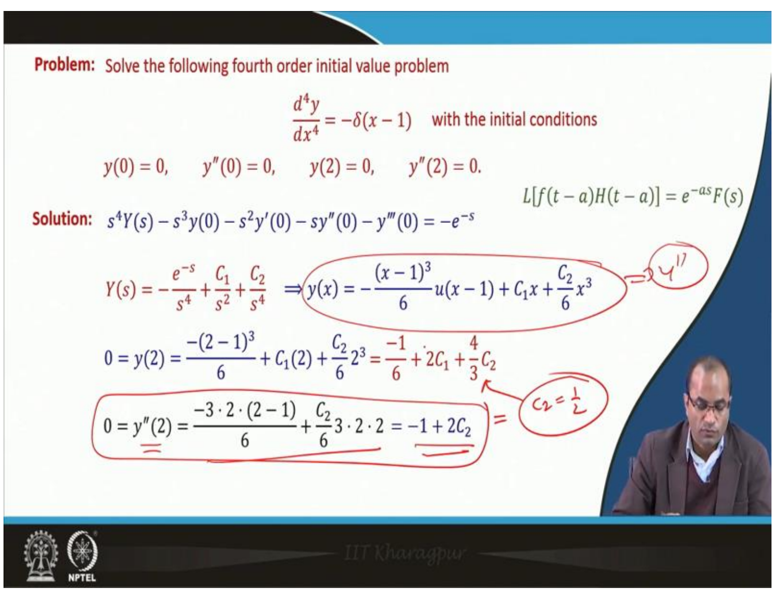

And now we can apply these two conditions which were not use so far. So y2 is 0, so we will just pass here x is equal to 2 and simplify this, so we will get one equation C1 C2. And the other equation we have to take the double derivative of the given solution, so after taking from here the double derivative we will set to 2, x is equal to 2, and we will get another equation which is just in C2.

So from this second equation we will get straight away C2 is equal to half for instance and that C2 can be plugged in in this first equation and we can get C1 straight forward. So having C1and C2 we can just substitute in that solution and we will get the solution of the given differential equation.

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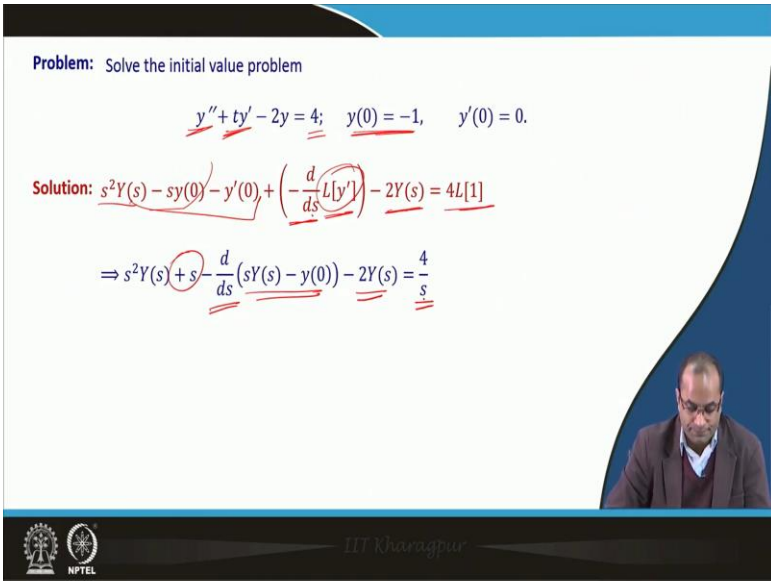

