Engineering Mathematics-II
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Lecture 56
Properties of Laplace Transform continue

So welcome back to lectures on Engineering Mathematics 2, so this is lecture number 56 and we will continue with the properties of Laplace transform.

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So in particular in this lecture we will be talking about the convolution theorem very important concept which is used for getting the Laplace Transform of or inverse Laplace transform of various functions. And then, some Limiting theorems they are also important for instance while solving the differential equations.

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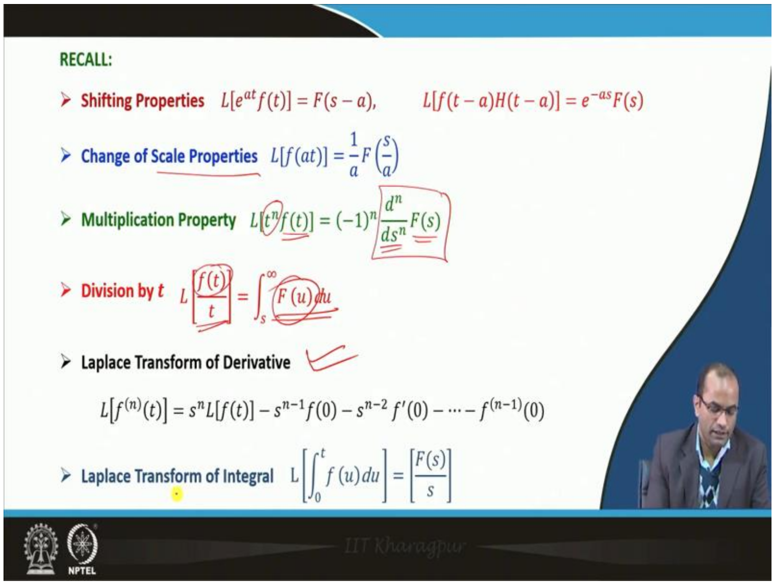

So just to recall what we have done we have gone through several properties including the shifting properties where these two versions of the properties were discussed. And we have also gone through the change of scale property, the multiplication property, so when we multiply by t power n to the function then we can just get by differentiating the Laplace transform of this ft function.

And the division by t property, so if we know the Laplace transform of ft then we can also get the Laplace transform of this ft divided by t by just integrating that Laplace transform. Also the Laplace transform of derivatives was discussed and finally we ended up with in the last lecture the Laplace transform of integral.

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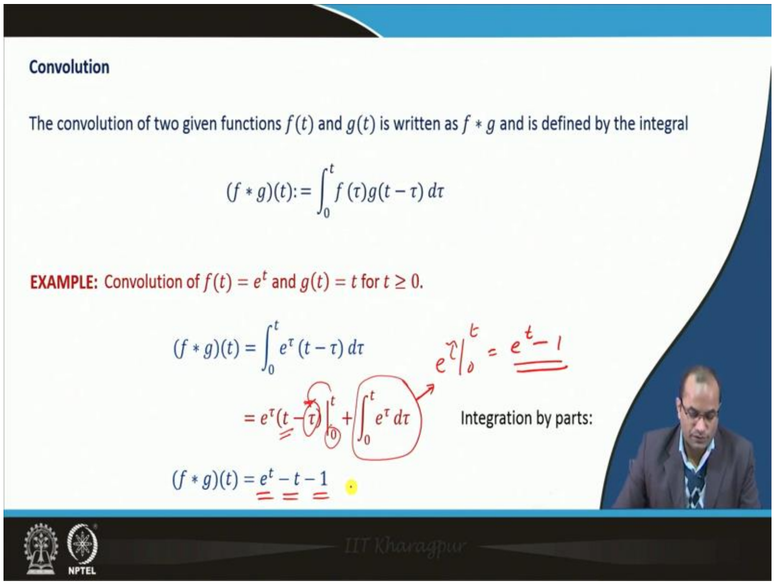

So today we will start slightly different properties that is one of them is the convolution property. So the convolution of two given functions ft and the gt is written as f star g that is the notation for the convolution. We have used similar concept while discussing the Fourier transform as well but the limit was taken from 0 to infinity. And we are considering this Laplace convolution integral as 0 to t f tau and g t minus tau so there is a shift here t minus tau.

And it is integrated over d tau. So the example we have for instance we want to get the convolution of this exponential t and the function t. So we can use this definition to compute this convolution integral. That means we need to get e power tau and then for the second function we have shifted here t minus tau. And we will see just in the next slide that there is a property that f star is g is equal to g start f that means that does not matter whether we shift in this g function or we make a shift in this f function.

The value of the convolution integral will remain the same. So here we have shifted in the second function g that is t minus tau and then we can integrate by parts because we have e power tau and this is another function. So there are two functions this we will take as first and this is second function. So this will be integrated and this one will be differentiated. So here by doing this integration by parts we have this t minus tau as it is.

The integral of this e power tau is again e power tau and then the limits 0 to t plus this 0 to t and e power tau and this d tau. So then we can substitute the limits of this tau here so if we put first the t there this will be 0. And when we put the 0 there so it will be e power 0 that means you get t from there. And again here we have e power tau and then the limit 0 to t which will be e power t and minus 1 and there will minus t from there. So we will get e power t minus t and then minus 1 as the value this convolution product or this integral f star g.

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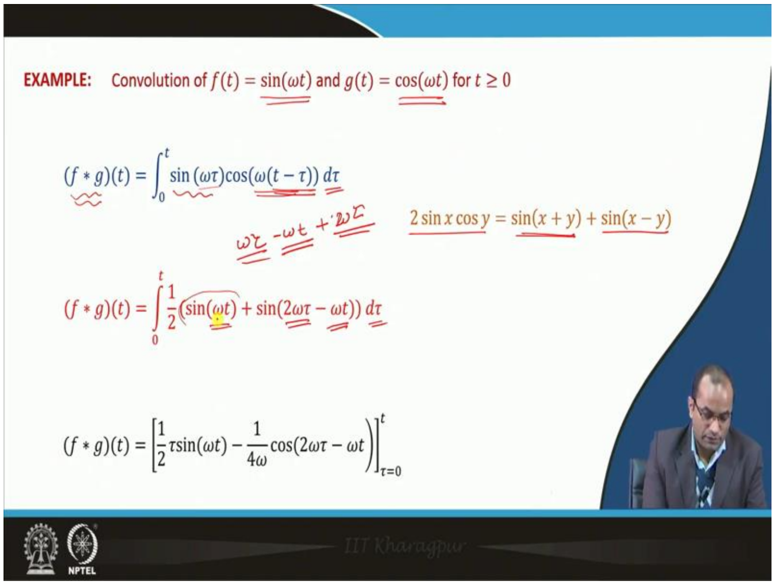

