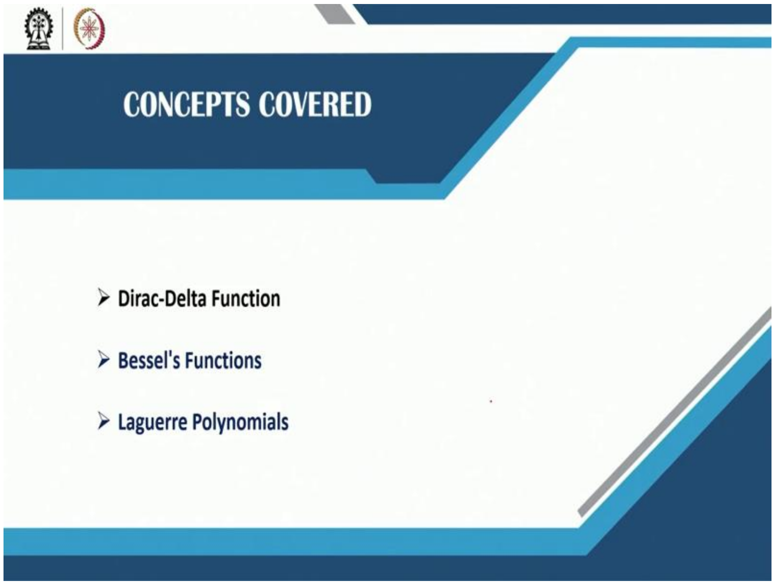
Engineering Mathematics-II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Science – Kharagpur
Lecture 58
Laplace Transform of Special Functions continue

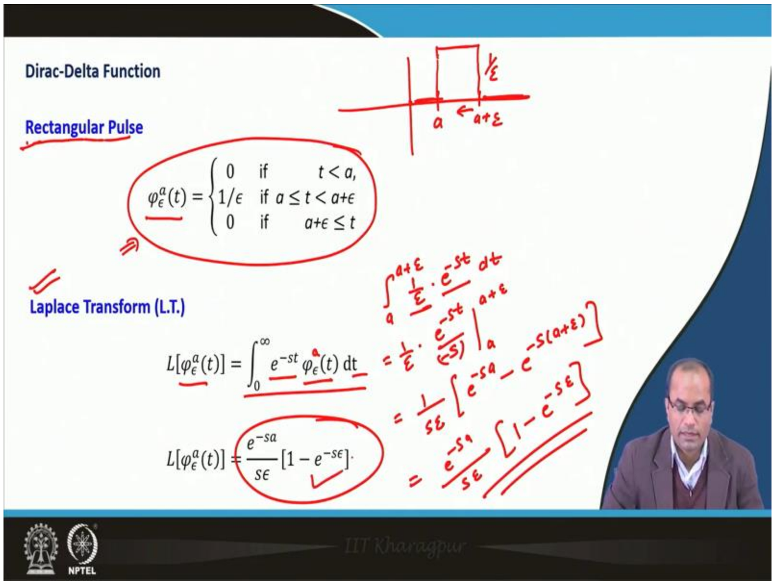
So welcome back to lectures on Engineering Mathematics 2. This is lecture number 58 on Laplace transform of Special Functions.

(Refer Slide Time: 00:21)



So in this lecture we will cover Dirac Delta function or in particular the Laplace transform of Dirac Delta function. Also the Bessel's function and these Laguerre polynomials.

(Refer Slide 00:34)



So coming to the Dirac Delta function we usually define through this rectangular pulse or there are several ways we can define such specialized or generalized function where with the help of this rectangular pulse so we define this rectangular pulse phi epsilon a as 0 if t is less than a and between this a to a plus epsilon we define as 1 over epsilon. So the height is 1 over epsilon from this a to a plus epsilon.

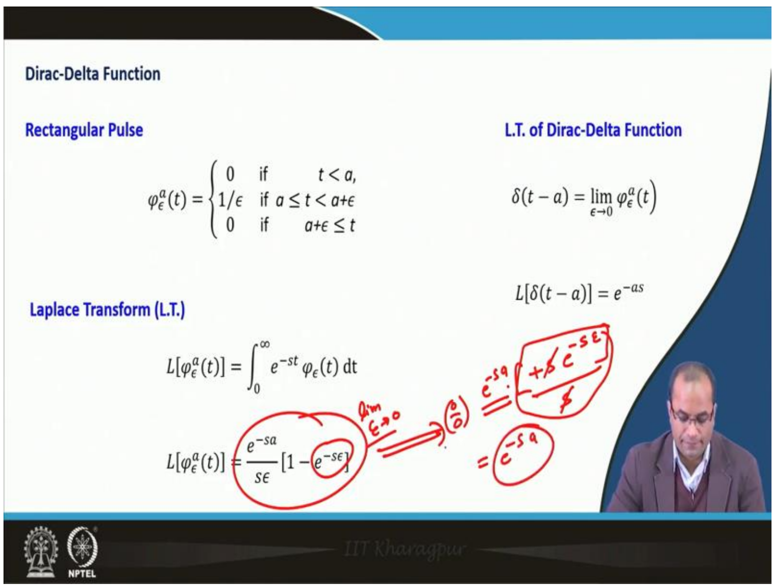
Otherwise so in this region and the region before it is 0 so this is the so called rectangular pulse function. Now the first we will get the Laplace transform of this rectangular pulse function because this Dirac Delta function is defined as or can be defined as the limiting situation of this rectangular pulse function when this epsilon is ver very small. So if this epsilon goes to 0 and this 1 over epsilon in that case will tend to infinity naturally and that is exactly the function which we will define here as Dirac Delta function.

So first we will take the Laplace transform for this given epsilon. So without considering the limiting situation first. So getting this Laplace transform with the definition we have e power minus st and then this phi epsilon t will be integrated over this function or this variable t. So here how to get this integral it is simple because 0 to infinity and it is define as from a to a plus epsilon this function phi epsilon t or at a also. So here we have 1 over epsilon and then e power minus st is there and then we have dt.

So because in the rest of the domain this 1 over this phi epsilon a is 0 so the integral will remain from a to a plus epsilon and e power minus st because of this Laplace transform. So here 1 over epsilon and then here we have e power minus st over s with minus sign it is integration and that will be integrate will be set for the limit a to a plus epsilon.

So here we have 1 over s epsilon the minus sign we will adjust now so first we will put the limit a so e power minus s a and then we have minus e power minus s and then a plus epsilon. So e power minus sa is common over this s epsilon then here we have 1 minus e power minus s and then this epsilon. So this is the Laplace transform of this rectangular pulse function which is written here e power minus sa over s epsilon and then 1 minus s epsilon.