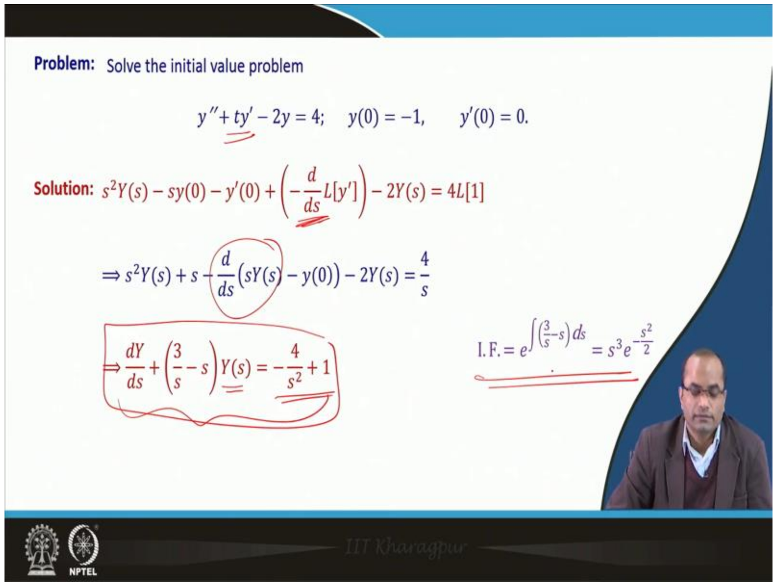

So here we have the initial value problem because this conditions are given at 0 only. And here the difference is that this is t y prime. So we have the variable coefficients, but we can deal again with the Laplace transform. So we have s square the derivative theorem for y prime goes up to this and there is a t times, so we have already gone through the properties of Laplace transform. So Laplace of this t y prime is minus d over ds Laplace of y prime and minus 2 y and then we have 4 times this Laplace of 1 right hand side. So here we have s square Ys and we have use this condition y0 as 1.

So there will be a term s here and then minus this d over ds and the Laplace of y, so again we have to y prime, so we have to use the derivative theorem minus this 2 Ys the right hand side we have 4 over this s the Laplace of 1 is 1 over s. And here now we can use these initial conditions. So indeed does not matter here because d over ds, so that will make this 0. So we have this d over ds of sy. So we can use the product rule.

Finally, we will get this differential equation in this case not just the algebraic equation because of this t there. And this derivative time derivative term has appear. So we will get a differential equation when we have these coefficients depending on the independent variable. So we have dy over ds plus this 3 over s minus s and Ys right hand side minus 4 over s square plus 1. And that this is a linear equation so we have to find the integrating factor, and then we can write down its solution. So the rest is just simplification and to get again this Ys, so you can go forward.

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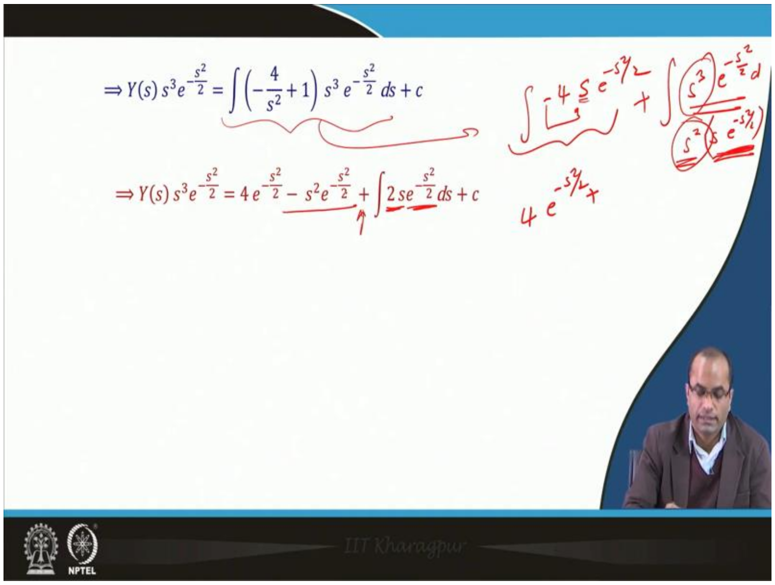