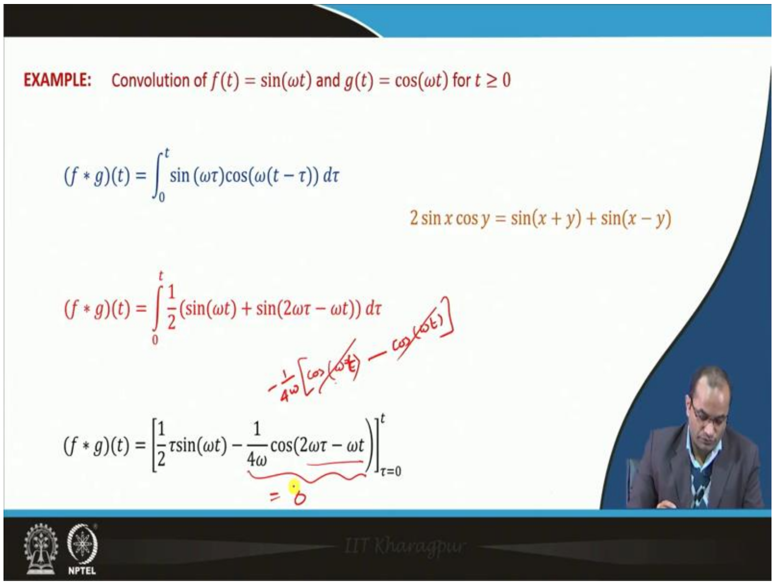

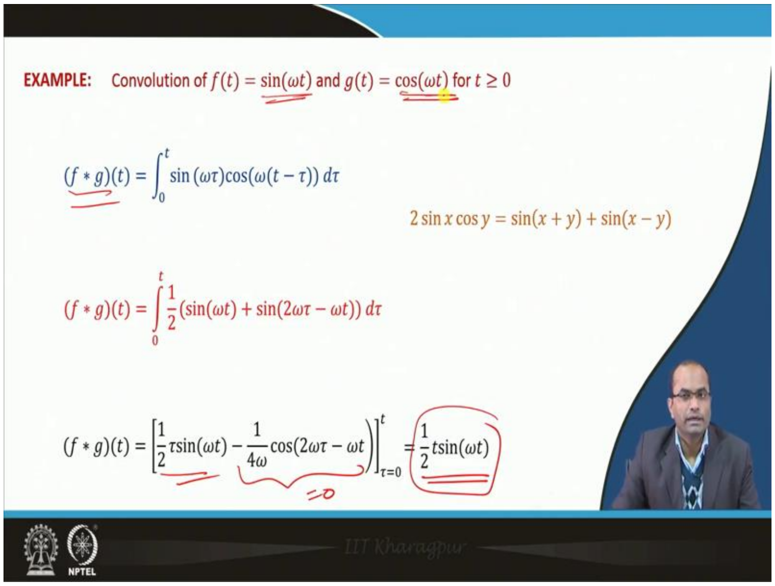

Another example where we will consider here sine function and the cosine function so what will be the convolution of these two functions. Sin omega t and cos omega t, so f star g again this product which is defined as sin omega t and cos omega and there is shift here in the argument. So we have replaced t by t minus tau and then this has to be integrated over d tau. We know this trigonometric identity that 2 sin x cos y is sin x plus y and sin x minus y.

That we can use here because we have sin x and then if we treat this cos y so we can do that. So half and we will get here sin the sum of the two so this omega tau will get cancell we have omega t and then the difference. So we have omega tau minus omega t and then minus minus again here this will plus so 2 sorry omega tau. So this omega tau omega tau will become 2 omega tau and minus omega t and then we have this d tau.

Which we can integrate easily because here we have this sin omega t we are integrating with respect to tau. So that will be treated as constant so we will get tau there and the upper limit t and minus this 0 will introduce here t there. So that will be substituted in a minute and then from here also we can integrate with minus sign this will be cos then 2 omega tau minus omega t and this 1 2 omega will come here in the denominator and this 2 there.

So therefore this has become 4 omega and then this limits for tau from 0 to t. So when we put t there this will t sin omega t and then 0 so this will be 0. And similarly here we can substitute this 2 omega t and then 2 omega 0. But what we should note that while discussing this limit the first term will be like cos omega tau when we replace this tau by t. So 2 omega t minus omega t will be omega t. So we have the first term cos omega t then the minus term so front this is sitting here 4 omega minus, when we put tau 0 this is cos 0 we have cos omega t with minus.

But cos will not read this minus so it is cos minus x is equal to cos x. So here again we have cos omega t and they will cancel so there will be o contribution from the second term and we will have only the first term there. That means the value will be half t sin omega t. So that is the convolution product of these two functions where we have sin omega t and cos omega t.

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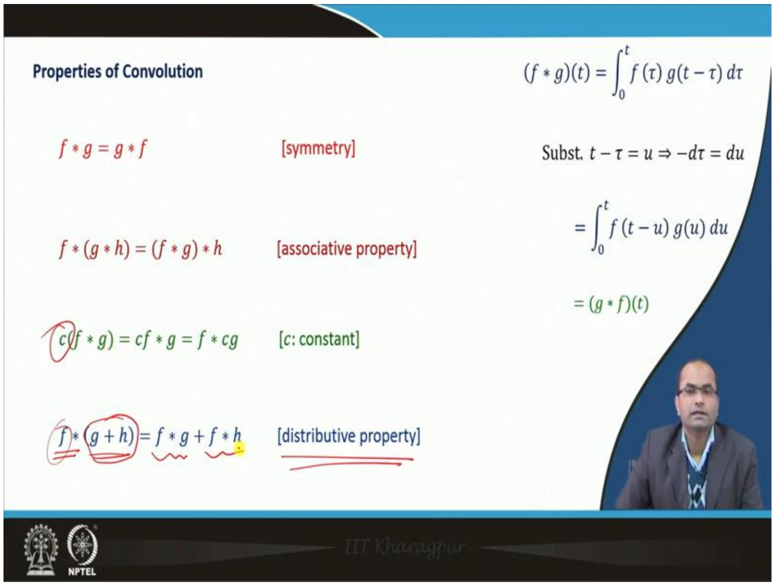

There are some interesting properties of the convolution some of them we will list here. So for instance the symmetry property that means f star g is equals to g star f and one can observe all these properties just by the property of the integral. So we have to take this f star g for instance just the prove we will go through for the first case and the others are actual trivial.

So f star g is 0 to f tau and g t minus tau dt and if we substitute this t minus tau as new variable u that means minus d tau as tu. So the limits this t tau is 0 u will become t so this limit here will be t and then the limit there when tau is t that u will become 0. So, we will have a limit t to 0 but since there is a sin here with du as minus d tau so that will be again changed from 0 to t. So the limit 0 to t we have f and this tau we have t minus u and g t minus tau that is again u so du.

So this is the integral we have this f star g, and this is equal to g star f in the sense considering this definition. So what we have seen here that f star g, this integral is equal to this g star f. So there is a symmetry and it does not matter whether we make a shift in f or we make a shift in g.

The value of the integral will remain intact. Another property we have the so called associative property that means if we make a convolution product of f with this another convolution g star h and then the convolution with f.

So in this order that first g and h will be convoluted and then its product with f convolution product with f that will be equal to that we can do this first with f and g and then later on with h this will be also equal. So similarly to the property the symmetry property we have proved here one can also easily prove this property. And then there is another kind of associative property that if there is a ca constant that first we here perform the convolution product of f and g and then we multiply this by this constant c.

Or first we multiply c to f and then make convolution with g or other way round that the c is multiplied to g and then we can convolute with f. This is going to be same because the c will actually not play nay role. So this is a constant and then we have this convolution integral so that will come out of the integral always in each case and finally we will end up with like c the product with f and g.

