Engineering Mathematics-II
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Lecture -55
Properties of Laplace Transform (Contd.)

So welcome back. This is lecture number 55 on Properties of Laplace Transform and in continuation to the earlier lecture we will be discussing some more properties of the Laplace Transform.

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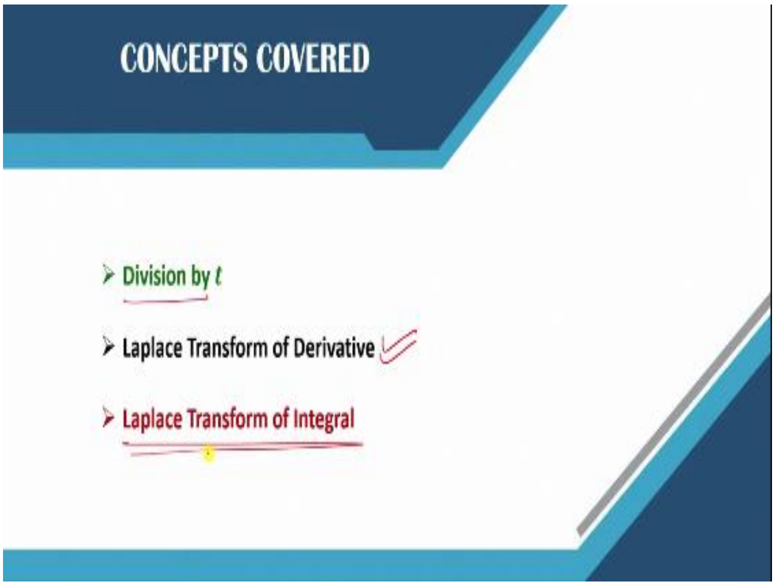

Mainly we will discuss this division by t that if Laplace Transform what t is given that what will be the Laplace Transform of ft by t and then the Laplace Transform of Derivative that is one of the important result in this chapter, in this lecture and then also the Laplace Transform of integral will be discussed.

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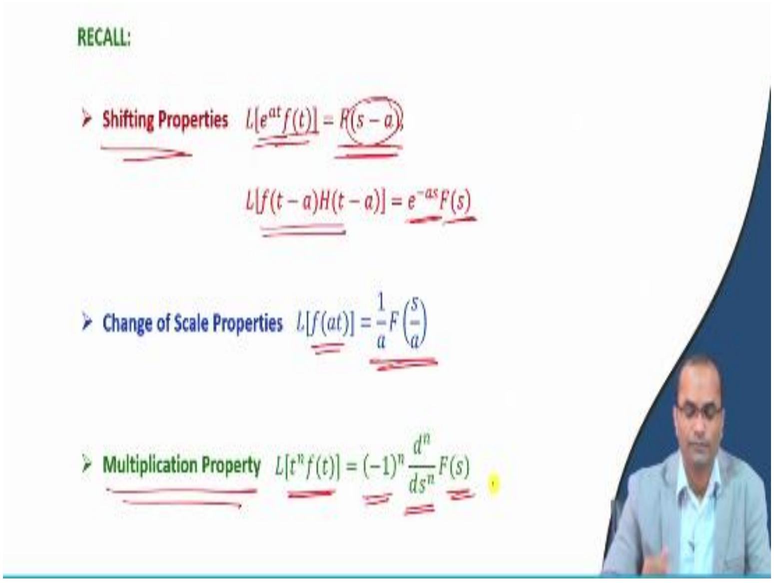

So coming to just to recall what we have covered, what properties we have covered in the previous lecture. So there were basically shifting properties so e power minus at if multiplied by ft then we have F s minus a there is a shift there, and if there is a shift in the function, then e power minus as comes in this transform Laplace Transform. Change of scale property so if Laplace of ft is Fs then f at is 1 over a Fs over a. And then we have also discussed this multiplication property that t power n ft is minus 1 power n and this nth derivative of this Fs.