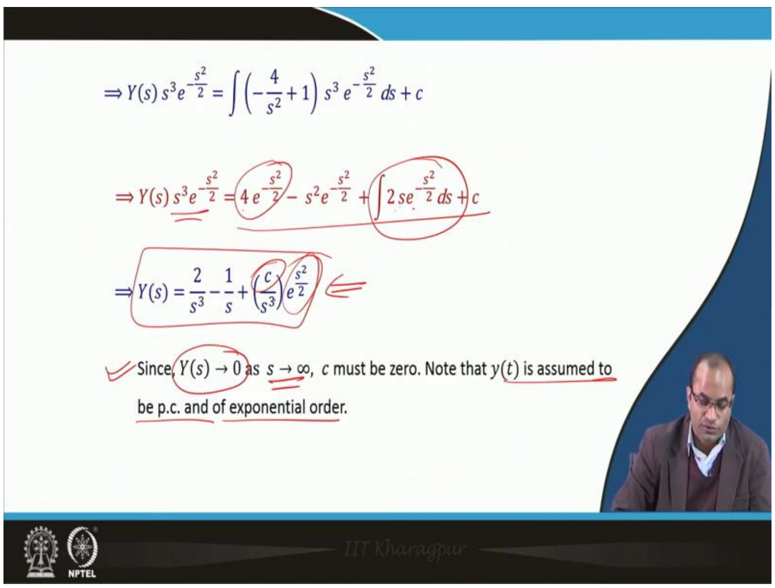

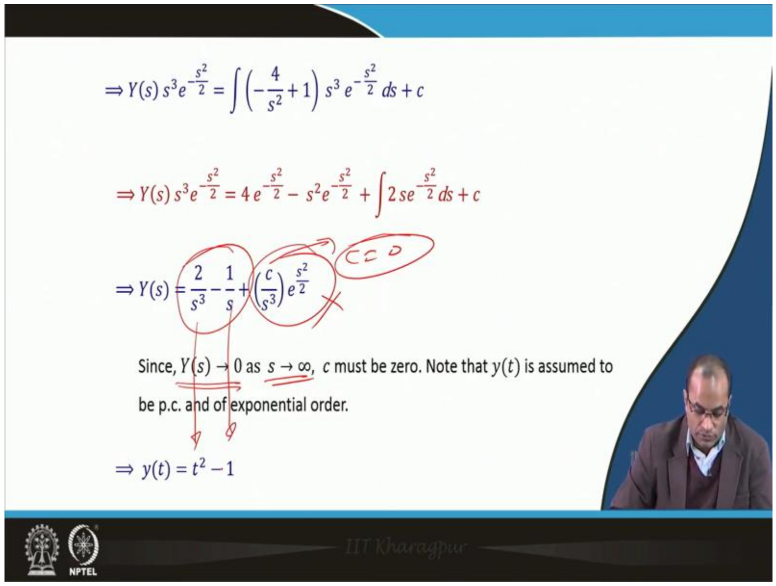

So after again here integration by parts because there will be at the terms. So the first integral will be like minus 4 se power minus s square by 2, and then there will be integral with this s cube and e power minus s square by 2 ds. So that is the integral of sitting here right hand side the first readily will give here because the derivative of this is sitting here with minus sign indeed so minus s.

So we will have 4 e power minus s square by 2 that is the integration. Here we have to apply the chain rule again. So the chain rule says that this can be written as s square and then s we can bring this with e power minus s square by 2. So this term can be integrated and this can be differentiated. So here with the minus sign because we need to integrate this and there is a minus sign there. So minus s square e power minus s square.

The derivative of s square will be 2s and again the derivative e power minus s square by 2 with this minus sign so it has become plus and then we have C. So this, so here we can integrate again this and this is again minus e power minus s square by 2, with the 2 here it was with 4. So the 4 will remain. And then the 4 and then minus 2 so this will be like 2 e power minus s square by 2. And then we can divide by this, so finally we get the expression for this Ys in terms of s.

And since we assumed already that yt be piecewise continued exponential order. And we know the property of the Laplace transform that its transform must go to 0 as s goes to infinity. So if this property need to be fulfilled here, then the C has to be 0. Because if C is non-zero because of this e power s square by 2, this will go to infinity. So to have this property of the Laplace transform that Ys the Laplace transform of this function must go to 0 as s goes to infinity in that case we got that the C has to be 0 otherwise this is not possible.

So then this term will vanish and then we can take the inverse, so this 2 over s cube will give t square and 1 over s will give 1. So we have t square minus 1 the function this y t.