(Refer Slide Time: 03:58)

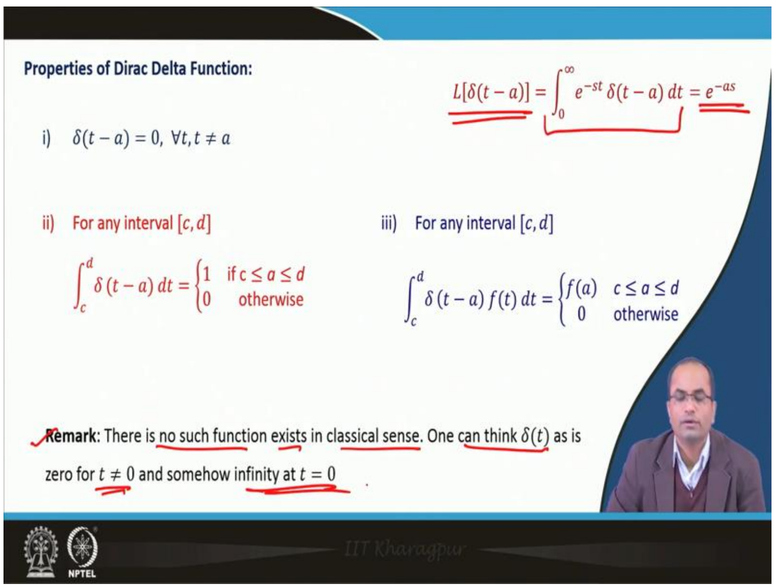


So now getting back to this Laplace transform Dirac Delta function which is limiting situation of this rectangular pulse that means this delta t minus a can be defined as the limit epsilon goes to 0 of this phi epsilon a t. So we already know the Laplace transform of this rectangular pulse function the only thing we have to deal now that what will happen now when we take this epsilon tends to 0 because that is precisely the Dirac Delta function.

So here in this Laplace transform we will let this epsilon to 0 and we will get the Laplace transform of this Dirac Delta function. So the Laplace transform of the Dirac Delta function will come as e power minus as how to get this so we will just set here the limit. So we will consider that what will happen to this when epsilon goes to 0. So to deal this here when epsilon goes to 0 e power minus s epsilon that is going to 1 1 minus 1 so 0 by 0 form we are getting.

When we consider this epsilon goes to 0. So 0 by 0 then we can apply the LHopital rule that means we have e power minus s a there we will differentiate here. So with minus and minus will be plus s we have e power minus s epsilon. And then divided by this s and this epsilon derivative will be 1. So this s and s will get cancel so we have e power minus sa and when we take this epsilon to 0 so this is 1 so the limit will be just e power minus sa. So here the Laplace transform of this delta t minus a is e power minus as.

(Refer Slide Time: 05:59)



There are some interesting properties of this Dirac Delta function and we will go through them and we will look an alternative way of getting this Laplace transform of Dirac Delta function through these properties. The first property says or sometimes it is defined or I means this is also treated as the definition of this Dirac Delta function.

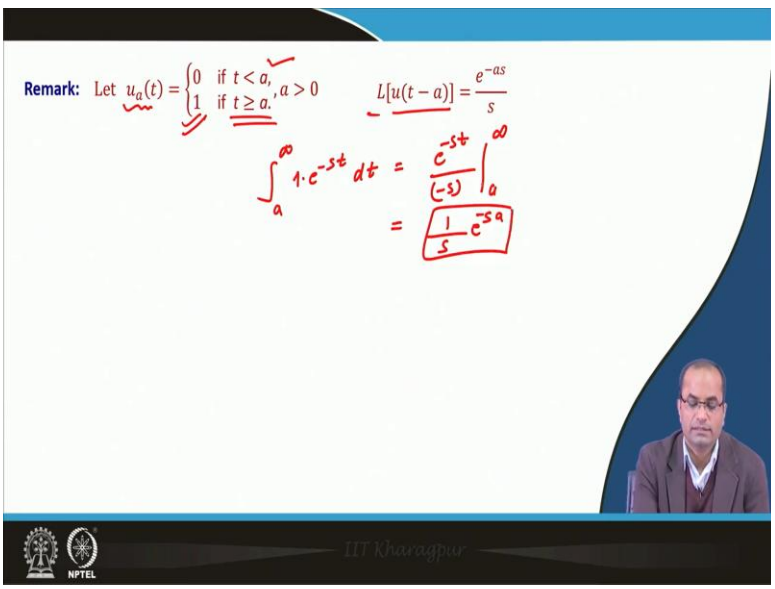
So we have delta t minus a is equal to 0 whenever t is not equal to a so whenever we are the point t which is not equal to a this is defined as 0. And when we integrate this delta t minus a over some interval c to d and if this c to d contains that point a than the value of this integral is going to be 1 otherwise the value of this integral will be 0.

Or more general situation when we integrate delta t minus a with this ft from this c to d in that case the value of this integral is going to be fa when a belongs to this interval c to d or it will be 0 when a is outside the range of c to d. So this a more general situation when we have this delta function under this integration with some function here ft tt the value will be just fa whenever a lies between c and d.

So having these properties of the Dirac Delta function we can indeed now directly apply this Laplace transform of this delta t minus a so as per definition we have e power minus st and delta t minus a dt and using this property here since this a will this t goes from 0 to infinity. And whatever a we take here a is certainly positively consider. So here in that case this a will lie in this interval of integration and the value of this integration is going to just the value of this function e power minus st at equal to a.

So that means e power minus as that is the value of this integral and which is the Laplace transform of Dirac Delta function. Just a short remark that there is no such function exist in the classical sense. So one can think this delta t just as 0 and t is not equal to 0 and somehow infinity when t equal to 0.