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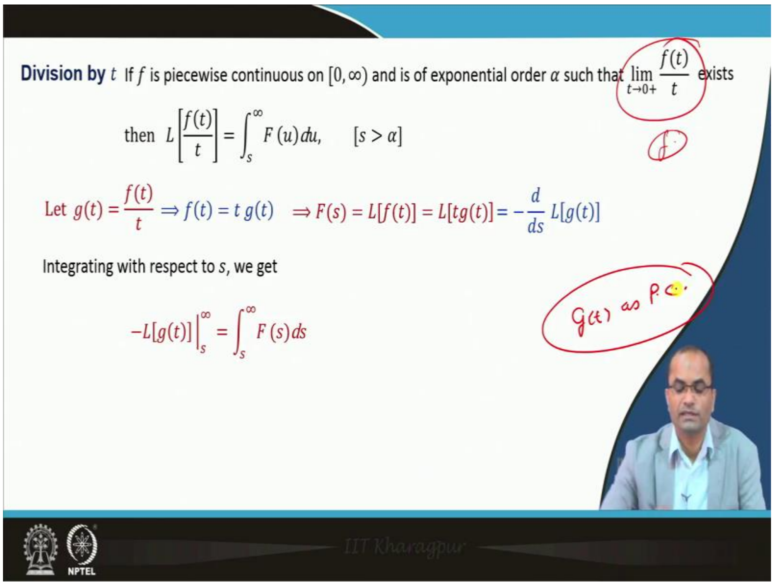

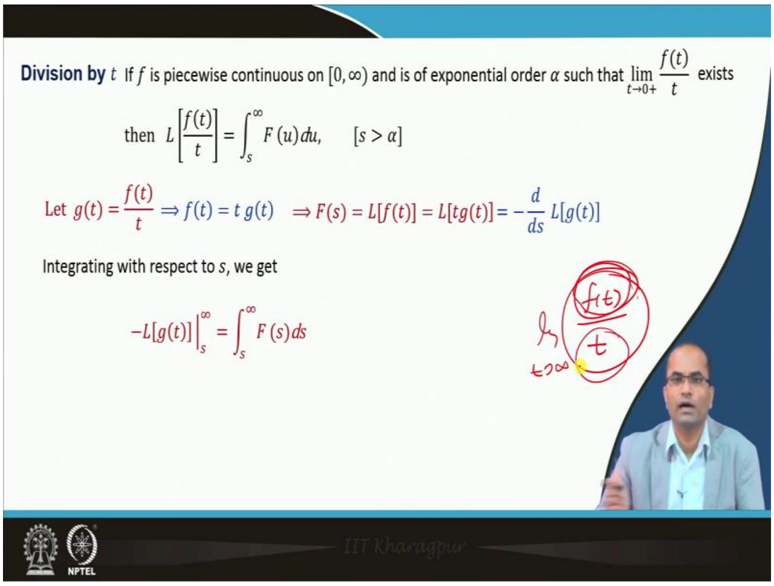

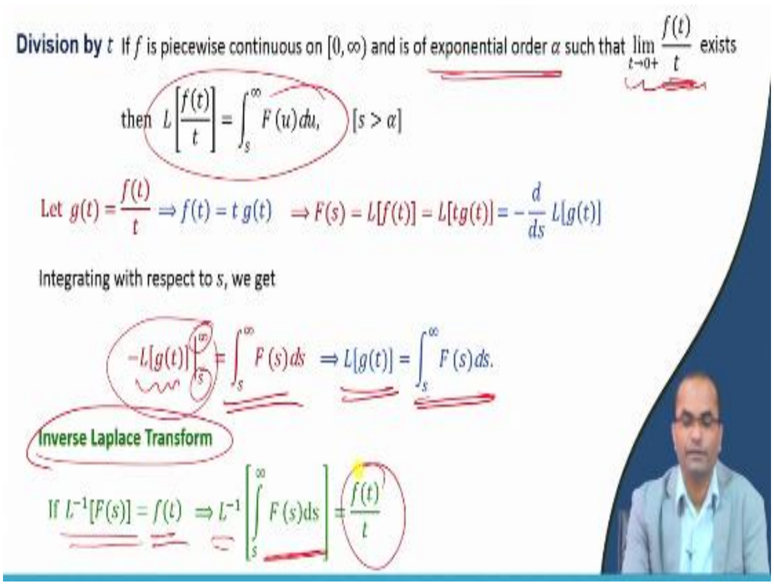

So now we will continue with the division property so what will happen if ft is given then ft by t. So here we assume that the f is piecewise continuous function and is of exponential order alpha and we have to have this additional condition which is very important to say that ft over t st approaches to this from right hand side to 0 this should exist then only we can talk about the Laplace Transform of ft over t.