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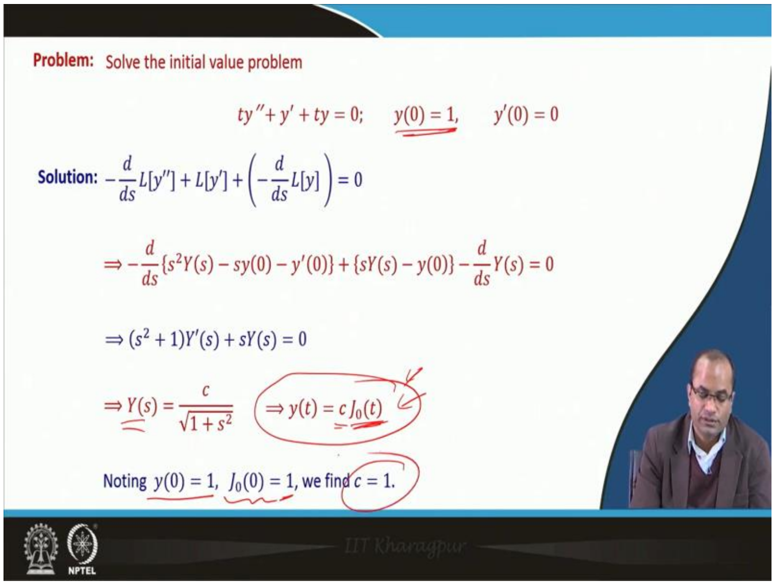

Solve the initial value problem, so here again we have the variable coefficient so t y double prime plus y prime plus t y equal to 0, and these initial conditions are given. So we will follow the same step, we will take the Laplace transform and because of t there will be a term d over ds.

So after this simplifying because here we have to there are two places where d or ds will appear and then the derivative theorem will be applicable here, then you have this L y prime again the derivative theorem and then we have minus t over ds there. So after simplifying what we observed because we have incorporated these initial conditions also here. So we will get this differential equation a simple differential equation.

And then whose solution can be written simply by this Ys C over square root 1 plus s square. And just to recall that if we take the inverse here, this is the inverse of this special function which we have discuss before J0 t. So there was a this was discussed in the earlier lecture and also we recall from there that J0 t was actually 1. And then here y0 is given as 1, so we can compute this C also. So with these two conditions which that function follows and also y0 1 is given here in the problem we find that C is equal to 1. So we have yt is equal to J0 t.

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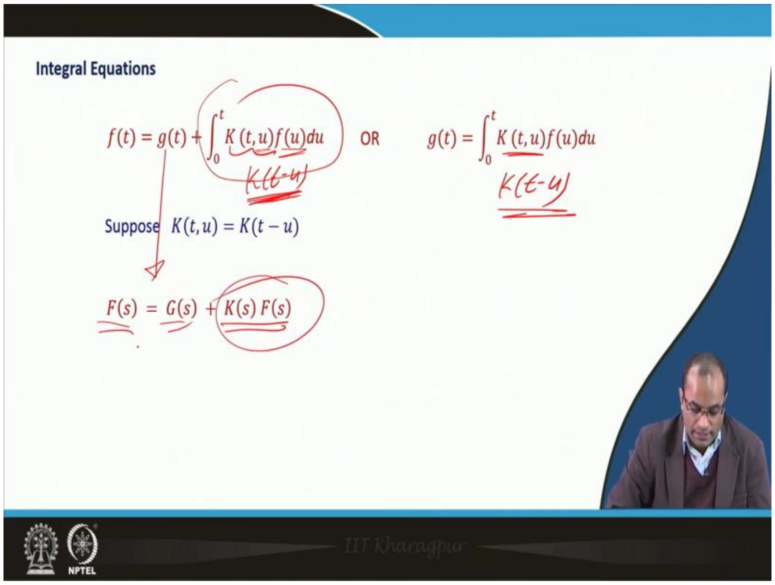