(Refer Slide Time: 09:01)



Remark: Let $u_a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geq a. \end{cases}, a > 0$ $\quad L[u(t-a)] = \dfrac{e^{-as}}{s}$ $\quad L[\delta(t-a)] = e^{-as}$

$$L[u_a'(t)] = -u(0) + s\,L[u_a(t)] \quad \Rightarrow L[u_a'(t)] = -u(0) + s\,L[u_a(t)] = e^{-as}$$

Laplace of derivative of Heaviside function $= e^{-as} =$ Laplace of Dirac-Delta function

Further $\displaystyle\int_0^t \delta(x-a)\,dx = u(t-a)$

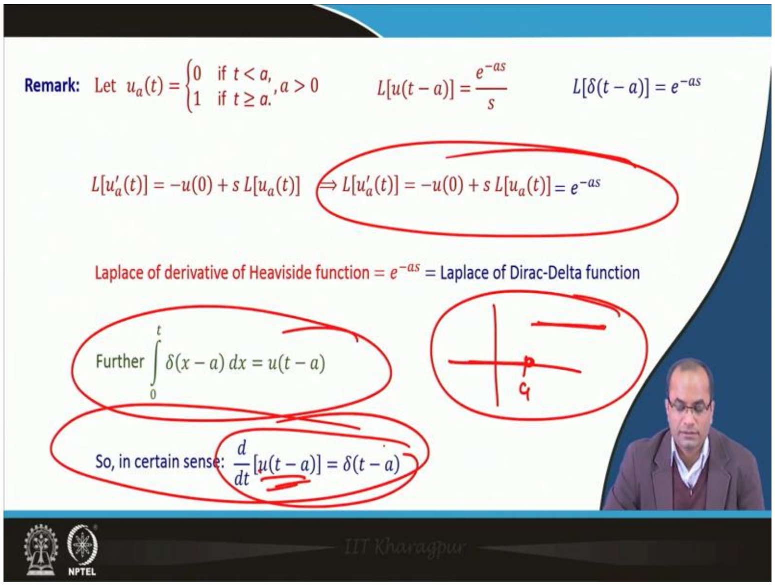
Remark: Let $u_a(t) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geq a. \end{cases}, a > 0$ $\quad L[u(t-a)] = \dfrac{e^{-as}}{s}$ $\quad L[\delta(t-a)] = e^{-as}$

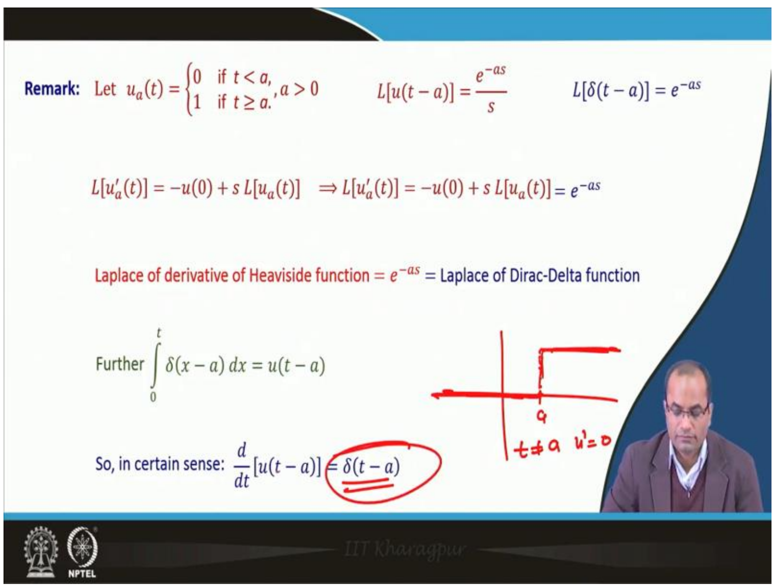
$$L[u_a'(t)] = -u(0) + s\,L[u_a(t)] \quad \Rightarrow L[u_a'(t)] = -u(0) + s\,L[u_a(t)] = e^{-as}$$

Laplace of derivative of Heaviside function $= e^{-as} =$ Laplace of Dirac-Delta function

Further $\displaystyle\int_0^t \delta(x-a)\,dx = u(t-a)$

So, in certain sense: $\dfrac{d}{dt}[u(t-a)] = \delta(t-a)$





So just another remark so we have already consider this heavy side function which is defined as whenever t is less than a it is 0 and t greater than or equal to a it is 1. And indeed we have computed earlier the Laplace transform of this function ut minus a because in this case this will be just integrated from infinity the function is 1. And then e power minus st and then we have dt which can be integrated so we have e power minus st over minus s.

And then a to infinity infinity so it will go to 0 then we have 1 over s and e power minus sa. So that is the Laplace transform of this function this heavy side function ut minus a or sometimes we denote here ua t whatever way. So and we also know that the Laplace transform of this delta function Dirac Delta function is e power minus as and we will try to give some intuition that what is the relation between this Dirac Delta function and this heavy side function.

Because if we take a closer look at there Laplace transform there are having e power minus as common in both. So we know already this derivative theorem that the Laplace of the derivative of this uaa will be minus ua 0 and plus s l uat. So Laplace of this derivative of ua will be minus u a0 plus s times this the Laplace of this. So we can substitute this Laplace here and what we will get because this is going to be 0 and then we have s times the Laplace of this uat. Which is just e power minus as.

So we have observe that the Laplace transform of this the derivative is equal to e power minus as and in that case the Laplace what we observe here that the Laplace of the derivative of this heavy side function because the Laplace of the derivative of this heavy side function is e power minus as and which is the Laplace transform of Dirac Delta function.

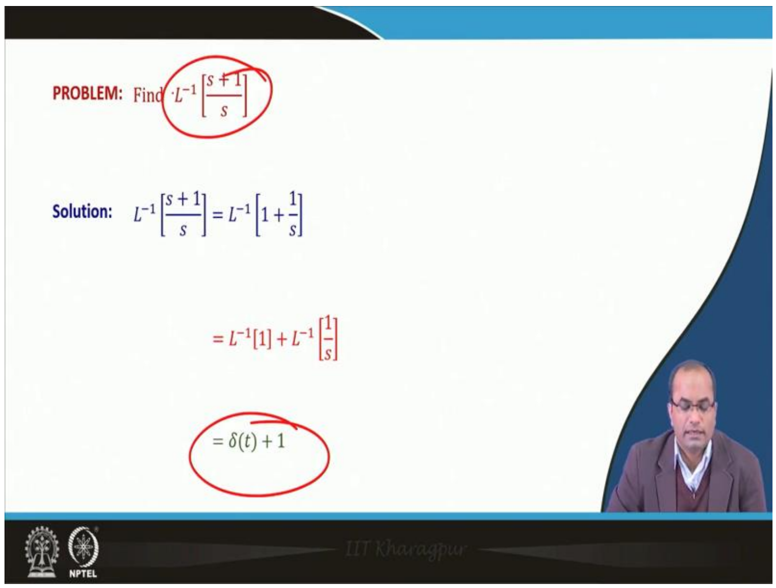
So the Laplace transform of the Dirac Delta function and the Laplace transform of the derivative of the heavy side function they are the same. So that it seems like the derivative of this heavy side function is the Dirac Delta function from this conclusion. But that is also justified because if we take a look at the definition so delta x minus a if we integrate from 0 to t what we will get we will get the heavy side function.