So this distributive property also holds in this case that f star g plus h so we have three functions again so g plus h we do and then we make a product with this f or we can do this f star g and then plus f star h. So in either situation if this will be equal to the left hand side. So these are some of the properties which we can use directly because they are not also difficult to prove because of this integration.

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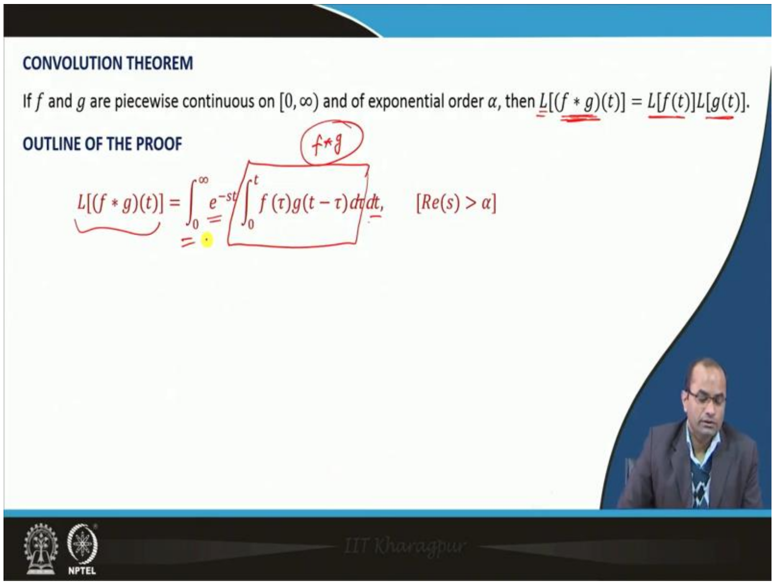

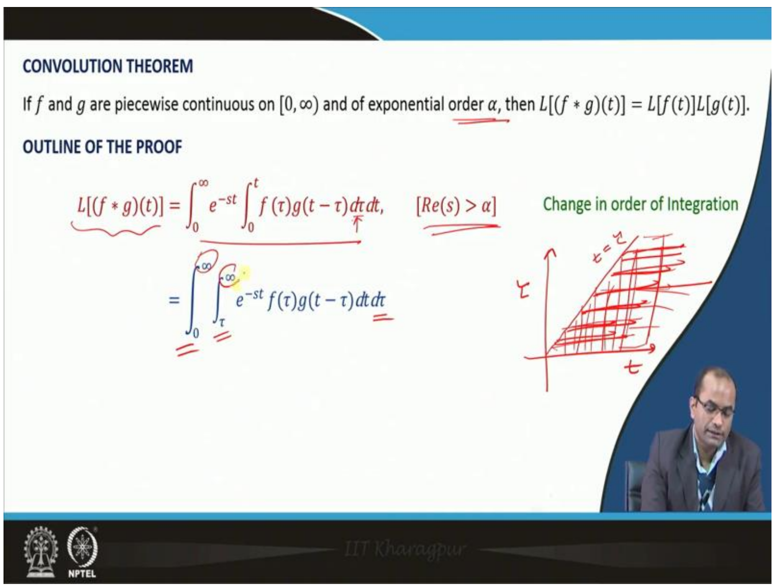

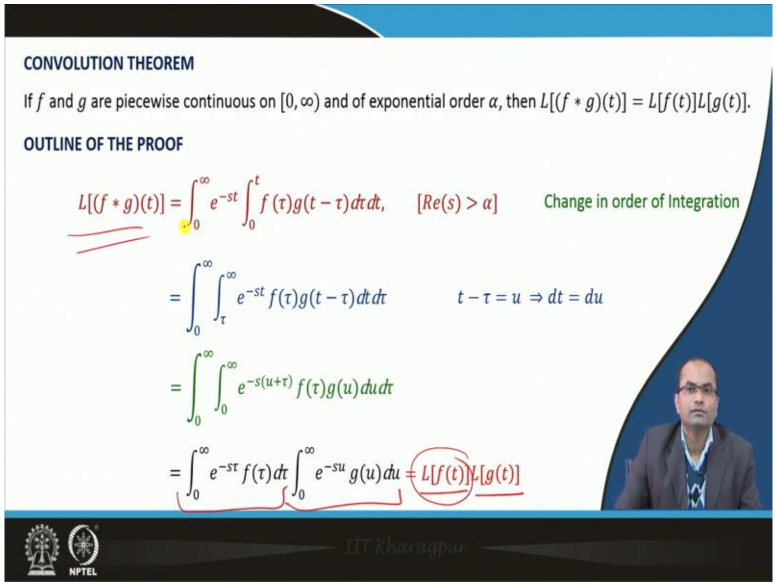

The convolution theorem which is one of the important results in this section and this says that if f and g are piecewise continuous on 0 to t of exponential order alpha so these are the sufficient conditions for the existence of the Laplace transform. And then it says that the Laplace of the convolution f star g is equal to the product the simple product of this l the Laplace transform of f product of the Laplace transform of g.

So the convolution is removed here and that is exactly the use of this convolution theorem that when we apply the Laplace transform to this convolution integral that becomes the Laplace of f into Laplace of g. Just to give some outline of the prove here we will start with this Laplace of f star g that is by the definition because this is exactly the convolution here f star g and this f star g we have taken the Laplace transform.

So e power minus st and dt will come with this integral 0 to infinity. So this is as per the definition we have the Laplace of f star g that is given by this integral and here it is given that f and g are of exponential order alpha. So naturally this will exist for all s greater than alpha and now if we change the order of integration here. So what it says the area of integration that if we assume this is t axis and this tau axis so this is the line t is equal to tau line.

So the integral here says that the t is varying from 0 to infinity and tau varies from 0 to tau. So this so this is the area of integration in this situation and he want to change it now the order of integration that means the first with respect to t and then with respect to tau. So first we have to fix let us say with respect to t that means this t will vary than this tau to infinity and the tau will vary from because that will be the outside limit so 0 to infinity.

So what we will have the limits of tau will be 0 to infinity now 0 to infinity at most will go to infinity and here we have for the other for t that will go always from this tau to again infinity. So these are the new limits and now what we can just observe here that if we take the substitution here t minus tau as u that for the inner integral.

So dt is du and we have the new integral now 0 to infinity e power minus s t here is u plus tau and then if we have f tau we have gu du and then d tau. So we can separate it into two integrals because we have e power minus su and e power minus st. So e power minus st with tau sorry so f tau d tau and e power minus su will go with gu so e power minus su gu and then we have this du.

So this is Laplace of this ft and the Laplace of gt these are the two separate integrals now. So it is a simple product of Laplace transform of f and the Laplace transform of g. So this is the convolution theorem which we will see the applications for getting for instance the inverse Laplace transform of various functions.