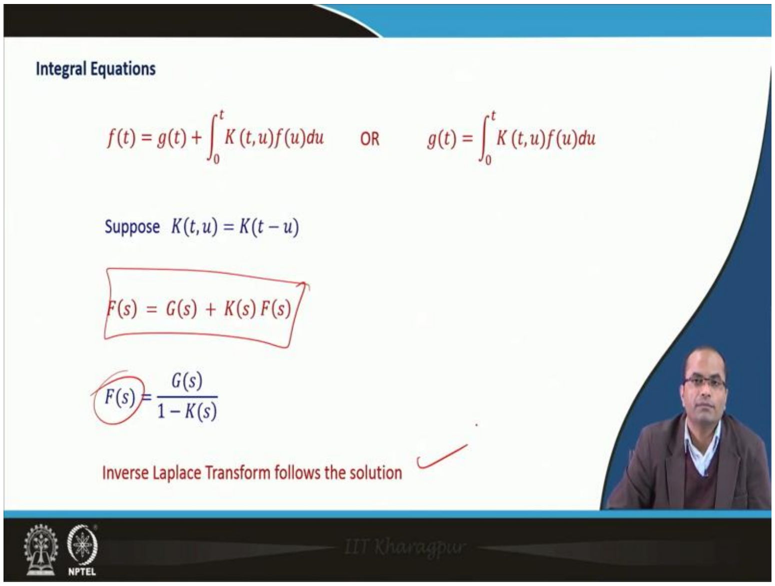

Now coming to the integral equations, so we have ft is equal to gt of this nature or we can have the differential equation of this nature that gt is equal to K t u fu du. And the question is what is f? f is unknown basically in this problem. How to get this f we will again use this Laplace transform. So suppose and under these condition this Laplace transform is very much useful. So if K t u if of this kind K function of t minus u here.

So this is K t minus u here also we have this K t minus u. So in that case we can easily apply the Laplace transform and so we have here Fs we have for instance from the first equation Gs and here we have the convolution theorem, because this became no convolution integral if we have instead of this K t u, K t minus u and Fu. So we will get Ks and Fs. Once we are here we can again find for this Fs, the expression for Fs and finally the inverse transform will allow the solution. So it is a simple steps again, but the similar one.

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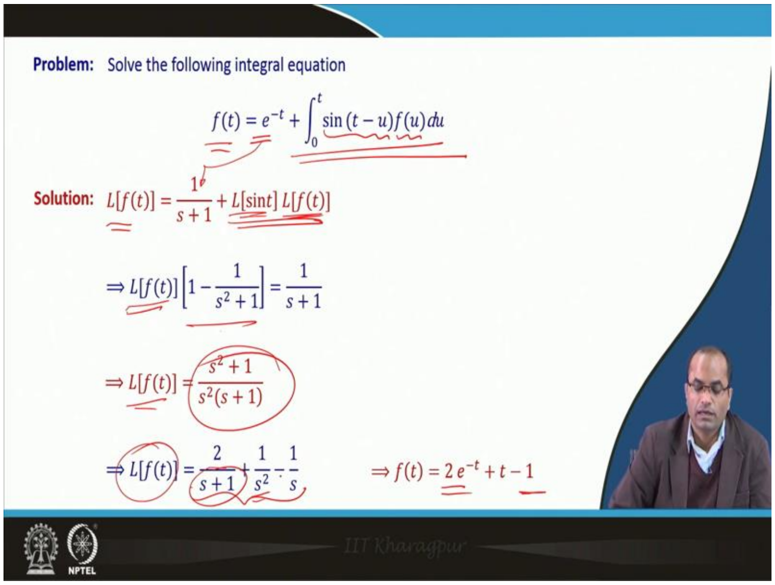

So let us go through some of the examples. So for instance we have ft is equal to e power minus t and then this integral which is the convolution integral. There is a function f and there is a function sin. So after taking the Laplace transform we have here the Laplace transform of e power minus t and then the convolution theorem. So this will be the product of the Laplace of sin t and the with the product Laplace of ft.

So taking now common Laplace of ft we get Laplace ft is equal to this, which again can be in inverted for the Laplace transform and having these partial fractions now we can take the inverse. So it is 2 e power minus t from here then t from here and the one from the last term. So we have 2 e power minus t plus t and then minus 1 is the ft.