Because if this t is greater than a so this will contain in the interval and the value will be 1 for t greater than a the value of this integral and 0 and t is less than a. So we are getting we getting here that the integral of this delta x minus a is equal u t minus a. That means in some sense this implies that the derivative of this heavy side function is the Dirac Delta function. And which is coming from this Laplace transform as well because the Laplace transform of the derivative of the heavy side function was equal to the Laplace transform of the Dirac Delta function.

And the same conclusion we are getting from this observation so in certain sense we can say because again not the classical sense because this heavy side function is a discontinuous function. It has a value 0 and t is less than a and then it has the value 1 when t is greater than a. So this is a discontinuous function at this t is equal to a and still we are talking about its derivative and its derivative is Dirac Delta function which again not a classical function.

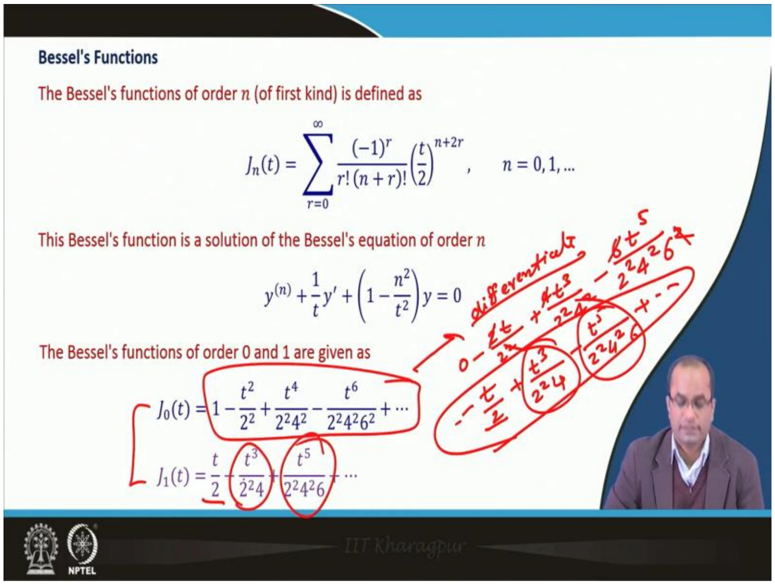
But what we observe that this heavy side function when we integrate the value because it has a constant value here 0 and this is 1. So its derivative is going to be 0 there and also 0 there. That means whenever t is not equal to a the derivative of this u hat is going to be 0 as per this implication that is our delta function. And when t is equal to a it is going to infinity because this Dirac Delta function takes very high large value where t is equal to a. So it is going to infinity which is again indeed the case here because the slope is infinity at this point. So the derivative of this heavy side function seems to be this Dirac Delta function.

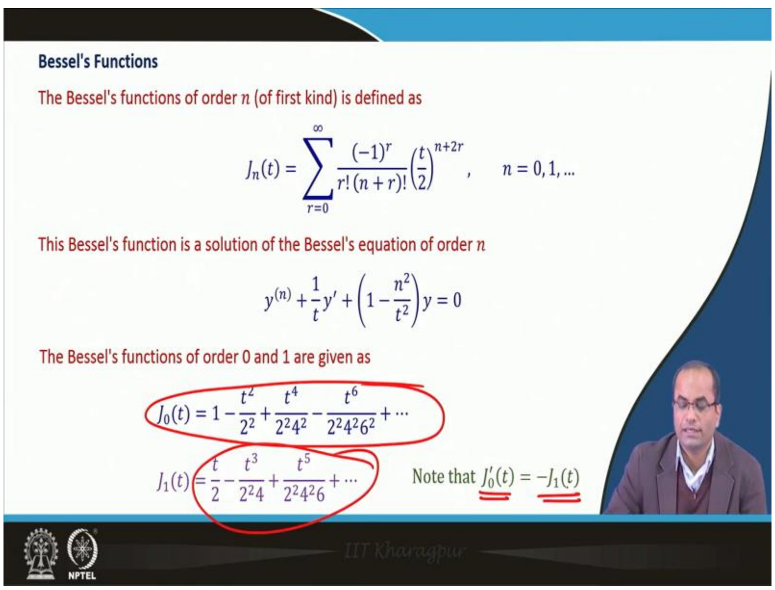
(Refer Slide Time: 14:32)



Just a problem here where we will find the Laplace inverse of s plus 1 divided by s. So this s plus 1 divided by s can be written as 1 plus 1 over s and then we can apply this Laplace inverse. So we have Laplace inverse 1 and then Laplace inverse 1 over s and the Laplace inverse 1 is the Dirac Delta function. So delta t minus 0 delta t and this l inverse 1 over s that is 1 the Laplace inverse of 1 over ss 1. So we have 1 plus delta t the value of this l inverse s plus 1 divided by s.

(Refer Slide Time: 15:20)





Now we will move to the next function which is the Bessel's function or rather Bessel functions because it is the series of sequential functions. So the Bessel's function of order n so that order is important and this is defined as with this infinite sum j and t the r goes from 0 to infinity minus 1 power r factorial r factorial n plus r. And then t by 2 n plus 2r and then n goes from 0, 1, 2, 3, etcetera. So this is how this Bessel function of order n are defined so of order Bessel functions of order n are define and this is of first kind.

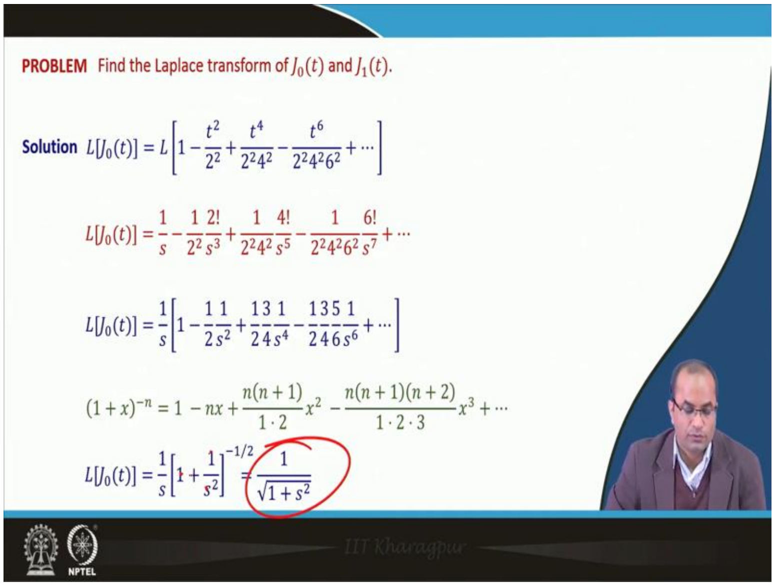
So the Bessel functions is just to give you more information that this Bessel's function is the solution of so the so called Bessel's equation of order n. And those ordinary differential equations are given as so here have the nth order derivative of this y plus 1 over t y prime. And then we have here 1 minus n square over t square y equal to 0. So the solution of this Bessel's equation come as this Bessel's function written in this written as infinite sum.