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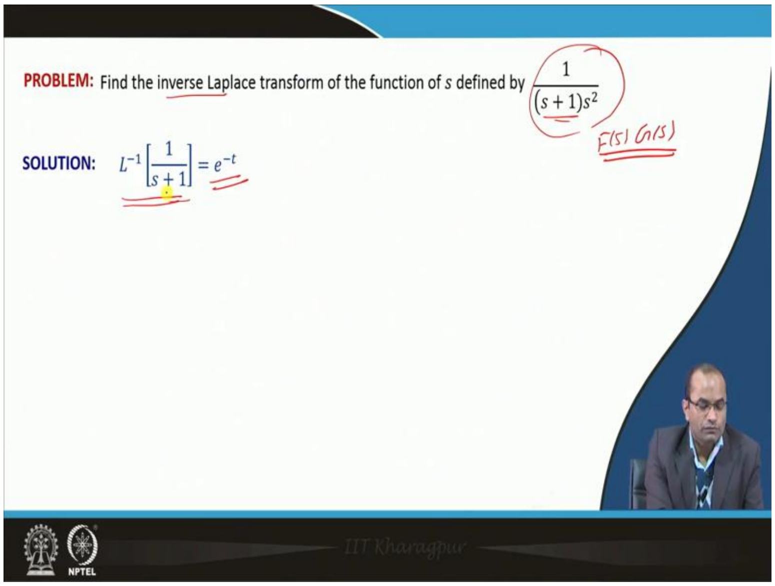

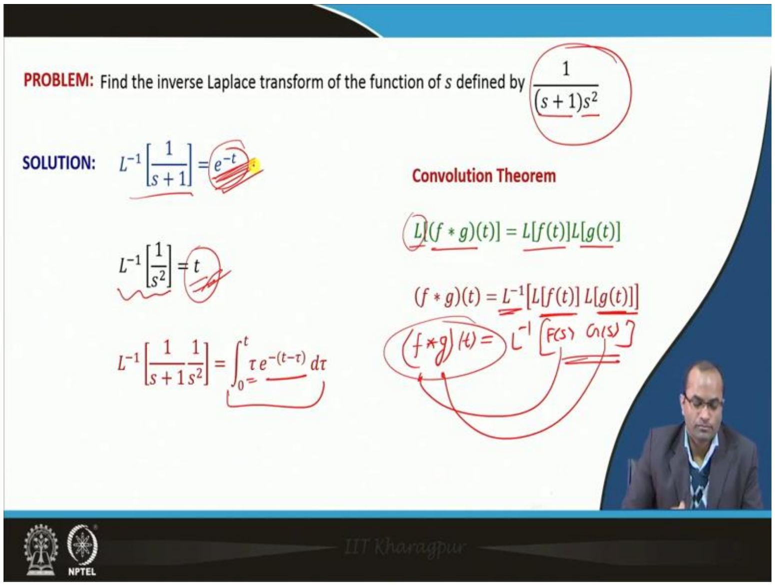

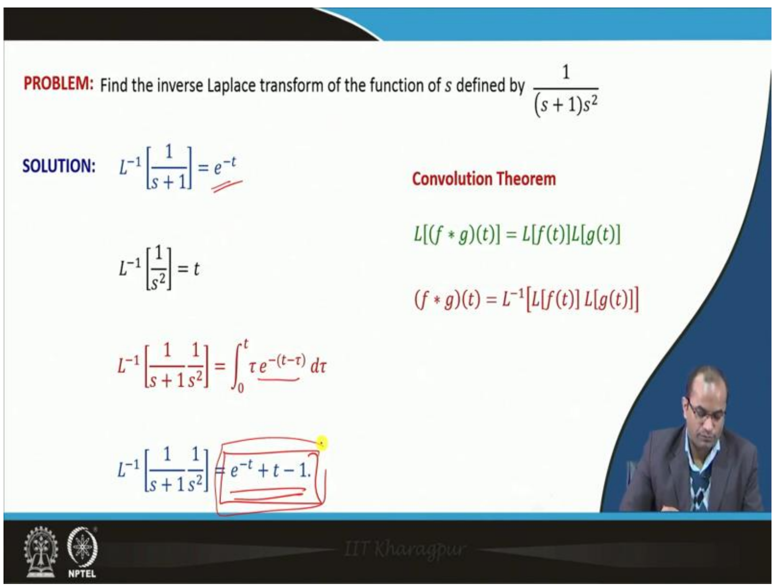

Like here so if you find the inverse Laplace transform of this function 1 over s plus 1 into this s square so what we have the Laplace inverse we know for 1 over s plus e power minus t. So there are two ways now to go with either we can go with the partial fractions and then we can get the inverse which we have already done before. The another approach we very often is simpler than getting this partial fractions would be this convolution theorem.

So that means we have the product of two functions here fs and let us say gs this is given as the product of two functions. The two functions are 1 over s plus 1 the other one is 1 over s square and we know the Laplace inverse of both of them because they are very elementary functions. So the Laplace of 1 over s plus 1 will be e power minus t and the Laplace inverse of 1 over 1 plus s square will be t. So we know the Laplace inverse of these two.

And now we can apply this convolution theorem which says the Laplace of f is star g is the product of these two Laplace transform. Or in other way round we can read this that the Laplace inverse of these two, so let us say this is Fs and Gs. So the Laplace inverse of this product Fs Gs would be equal to the simple product of these two functions whose inverse are taken here.

So Fs inverse is ft Gs inverse is gt and the this L inverse of the product is simply the convolution product. So here we have the product of two functions whose Laplace inverse inverses are known to us e power minus t and here t. So we can apply this theorem the convolution theorem now, and results says that the Laplace inverse given by this one over s plus 1 s square will be simply the product of this f and g.

So this is the product of f and g 0 to t then here we have taken tau and shift we have taken in exponential function we can do the way round as well. And then here e power minus t we can take out and we have this tau e power minus tau which can be integrated. And we have done this similar problem before where we have just evaluated the convolution of such two functions exponential t and t, here we have exponential minus t. So we the result will be also similar to what we have obtained before that is e power minus t plus t and then minus 1.

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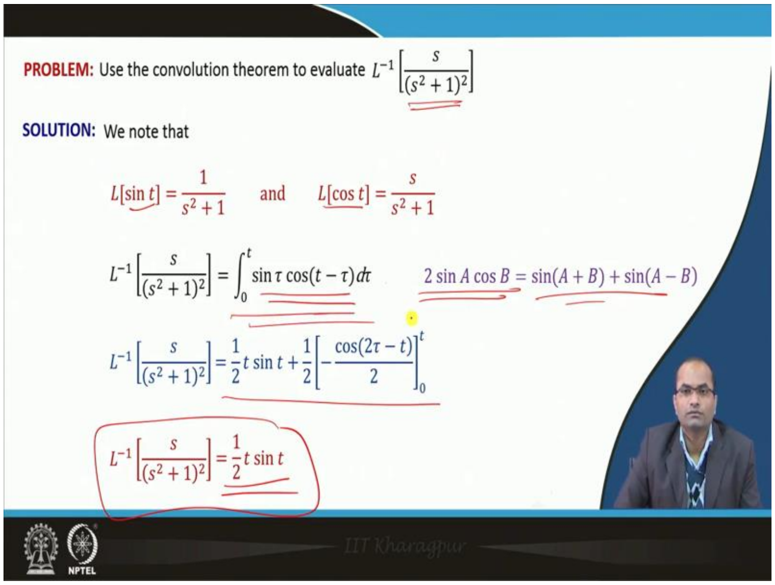

If you want to apply this convolution theorem to evaluate this here, s over s square plus 1 whole square. Again either we break into the partial fractions or we do with the help of this convolution theorem. So again here also the convolution theorem will be useful because we know that the Laplace transform was sin t is 1 over s square plus 1 which is seated here and then other function is multiplied by there s over s square plus 1.