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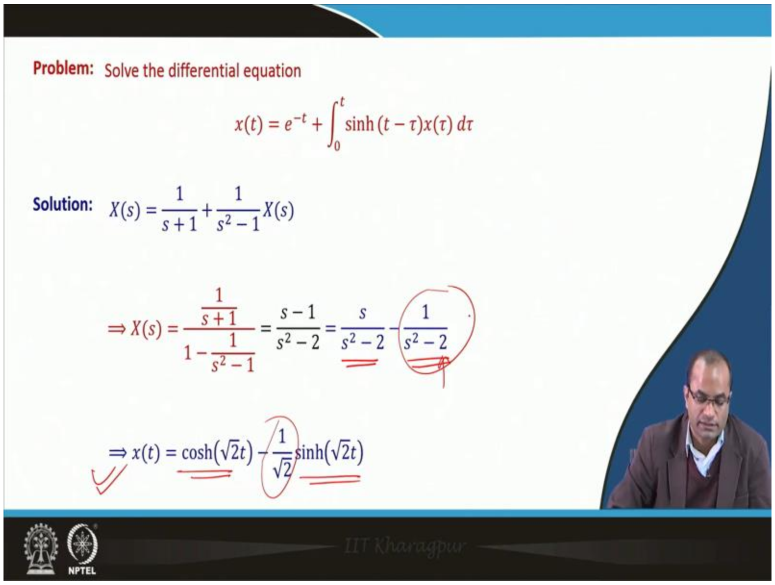

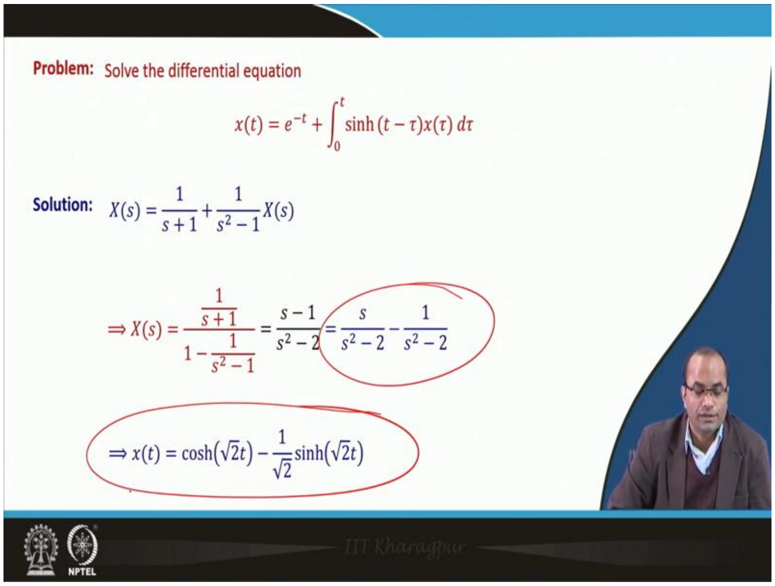

Here we have the xt is equal to e power minus t and sin hyperbolic function t minus tau x tau d tau. So again we have a similar situation that this is a product of these two functions. So the convolution product and then we can apply the convolution theorem. So that will be the product of the Laplace transform. So having that we have the access here and then for e power minus t we have 1 over s plus 1 and in this case this will be the product. So the Laplace of sin hyperbolic will be 1 over s square minus 1 and the product of the Laplace of x tau that is Xs.

So we have Xs is equal to this 1 and that means this s minus 1 over s square minus 2. And we have s over s square minus 2 minus 1 over s square minus 1, and taking this Laplace inverse we will get cos hyperbolic square root 2 t sin hyperbolic square root t with this minus 1 over root 2 that will be just adjusted. Because s square minus this a square will be coming there. So that is the just the after getting this inverse we get the Laplace we get the solution of the given differential equation.