So we will consider in this lecture now the Bessel's function of order 0 and 1. And for the order 0 when we expand this, this will becoming as 1 minus t square over 2 square t 4 2 square 4 square and t 6 then 2 square 4 square and 6 square and so on. And J1 if we write it will be t by 2 and t cube by 2 square 4 and then we have t 5 by 2 square 4 square and 6.

And if we take a closer look at this two functions here, if we differentiate this so if we differentiate if we differentiate this function, so one the derivative will be 0 then we have minus 2 t and then 2 square then we have 4 t cube and then 2 square 4 square then we have here 6 t 5 2 square 4 square and 6 square. And so this 2 will get cancelled this 4 will also get cancelled and this 6 will also cancelled, this is square.

So what we are getting then? Minus this t by 2 plus here we have t cube 2 square 4we have t 5 and 2 square 4 square 6 and so on. So here if we take the minus sign out we have t by 2 we have minus t cube by 2 square 4, we have t 5 2 square 4 square and 6. So what we observe that the derivative of this J0 t is nothing but minus of this J1 t. So we will note now that the derivative of J0 t is nothing but J1 t.

(Refer Slide Time: 18:51)



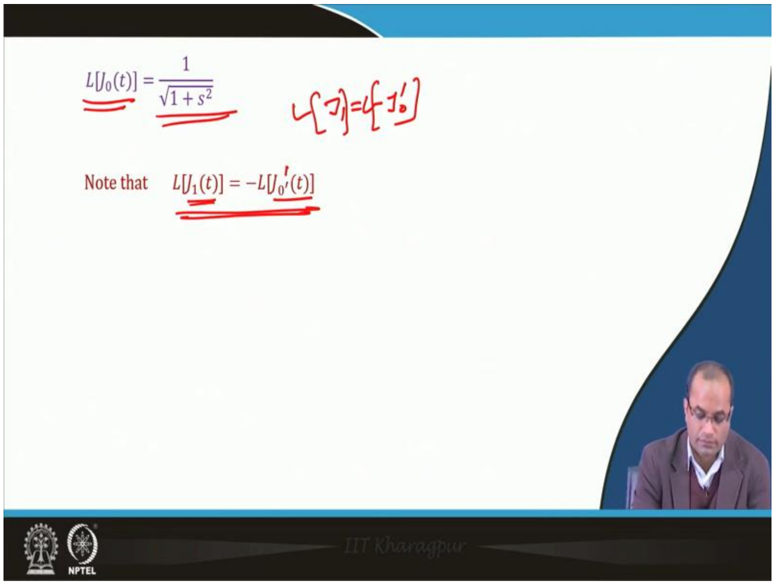
So we will now find the Laplace transform of J0 t and J1 t. So the Laplace transform of J0 t that is the Laplace transform we have 1 minus t square by 2 square then we have t 4 2 square 4 square, we have minus t 6 2 square 4 square 6 square that was the definition of the J0 t. And now we will apply this Laplace transform this term by term and, so we will get the Laplace transform of 1 which is 1 over s, the Laplace transform of t square which is factorial 2 over s cube the Laplace transform of t 4 so factorial 4 over s 5 and the Laplace transform of t 6 which is factorial 6 over s power 6.

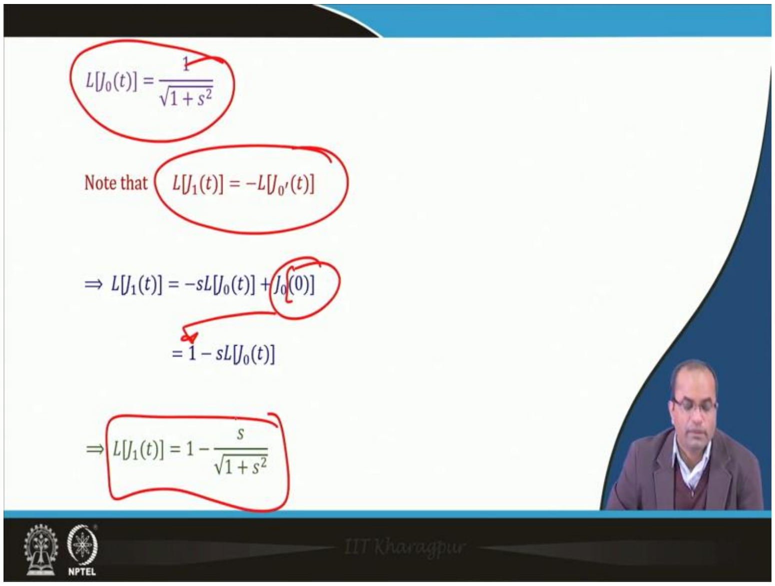
And then we will just rewrite so if we take common this 1 over s, we will get 1 minus this 1 by 2 1 by s square because its factorial 2 will get will cancel this square term. So we have 1 by 2 1 over s square. Here in this case we will get so here we have the 4 3 and 2, so 1 2 will cancel this square the 4 will cancel this square and we have this 3 so we have 1 by 2 we have 3 by 4 and then we have 1 by s power 4.

Similarly here we will get 1 by 2 3 by 4 and this 5 by 6 and 1 over s power 6. So taking note on this expansion binomial expansion what we have 1 plus x power minus n is 1 minus nx and then we have n n plus 1 over factorial 2 x square then we have n n plus 1 over n plus 2 and then we have here 1 product 2 product 3 and then we have here x cube. So taking note on this if we try to look at this and take this n is equal to half, so we will precisely get this because here we have this half it is matching there.

So when n is half so n is half, we have 1 minus this 1 by 2 n is half then half and then 3 by 2, so we will get precisely this one. Here also we have n so half we have 3 by 2 and we will get also here 5 by 2. So that can be a now written this term can be written as that 1 over s and this everything is 1 plus 1 over s square and power this minus half. So more compact form, so it is still we can simplify this we have s square plus 1 over s square and this s square will go up with s, so s s gets cenacle and we will get juts 1 over 1 plus s square and the square root of it.