And the result will be s to infinity Fu du that means when we divide by this t we have to basically integrate this Laplace Transform of f and this result is valid for all s greater than alpha, alpha was the exponential order of f there. Just looking at a proof so we take here gt as ft over t whose Laplace Transform we are going to compute now. So from here we have this relation that ft is equal to t times gt.

And then Fs which is the Laplace Transform of ft and ft is given as this t gt. So the Laplace of t gt and we know this from the previous lecture the property that when we multiply by t it is like minus d over ds the Laplace Transform of gt. Well if we integrate now with respect to s this relation here Fs is equal to minus d over ds Laplace of gt then we will get so this side first. So we will get here the Laplace of gt.

Because this was a derivative and we integrate so this derivative will be removed we have the limit s to infinity and then the right hand side this Fs will be integrated from s to infinity. Now we have to look at these limits here. So we have this Laplace Transform of gt and we are looking at that what will happen here when this s goes to infinity. This is a function of s and we are looking at what will happen when s goes to infinity.

And remember in the result where we discuss the existence of the Laplace Transform we notice that if a given function ft is piecewise continuous function of exponential order, then its Laplace Transform goes to 0 as s goes to infinity. So here we have this function gt now we have to see whether gt is piecewise continuous and it is of exponential order alpha. So concerning the continuity piecewise continuity of gt, so here gt is ft over t.

And f is piecewise continuous so we have the piecewise continuity of t already and t is anyway continuous. So other than this because there is a point here when t approaches to 0 so other than that there is no issue about the piecewise continuity of this quotient here which is we have denoted by gt, but the question is that when t approaches to 0 this limit may not exist that means the limit t approaches to 0 plus this ft over t may not exist.

Because that is not coming automatically from the piecewise continuity of f. The piecewise continuity of f the piecewise continuity of ft says that the limit ft when t approaches to 0 that exist, but here we are dividing by t so this may not exist in general. So therefore we have to say that this limit ft over t exist and this is exactly to make this function gt as piecewise continuous. So the function gt is piecewise continuous because this is a ratio of this function ft divided by t and as t approaches to 0.

We have this additional assumption that ft over t exist. So this is very important for the piecewise continuity of function gt. Regarding the exponential order since f is of exponential order and here we are talking about this division by t and normally we look at this limit t approaches to infinity. So if this ft has this boundedness by the exponential function definitely here we are dividing even by t.

So this ft over t will definitely have the boundedness for large values of t. So the exponential order alpha is not distributed by dividing t and this piecewise continuity is maintained because we have this additional assumption that ft over t limit t approaches to 0 plus exist. So having this gt piecewise continuous function of exponential order this its Laplace Transform will go to 0 as s approaches to infinity.

So the first limit here will be 0 and then when we substitute this s so we have simply the Laplace of gt and the right hand side we have s to infinity Fs ds and this is the result we want to prove. For its counterpart the Inverse Laplace Transform so we say that if L inverse Fs is ft then the L inverse of this integral s to infinity Fs ds will be ft divided by t.

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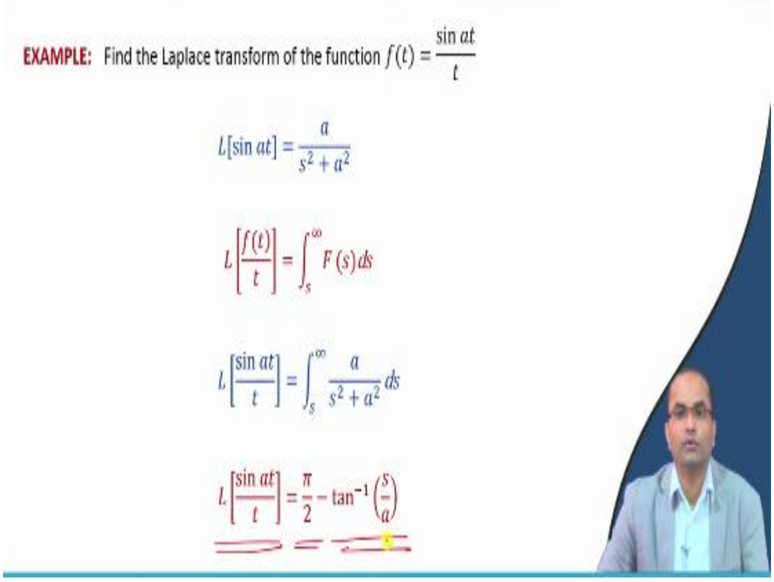