So we know that this product here is nothing but the Laplace of sin t Laplace of cos t. And then we can apply the convolution theorem which says that the value of this inverse would be equal to the convolution of these functions sin t and cos t. And this convolution indeed for less slightly more general functions sin omega t and cos omega t we have done before, so we can quickly go through this now.

So we apply this 2 sin a cos b formula here again can and then we need to integrate this and finally what we will obtain that will be just half t and sin t. So that is the inverse here which with the help of this convolution theorem we have evaluated if we do the partial fractions so they are the first and often time taking step would be just to do the partial fractions. So therefore here we do not have to do the partial fractions only we need to evaluate this convolution integral. So it may be easier in many cases.

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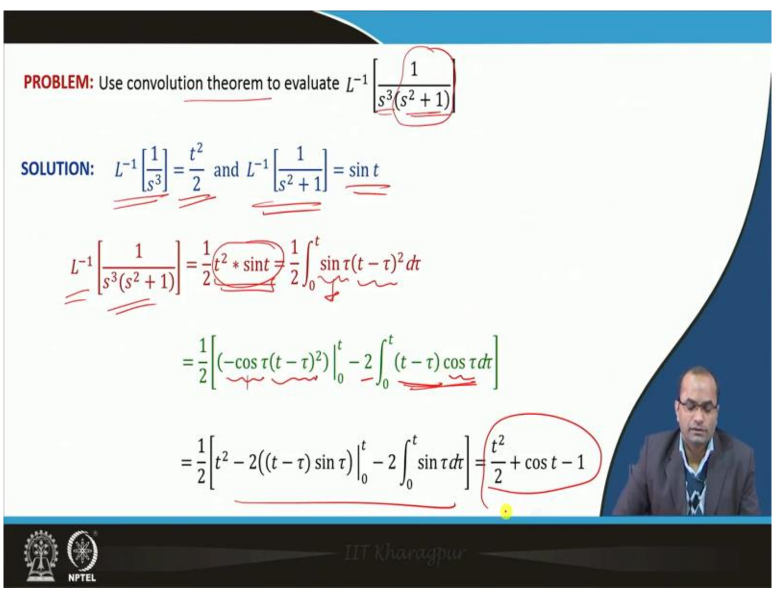

Using convolution theorem obtain 1 over s cube and s square plus 1 for example. So we have L inverse 1 over s cube whose Laplace inverse is t square by 2. And the another function here is 1 over s square plus 1, whose Laplace inverse would be sin t. So to get the Laplace inverse of the product we will apply the convolution theorem which says that this will be equal to the convolution product of this half t square sin t.

So we need to get this convolution product here t square sin t which as per the formula we have sin tau and t minus tau whole square d tau. And we can do the integration by parts, so here this sin tau will be minus cos tau the second this first function as it is , t minus tau whole square and here it will be differentiated 2 times t minus tau and then this cos tau for the sin tau.

And then we have because the minus will be coming from this minus tau and the one from the this cos, so it will be plus and this will remain minus as it is. And now we can substitute those limits by doing here also once again this pi parts what finally we will get here t square by 2 plus cos t minus 1 as the Laplace inverse of the given function.

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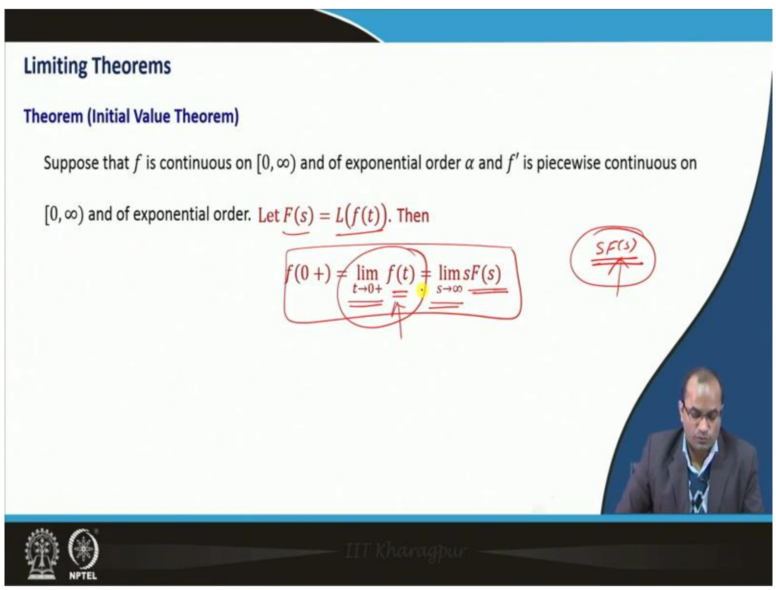

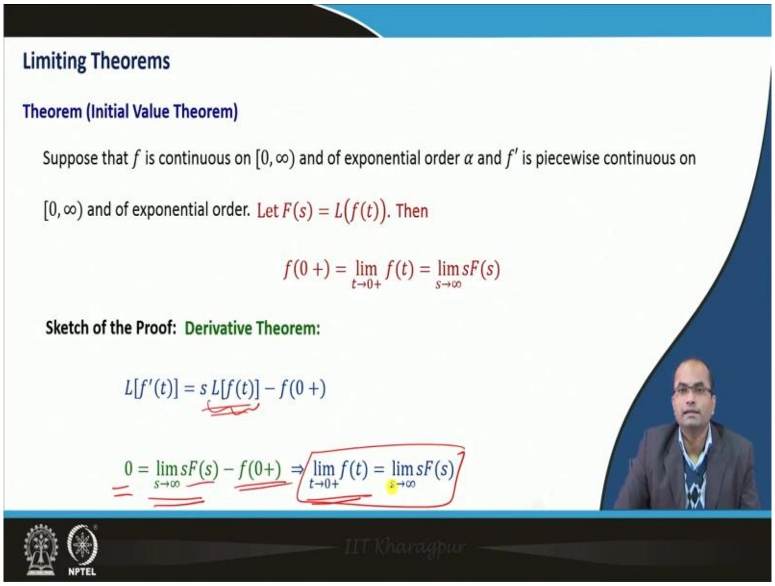

Now we will move to the another property of Laplace transform and these are the limiting theorems, so they are also important to get the behavior of the function as t approaches to 0, t approaches to infinity from the information of the Laplace transform. So these first we will go through the initial value theorem, which says suppose f is continuous on this 0 to infinity and of exponential order alpha moreover it is given that f prime is also piecewise continuous and of exponential order.

So all the Laplace transform of f and f prime exist. So suppose this Fs is Laplace transform of ft then this initial value theorem says that the limit of ft as t approaches to 0, so the limiting value of this ft as t approaches to 0 would be equal to limit s approaches to infinity sFs. So we need to compute the limit for example this sFs and we can get that is exactly the behavior of the function as t approaches to 0.

So this is a direct result that with the help of the given transform given Laplace transform we can also predict or we can get the value of the function as t approaches to 0. So this is so called initial value theorem and just to go through the proof of this is a quick proof with derivative theorem. So we know already the derivative theorem that the Laplace transform of f prime t is s into the Laplace transform ft minus f 0 plus when the function is not continuous for instance at 0.