(Refer Slide Time: 22:01)



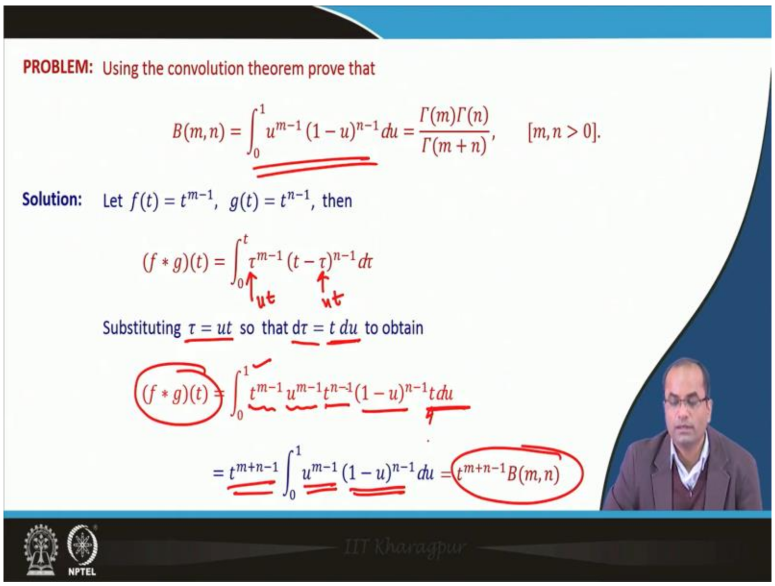


So having this the Laplace transform of J0 t is 1 over 1 plus s square square root t and we know that the Laplace transform so we know that J1 is equal to minus J1 is equal to minus J not derivative. So we take the Laplace on both the sides here, the Laplace both the sides so we are getting this Laplace of J not this hat the derivative t. And then we have the Laplace of J1 t is equal to use the derivative theorem here so it is a derivative.

And minus s Laplace of J0 t and plus J0 0. And then so this Laplace of this J0 will be 1 minus s and the Laplace of J0 t that is going to be 1 over 1 plus s square, so that we will put it, because J0 is going to be 1 over this square root 1 plus s square. So we can compute from there when t is 0 that means J0 0 that is going to be 1 only.

And then here s the Laplace transform of J0 t that is coming from here so you can substitute now. So we have 1 minus s over square root 1 plus s square that is the Laplace transform of J1. So given the Laplace transform of J0 t we can apply this derivative theorem and noting down that J0 0 is 1 so we can get that the Laplace transform of J1 t is equal to 1 minus s over s square plus 1 square root.

(Refer Slide Time: 23:59)



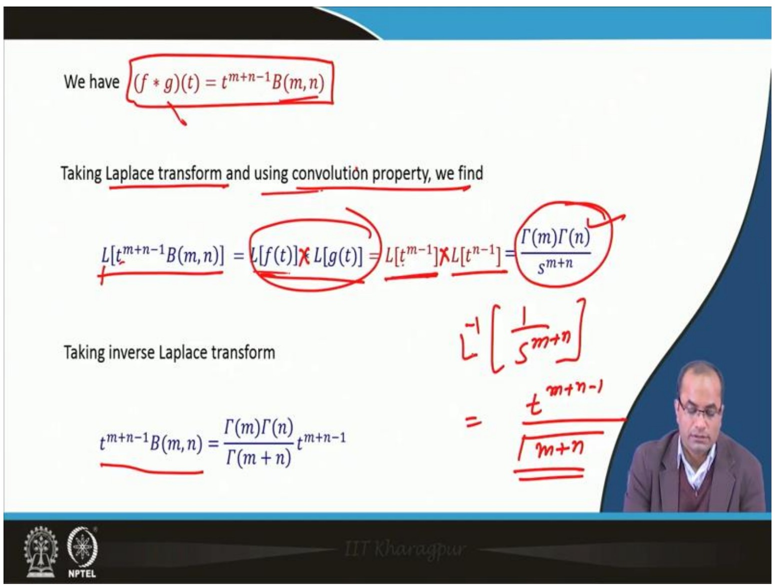
So using the convolution theorem, we will prove now that this beta mn which is defined as 0 to 1 u power m minus 1 1 minus u power n minus 1 du is given as gamma m gamma n divided by gamma m plus n. So this result we have already proved in integral calculus and now we will see it is much easier using this convolution theorem and the Laplace of the beta function we can get this relation easily.

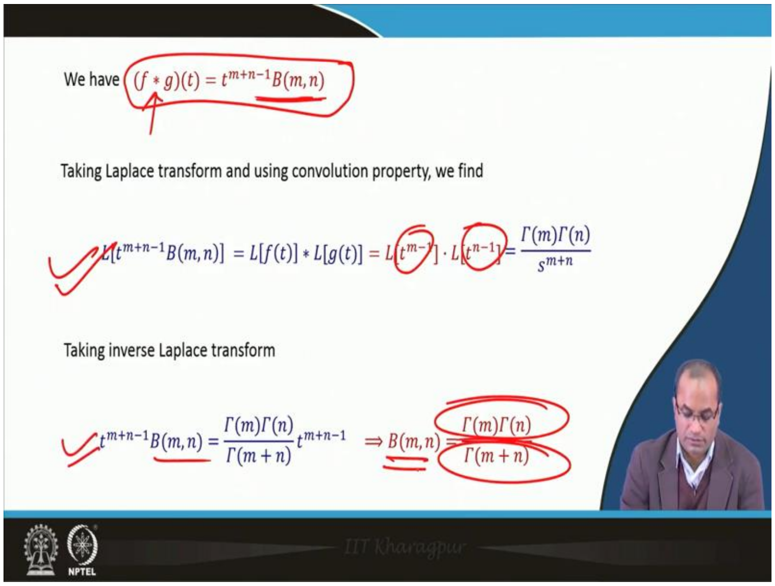
So if we assume this ft is equal to t power m minus 1 and gt we take t power n minus 1, so we defined these two functions. And then, consider their product their convolution product so 0 to t and then we have tau power m minus 1 t minus tau power m minus 1 t minus tau power n minus 1 d tau. So this convolution is defined in this way and then we substitute here tau is equal to ut. So this tau ut and here also ut.

So if we define this tau is equal to ut and d tau will be t times du. So our limits will be again because when t is 0 we have 0 and then t is this tau is t so u will be 1. So we have 0 to 1 the limits of this new integral tau is replaced by ut, so we have t power m minus 1 u power m minus 1 from there also because we can take this t power n minus 1 common and then we will get 1 minus u power n minus 1 and d tau will be t times du.

So this t we can collect we have m minus 1 and n minus 1 and then we have t there so m plus n minus 1 that is t. And we have 0 to 1 u power m minus 1 1 minus u power n minus 1 du which is the beta function. So this product the convolution of these two functions f and g is coming as t power m plus n minus 1 and the beta function.