So we will discuss are some examples now. So we want to find for instance the Laplace Transform of this function ft as sin at over t. So the Laplace of sin at we know a over s square plus a square and then we can apply this property of the Laplace Transform that means we have to just integrate this Laplace of sin at that is integral s to infinity a over s square plus a square ds and that means this Laplace of this sin at over t is pi by 2 minus this tan inverse s over a.

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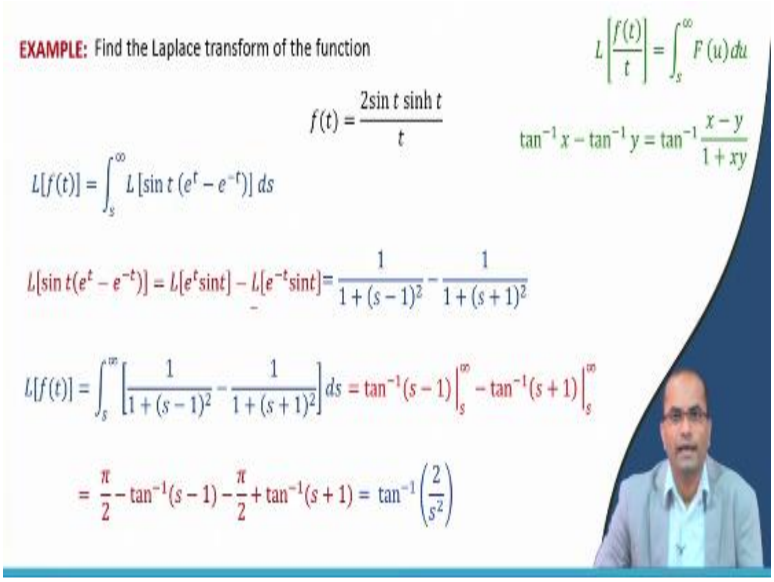

The next example we want to find the Laplace Transform of this function again the similar situation it is being divided by t here. So if we find the Laplace Transform of 2 sin t sin hyperbolic t and then we can apply this property division by t and we can get this Laplace Transform of this given function. So Laplace Transform of this ft by t is s to infinity Fu du. So we have the Laplace Transform of ft Laplace of sin t e power t minus e power minus t ds.

And the Laplace of this sin t e power t minus e power minus t that we can compute as Laplace e power t sin t and minus Laplace of e power minus t sin t. So here both the places we can use these properties. So we have here 1 over 1 plus s square s minus 1 square and then minus 1 over 1 plus s square plus 2 and here we have the Laplace of this ft then which is given already there.

So Laplace of ft will be integral s to infinity the Laplace of sin t and e power t minus this t and this has to be integrated over s which is given here now. So by the direct integration we have this tan inverse s minus 1 and then the limit s to infinity minus this tan inverse s plus 1 the limits again this s to infinity and then we have here pi by 2 and minus this tan inverse s minus 1.

Because when s approaches to infinity this tan inverse infinity will be pi by 2 and we have this tan inverse s minus 1 and then minus again in the same situation (())(09:50) tan inverse this infinity will be pi by 2 and then tan inverse s plus 1. So this pi by 2 and pi by 2 will cancel out and then using this property that tan inverse x minus tan inverse y is equal to tan inverse x minus y over 1 plus xy. We can further simplify to get tan inverse 2 over s square.