So we have to take the limit as 0 plus which will always exist because of this assumption here that this f is piecewise continuous normally of exponential order. In this case it is given continuous at 0, also this f0 will this value will be also equal to f0 in this particular case. So then we have using this derivative theorem, the Laplace transform of f prime tsL ft minus this f0 plus.

And if we take the limit here as s approaches to 0 what we will get? We will get Laplace of this f prime t we know already that f prime t is piecewise continuous and of exponential order and when we take this s approaches to infinity, we already discussed this before that the Laplace transform will go to 0.

So here this result follows form the earlier discussion that the Laplace transform of a piecewise continuous and of exponential order functions that goes to 0. So here that result we have used that this is 0 and then we have the limit here s Fs and this is Fs here. So s Fs as goes to infinity minus this f0 plus and this is the desired result that the limit here f this is f0 plus that t goes to 0 plus ft is equal to limit s goes to infinity and s Fs.

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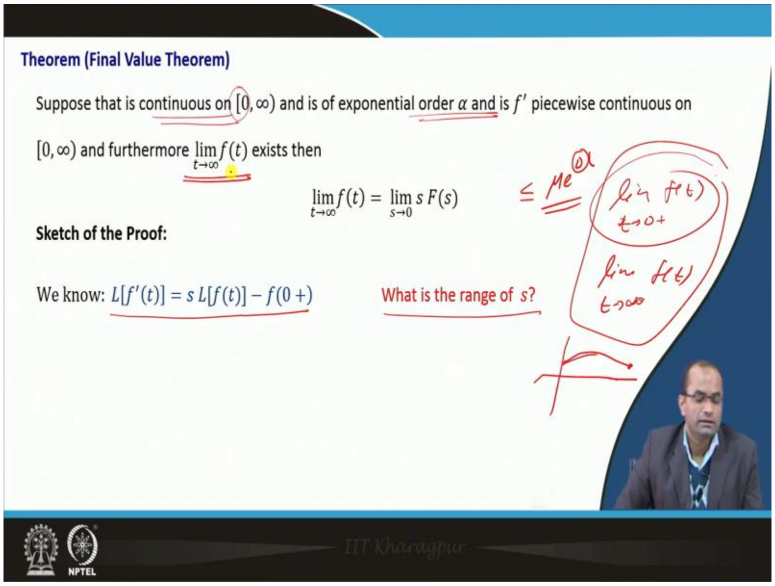

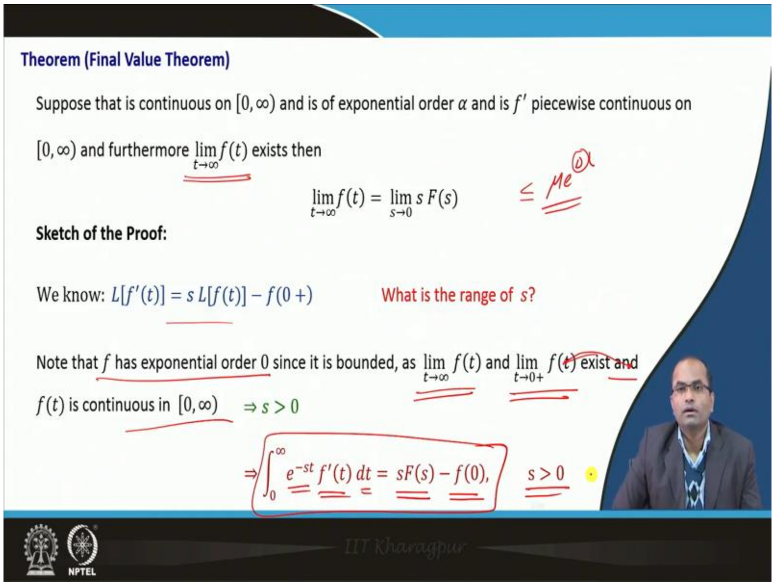

Coming to the final value theorem, there we have suppose that the function is continuous that f is continuous on this 0 to infinity of exponential order and f prime is piecewise continuous. And there is an additional condition that ft as t approaches to infinity that also exists. So in that case, the result is that when we take this s Fs again and get the limit s goes to 0 that will exactly be the value an ft t goes to infinity and the existence of this we have assumed already.

In that case this is the result of the final value theorem. Again going to the proof, so we know this theorem which is a the derivative theorem. And now we must take a look that what will be the range of s where all these Laplace will be valid. So here it is already given that this f is a for instance the exponential order alpha but what we should notice that this exponential order will be actually 0.

Because the function is continuous including at 0 or if it is even piecewise continuous, in that case also the limit as at t goes to 0 plus will exist. So this limit exist and more over the limit as t infinity ft that also exist and the function is continuous. So it will remain from 0 we know that the limit exist or the value exist and then as t approaches to infinity also this is finite.

And then the function is continuous in between, so it is actually a bounded function and once we know that it is a bounded function, it can be bounded by this Me power 0 t that means the order will be just 0 for this function because it is given that the limit e tends to infinity ft exists.

So actually the result here is valid for all s, the f is exponential order 0, since it is bounded as I discussed that this limit and this two limits say as t approaches to 0 and t approaches to infinity they exists, and ft is continuous. So it will be bounded, that means the exponential order 0. So hence this, expression here that the integral e power minus st f prime t dt this is equal to s Fs minus f0. This exists for all s positive.

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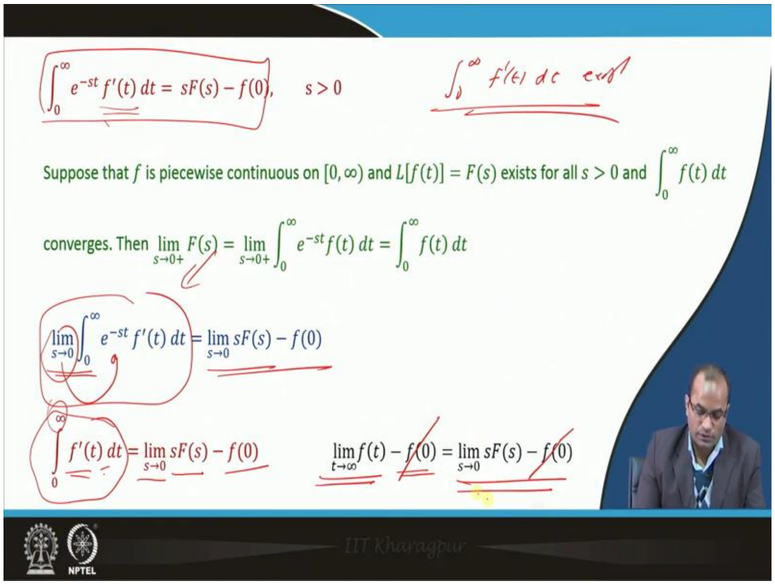

And then having this we can apply this result which is also a part of this a discussion here. So suppose this f is piecewise continuous indeed on this interval and we have this ft is equal to Fs exist and moreover this integral converges then we can actually because this is valid for all s positive we can take this limit there. S approaches to 0 plus Fs is equal to limit s 0 plus of this is just the Fs here. And this is equal to because this limit we can take.