(Refer Slide Time: 26:17)





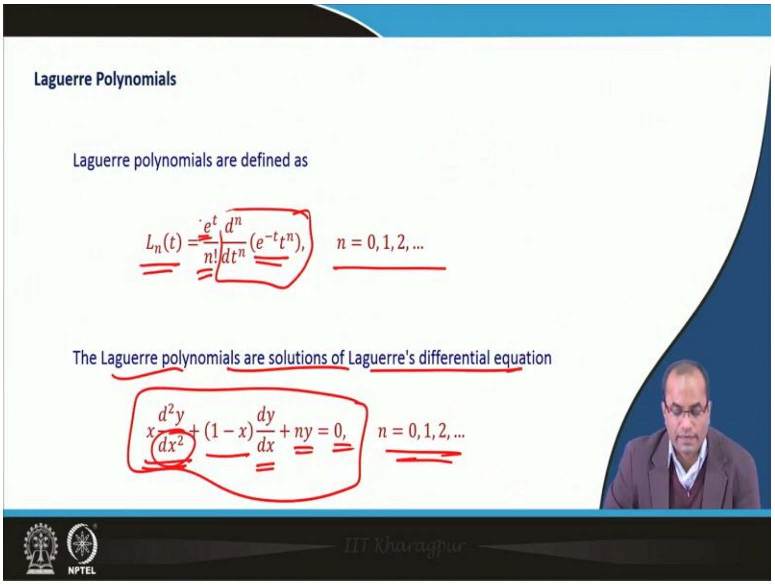
So having this result that the convolution of these two functions gives us the beta function with this t power m plus n minus 1. We can make use of the Laplace transform and in particular the convolution property of Laplace transform, which we have discussed before, that means the Laplace transform of this t power m plus n minus 1 B m is equal to the Laplace of this convolution and that is a convolution property that the Laplace of the convolution there will be equal to the product will be equal to the product of the Laplace of f and g.

So having this product, so L we have t power m minus 1 the Laplace of t power n minus 1 and we have the product there. So this Laplace of t power m minus 1 we know already say as gamma m over s power m. And for t power n minus 1 also we know gamma n over s power n that means we have here gamma m gamma n over s power m plus n and here we have the Laplace. So now if we take the inverse Laplace transform what will happen?

We will have here t power m plus n minus 1 and this beta mn the Bessel's function mn, and then here we have the right hand side, this is gamma m gamma n and the Laplace inverse of 1 over Laplace inverse of 1 over s m plus n, which is t power m plus n minus 1 over the gamma m plus n. So this gives us now that this beta mn is equal to gamma mn divided by gamma m plus n pus 1.

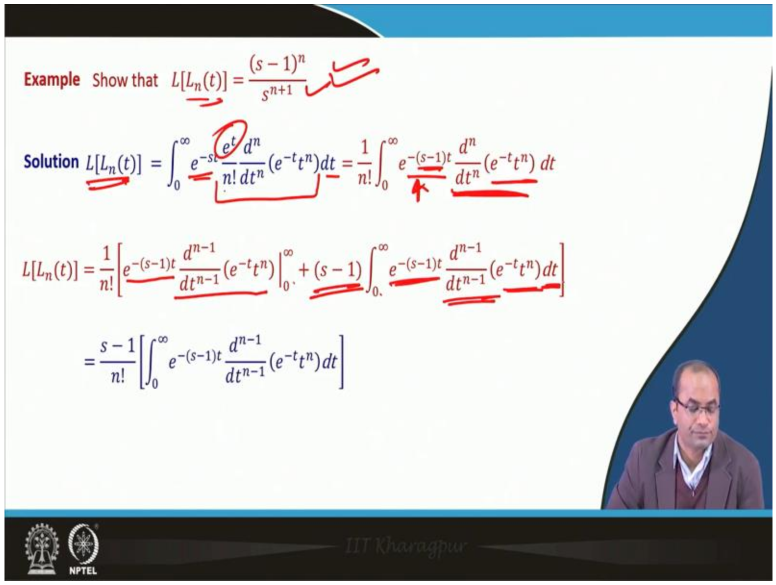
So considering these two functions t power m minus 1 and t power n minus 1, we have establish this relation that this convolution product is equal to this beta function with the multiplication of this t power m plus n minus 1. Then we take the Laplace transform and then the inverse Laplace transform and we readily gap this beta mn in terms of the gamma mn.

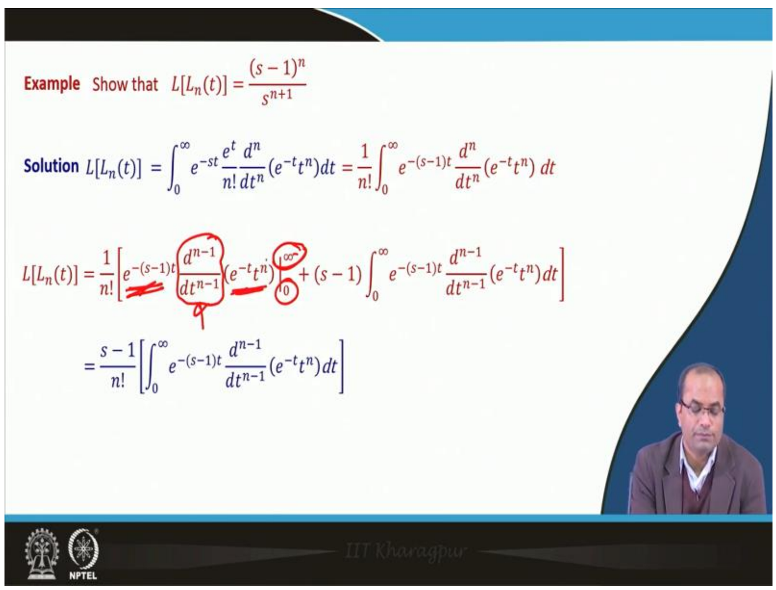
(Refer Slide Time: 28:40)