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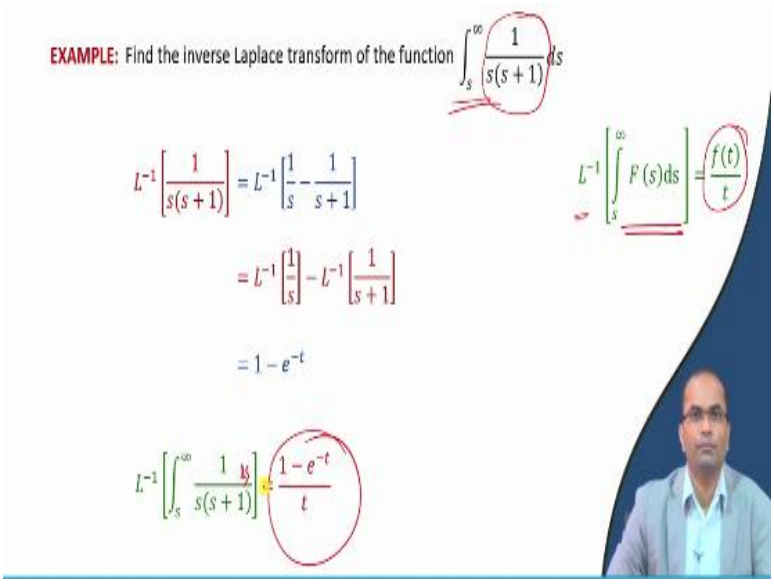

So now the next example we have to find the inverse Laplace Transform of this function s to infinity 1 over s s plus 1 ds that means this L inverse of this integral we know this property so we need this ft over t. So what is ft now that is the Laplace inverse of this integrant 1 over s plus 1. So we have to find the Laplace inverse of 1 over s s plus 1 which can be done using this partial fractions.

So we have 1 over s minus 1 over s and then the Laplace inverse using the linearity property we have here L inverse 1 over s L inverse 1 over s plus 1 which is 1 here and then 1 over s plus 1 is 1 e power minus t. So we have the Laplace inverse of this integrant and then this property says that L inverse for this integral will be just ft divided by t. So the L inverse of this s to infinity 1 over s s plus 1 ds that will be 1 minus e power minus t divided by t.

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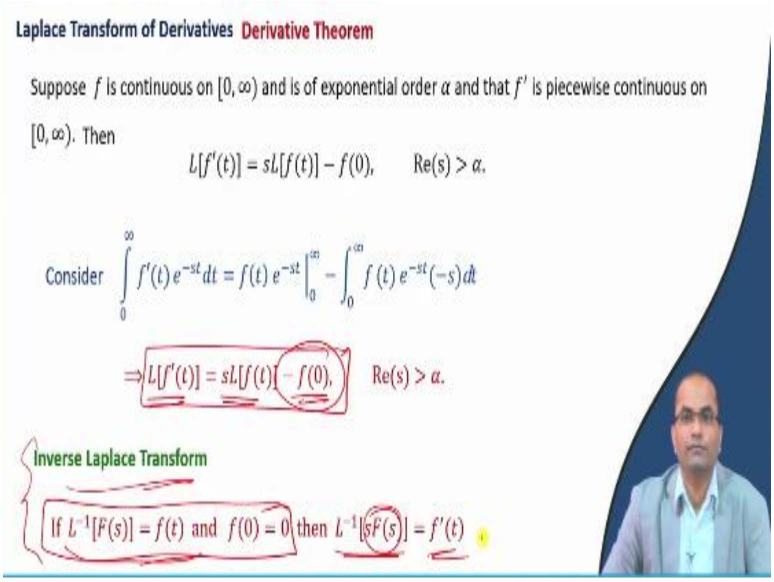

Coming to the Laplace Transform of derivatives so we have this the so called derivative theorem and indeed this is one of the results in this transform calculus because it has several applications for solving integral equations differential equations including ordinary and partial differential equations. So suppose this f is continuous and is of exponential order alpha and f prime is piecewise continuous.

So under these conditions we have this nice result that the Laplace Transform of f prime t is s times Laplace of ft minus f0 and this is valid for real s greater than alpha and this property again says similar to what we had in Fourier Transform that the Laplace Transform of the derivative is equal to s the Laplace Transform of ft and minus this f0. So we consider this integral 0 to infinity f prime t e power minus st dt.