So the results says that this limit we can take inside provided these conditions hold then this e power 0 will become 1 and this value will be just 0 to infinity ft dt. So in our case also this result is valid for all s positive and all other conditions are also valid if we assume that the integral this 0 to infinity and this f prime t dt exists. So having this what we can do now? We can pass the limit here s tending to 0 from the right hand and doing so.

So we have pass the limit both the sides here, and now this theorem the above result says that we can take this inside and apply to this s there. That means we have 0 to infinity f prime t dt, the result is limit s goes to 0 s Fs minus f0 and this one here this f prime that is f. So ft where t goes to infinity and minus f0 again. So ft t goes to infinity minus f0 that is the left hand side this is right hand side this will cancel out and we will get exactly this result here that t approaches to infinity ft is equal to s Fs and s goes to 0.

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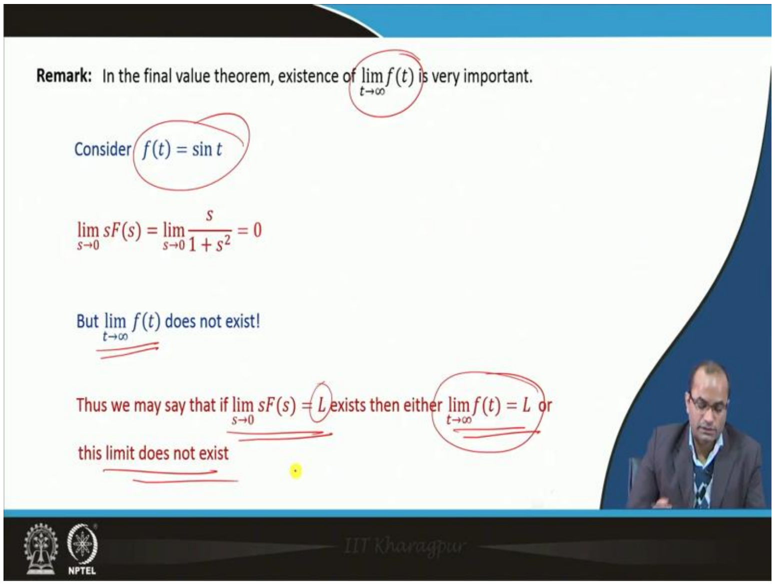

Just a remark that in this final value theorem the existence of this is very important that ft t approaches to infinity, why? If we consider this ft as sin t we know that s Fs because the Fs the Laplace transform of sin t is 1 over 1 plus s square. So this is equal to 0, if we take this limit here and in this way s Fs exists as s goes to 0 and we may conclude that the limit ft t tends to infinity exists, but here it does not exists.

So this final value theorem has importance when this exists then only the limit sFs s goes to 0 is equal to this limit t tends to infinity ft. Otherwise it has no meaning. So thus we can say that this limit is equal to L then either this t tends to infinity ft exists and this will be equal to L, or this limit does not exists. So if we find out that the limit s Fs exists then there will be two cases that either that limit itself this t approaches to infinity ft is equal to this. Or if it is not the case then limit does not exists in that case.

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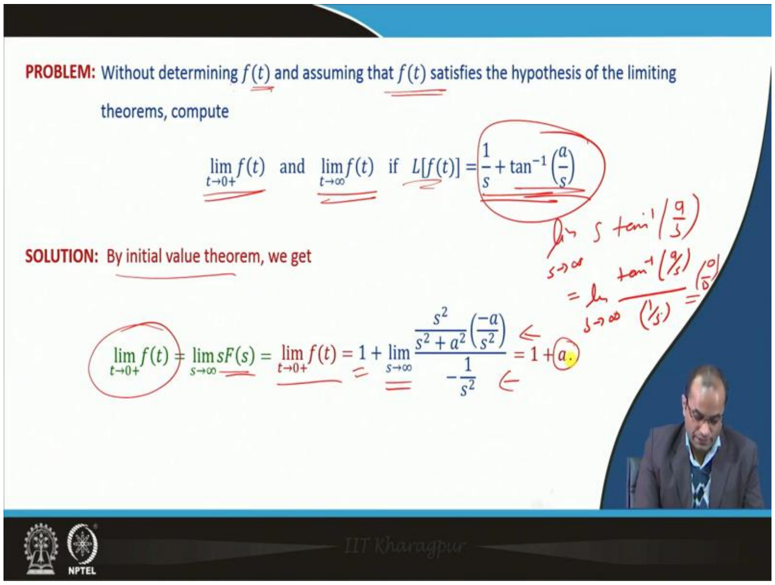

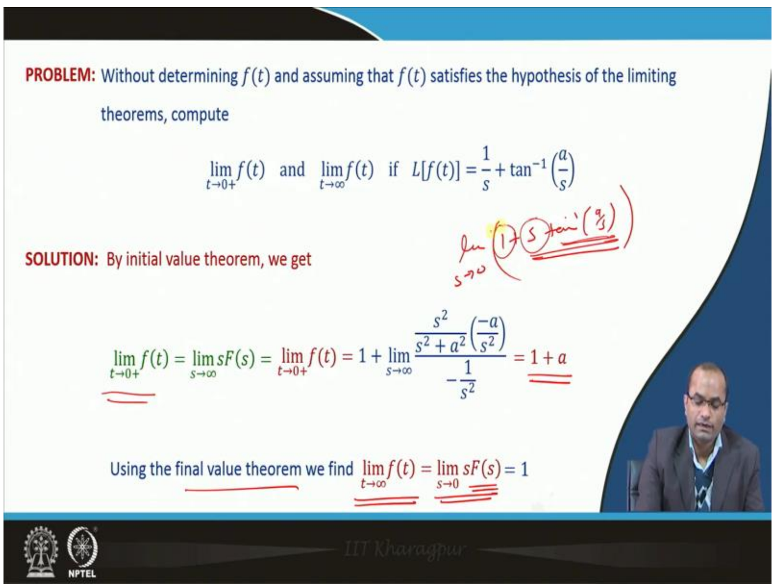

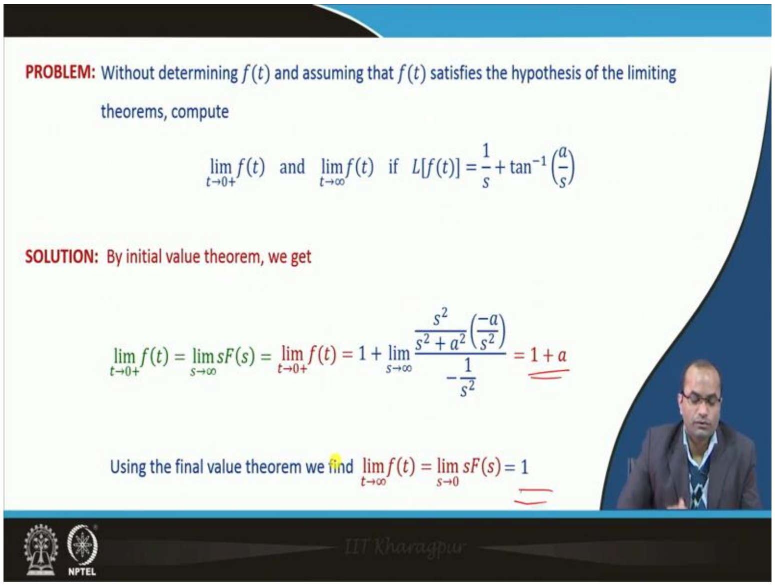

Just a problem that without determining this ft and assuming that ft satisfies all these hypothesis of the limiting theorems we want to compute that what is the limiting value as t approaches to 0 and t approaches to infinity given this Laplace transform. So we can apply directly the limiting theorems and by initial value theorem we can get this limiting value as t approaches to 0 of this s Fs and this will be 1 plus so sFs if we multiply this s here so we will have 1 and then here also we will multiply the s there and we will observe so this s and the tan inverse this a over s and we have to find the limit as s approaches to infinity.