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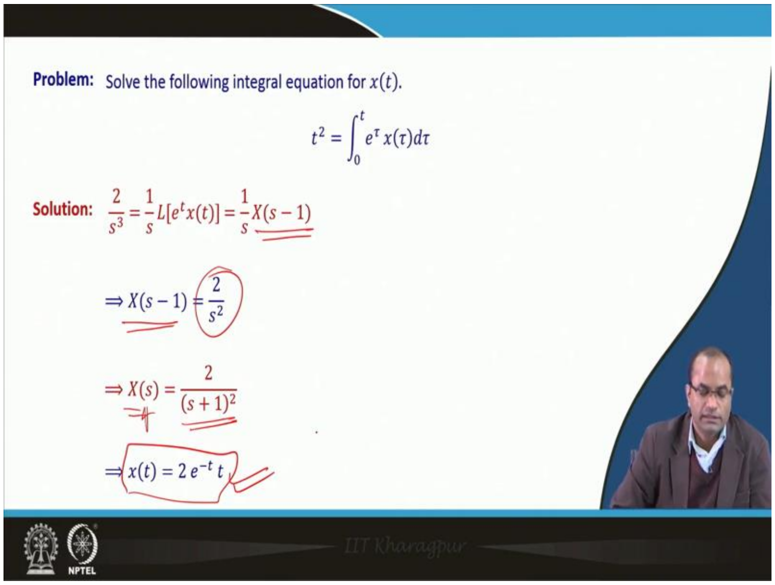

Here we have t square is equal to e power tau and x tau d tau. So here again we will apply the same principle. So 2 over s cube that is the Laplace transform of t square, then we have this integral there and we have already learn that the Laplace transform the integral only 1 over s factor will come there and the Laplace transform of this integral that is e power t and xt. So e power t xt the Laplace transform because of the integral we have 1 over s term.

So this is nothing but 1 over s and x s minus 1. So again the shifting theorem here because the Laplace of e power t xt that will be the Laplace transform of xt is Xs and there will be a shift because of this exponential function. So we have Xs minus 1. So we have Xs minus 1 s 2 over s square or Xs as 2 over s plus 1 whole square. So s is replaced by s plus 1 and then if we invert this we have xt is equal to 2 e power minus t into t.

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So that is all, this is these are the references we have used for preparing this lecture.

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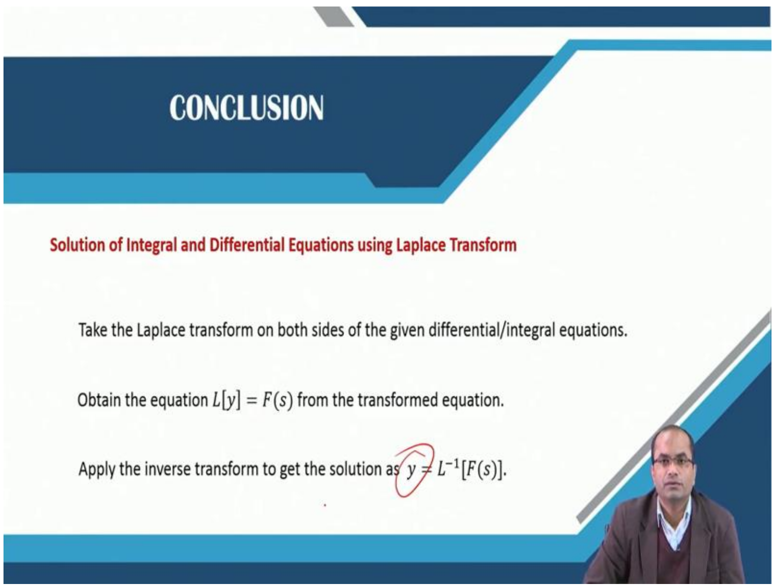

And just to conclude, we have discuss the solution of integral and differential equations using Laplace transform. The key steps were we have to take the Laplace transform and then we have to solve for Ly is equal to Fs in this case if we have the constant coefficient equations we directly get algebraic expression and we can easily get into this form.

But we have the variable coefficient then we need to solve some simple differential equations to again get this Ly is equal to Fs form. The same steps we have for the integral equations after applying the Laplace transform we can directly get this Ly is equal to Fs. And then final step would be taking the inverse from this so that we can get the solution of the given differential equation. Well so that is all for this lecture, and I thank you for your attention.