And then apply this idea of the integration by parts that means we have ft there e power minus st and this limit and then again here this ft will be there and this minus (12:44) minus st. So here now we have to put the limit as this t approaches to infinity and this t approaches to 0. So when t approaches to infinity this e power minus st will make this term 0 and then when t approaches to 0 we will get this f0.

So we have this with minus f0 and at this place this s and minus s minus sign and this minus s will make it plus s and then the result is Laplace Transform of ft. So we have this result that the Laplace Transform of f prime t is equal to s Laplace of ft minus f0. Its counterpart for the

Inverse Laplace Transform we usually use this result in a slightly different form that if we know that the L inverse Fs is ft.

And in addition to this we also require that f0 is 0 so if there is no this term here we can just invert it. So we have this L inverse of this s Fs is equal to the derivative of f with respect to t. So we will apply to one problem this Inverse Laplace Transform property.

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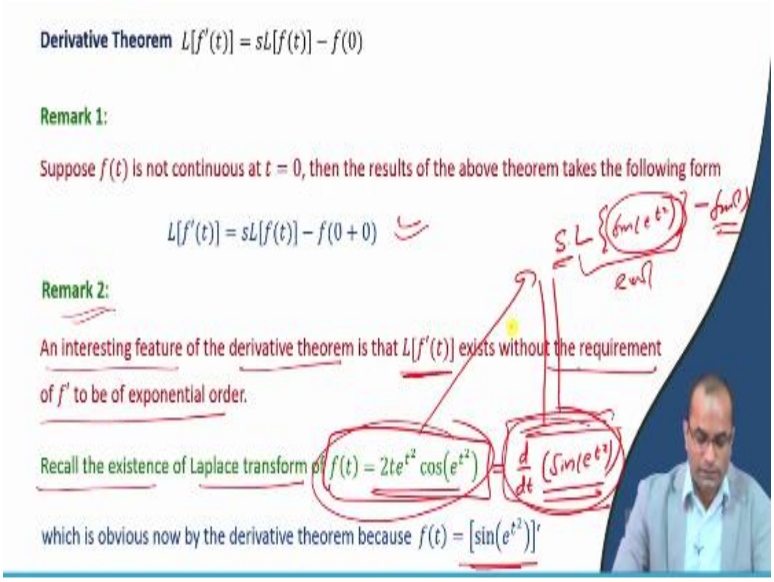

This derivative theorem we have some remarks now that if ft is not continuous at t equal to 0 for instance because this continuity we assume then we have taken here f0. Otherwise this can be replaced by this limit f0 plus that means the limit of this function f as t approaches to 0 from the right hand side and here we have this s Laplace of ft here we have the Laplace of f prime t.

So this is slightly more general than the earlier result. So coming to the remark 2 an interesting feature of this derivative theorem is that this Laplace of f prime t exists without the requirement of f prime to be exponential order. We have in the theorem not assumed that f prime to be of exponential order, but we got the Laplace Transform of f prime t and just to recall the existence of this Laplace Transform we have discuss in one of the lectures that the Laplace Transform of this function exist.

Whereas this function is not of exponential order and now it is clear which is obvious from this derivative theorem because this given function is the derivative of this sin e power t square and then if we apply the derivative theorem which says that the Laplace of this would be now s times the Laplace of ft so Laplace of sin e power t square and minus f0 which is sin 1.

So this existence of the Laplace of this function now it is clear from here which says that it is s times the Laplace of this function which is continuous function of exponential order. So the Laplace exist here and then minus sin 1. So that is obvious now from this derivative theorem the existence of for instance the Laplace of this function.