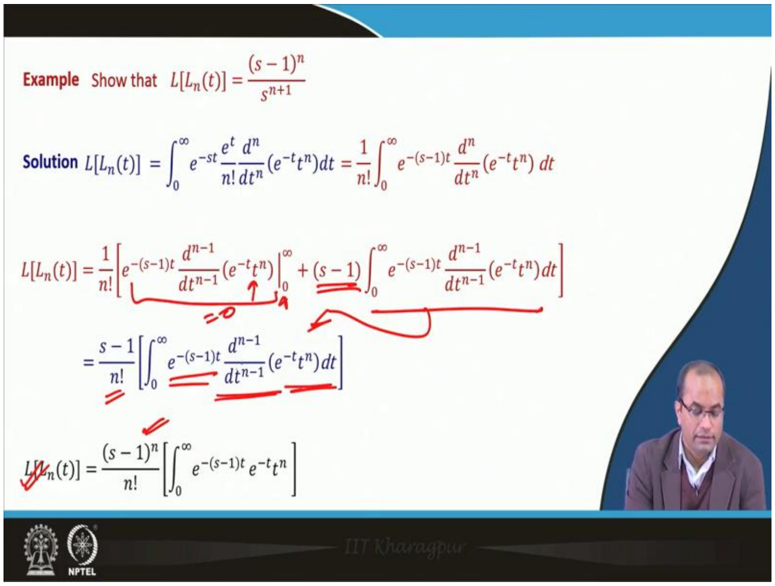


The last one we have the Laguerre Polynomials which are defined as Ln t e power t over factorial n and the derivative n the nth order derivative of e power minus t t power n n is 0, 1, 2, and so on. The Laguerre polynomials are the solutions of this Laguerre differential equations given in this form x d 2 y dx square 1 minus x dy and ny is equal to 0. So this is the Laguerre differential equation and solution will be given this in the form of Laguerre polynomial. So here it is written in terms of x, so naturally this Ln can be also written in terms of x.

(Refer Slide Time: 29:30)







Here we will show that the Laplace transform of these Ln t is nothing but s minus 1 power n over s power n plus 1. So if we consider with the definition so we have e power minus st. Here these are the Laguerre polynomial and we have to integrate this with respect to t. So one over factorial n that we can take out and then exponential we have e power minus st and e power t this one. So we have merged these two, and then the nth order derivative of e power minus t t power n.

And now we can integrate this by parts that means, e power minus s minus 1 t as it is and we will integrate this one. So one order will be reduced here then the limit 0 to infinity. Now we will differentiate this there so s minus 1 factor will come with negative sign which will make this sign positive here.

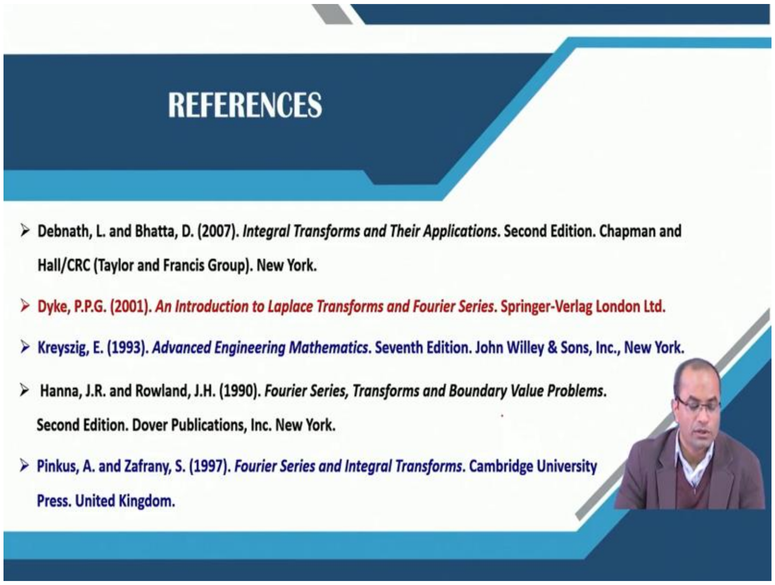
0 to infinity e power minus s minus 1 t then we have n minus 1th with e power minus t t power n and then dt. This first term if we see here when t goes to infinity because of this exponential term this will vanish, and when t goes to 0 because we have this n minus 1th derivative of this e power minus t t power n. So in each term when we differentiate this the t will survive and when t goes to 0 this will be 0.

So this because of infinite the infinity because of this exponential term this term will vanish and because of this t power n and we are talking about n minus 1th order derivative. So in each term t will appear and when t goes to 0 this will go to 0. So the whole term this will go to 0, so we will remain with s minus 1 over factorial n and then the integral here which 0 to infinity e power minus s minus 1 t we have n minus 1th order derivative of this e power minus t t power n dt.

So the Laplace of this will be s minus 1 power n and because we have to now continue this process of integrating so every time we will have a reduction here till we got a this derivative power 0 that means, no derivative. So here parallelly this power will increase so we will have this n times differentiation or this process. So we will get here power n and then we have these term without the derivative there.

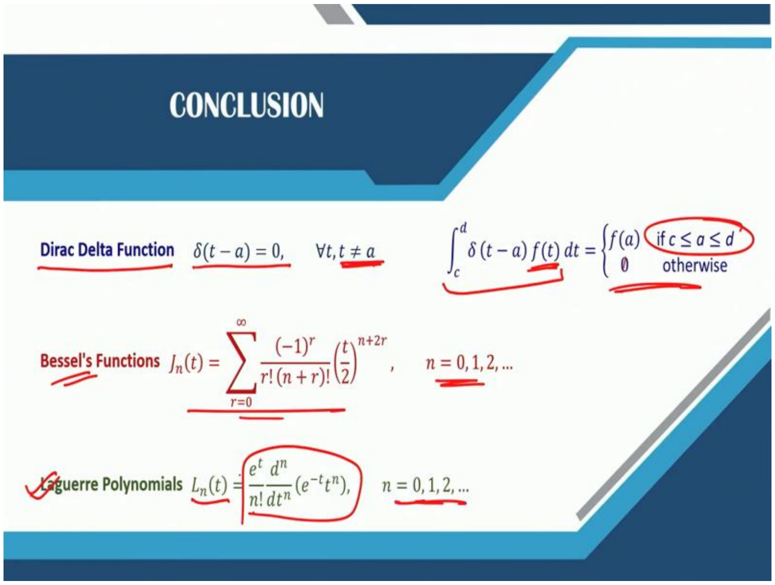
Which can be written now the s power s minus 1 power n over factorial n, and here this is e power minus s t. Because this is minus t there and plus t will cancel out, so e power minus s t and t power n which is the Laplace transform of t power n and which we know the result that is factorial n over s power n plus 1. And then we can combine or we can cancel these two so we got this s minus 1 power n over s power n plus 1.

(Refer Slide Time: 32:54)



So these are the references we have use for preparing this lecture.

(Refer Slide Time: 33:58)



And just to conclude, so in this lecture we have considered the Dirac Delta Function which was defined as that the delta t minus a is equal to 0 whenever t is not equal to a. and this integral t delta t minus a f t dt is always equal to f a and 0 if this a lies between c and d otherwise it is 0.

The Bessel's function we have discuss and we have computed its Laplace transform for the case when J is 0 and 1. We have also discuss this Laguerre Polynomial and Ln t so here also we have computed its Laplace transform for this general case and that is all for this lecture. And I thank you for your attention.