So this can be written as the limit s approaches to infinity tan inverse a over s and 1 over s. What we observe now when s approaches to infinity, this is 0 by 0 form. So tan inverse 0 is 0 and 1 over s as s approaches to infinity is also 0. So we have 0 by 0 form we can apply the LHopital rule, which is done so the numerator is differentiated denominator is also differentiated and then where this limit when we take s infinity, you will get only a there.

So this value is 1 plus a of this limiting value of ft as t approaches to 0 plus. Similarly using the final value theorem we can say that the limit ft t approaches to infinity will be limit s approaches to 0 sFs and the value will be 1, because here s Fs will be 1 plus this s and tan inverse a over s and then the limit here s approaches to 0. So the second term will vanish when s approaches to 0, 0 into something finite that will be 0, and we will get just the value 1. So we can apply these theorems for such functions for instance.

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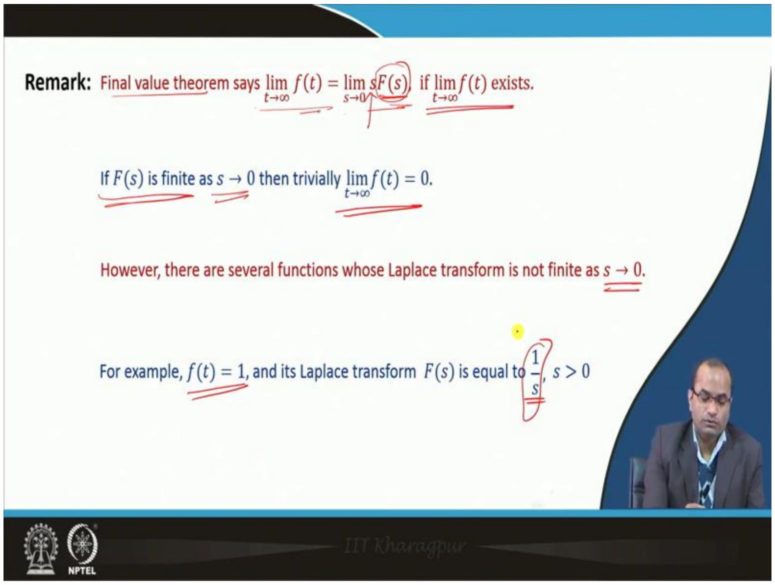

Just a final remark that the final value theorem says that the limit t approaches to infinity ft is limit s approaches to 0 s Fs, if this limit exist. If Fs is finite what we should note that here if Fs is finite and s approaches to 0 trivially this is 0, because s approaches to 0 and Fs is finite so trivially this is 0, but why the why this has some meaning because there are many functions whose Laplace transform is not finite as s approaches to 0.

For instance, if you take ft is equal to 1 itself whose Laplace transform is 1 over s, so it is not finite as s approaches to 0. So therefore this s Fs limit s goes 0 which not misunderstood that this will be 0 most of the time. Know for various functions this is not going to be 0, for instance, here we have 1 over s and s Fs will be just 1 and s goes to 0, so the value will be 1 which is indeed the function in this case.

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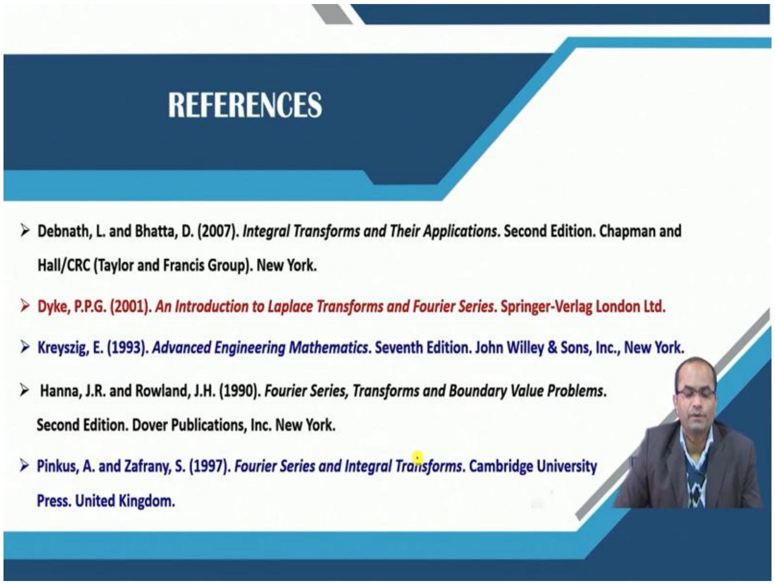

So these are the references we have used for preparing this lecture.

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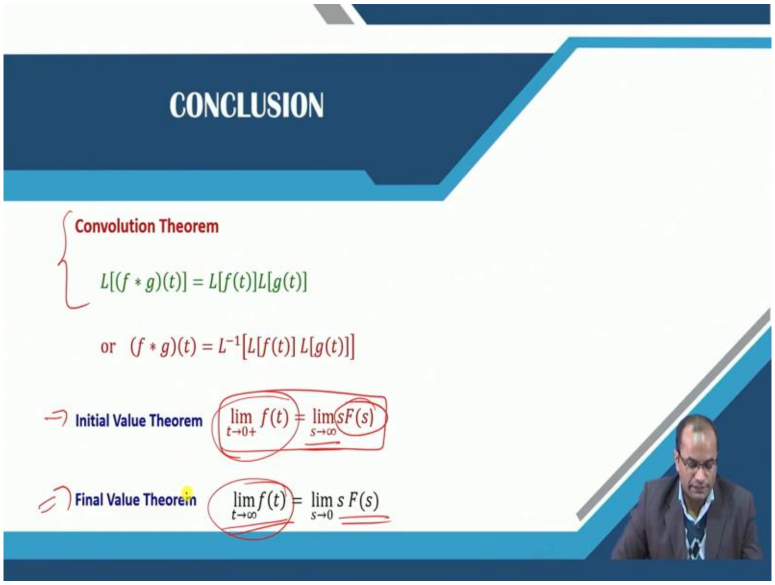

And just to conclude so we have discussed to convolution theorem and some of its applications for evaluating the inverse Laplace transform. And we have also discussed the initial value theorem which says that just computing this limit s Fs as s approaches to infinity we can predict the behavior of the function as t approaches to 0.

And the final value theorem says that by computing this s Fs and taking limit as s goes to 0, we can get this ft as t approaches to infinity. Well, so these are some of the important results which will be applied to various problems in applications. And that is all for this lecture, I thank you very much for your attention.