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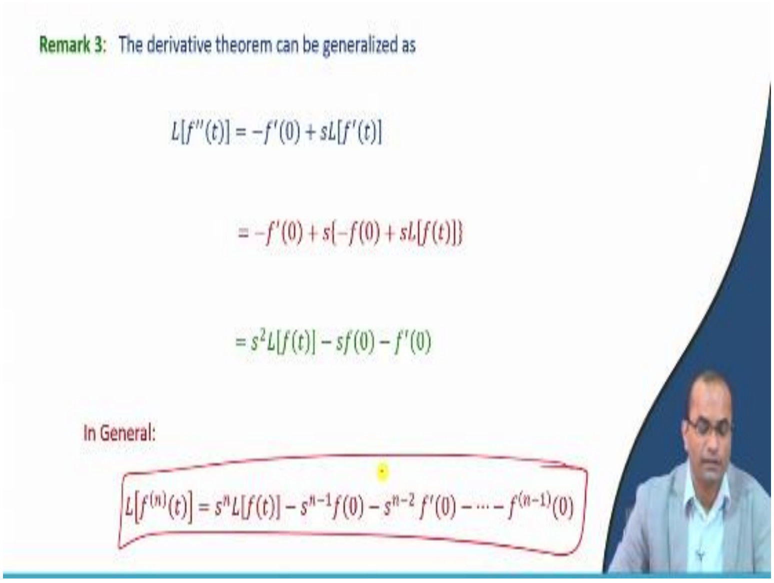

This derivative theorem can be generalized that if we take the double derivative that means we have minus so this is the derivative here of ft and then we are talking about the derivative. So here our function is this f prime so the direct application of the derivative theorem says minus f prime 0 plus this s the Laplace of f prime t that means this minus f prime 0 plus this s and then here we can apply the derivative theorem again minus f0 plus s the Laplace of ft.

So we have here this s square the Laplace of ft minus this sf0 and minus s prime 0. So in general we have this result now for any derivative we can get. So Laplace of the nth order derivative of this f is equal to the s power n the Laplace of ft and then keep on reducing the power of this s.

So we have s power n minus 1 f0 s power n minus 2 f prime 0 here the derivative will keep on increasing and the power of this s will be decreasing. So in that way this is a very general result where we can deal the secondary derivative, third derivative or any order derivative of the function.

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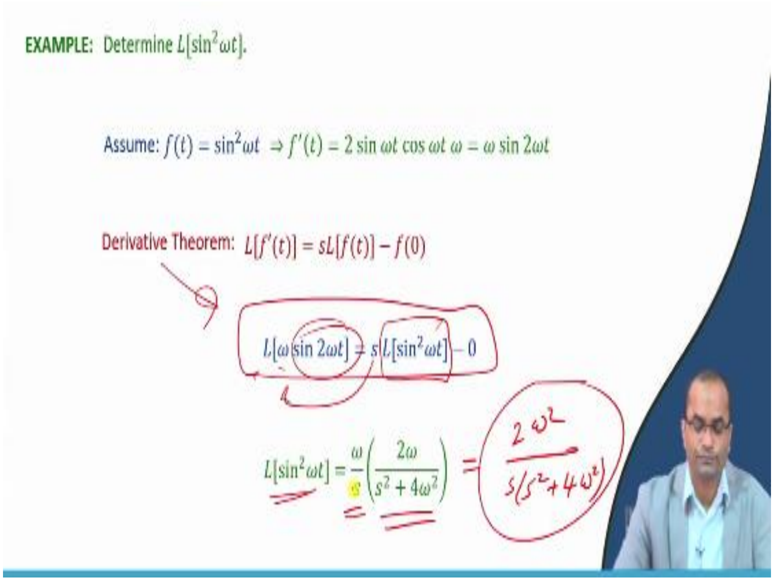

So some applications so if you want to find for example this sin square omega t and we will naturally want to use this derivative theorem. So we have let us take this ft sin square t and if we get this derivative so directly we do not know the result of this sin square t the Laplace Transform of sin square t, but if we take the derivative we are getting 2 sin omega t cos omega t and this omega factor which is omega times this sin 2 omega t.

And this sin at we know the derivative we know the Laplace Transform of sin at. So we can use now the derivative theorem which says the derivative of this omega sin 2t which is a derivative of this. So the derivative of this the Laplace Transform is s the Laplace Transform of the ft sin square t minus this f0. So f0 is sin 0 is 0.

So we have this due to the derivative theorem and then this Laplace of sin square t we can get this w this s will go to the other side. And then the Laplace of sin 2 omega t which is 2 omega over s square plus 4 omega square which we can write like 2 omega square over this s and then we have s square plus this 4 omega square.

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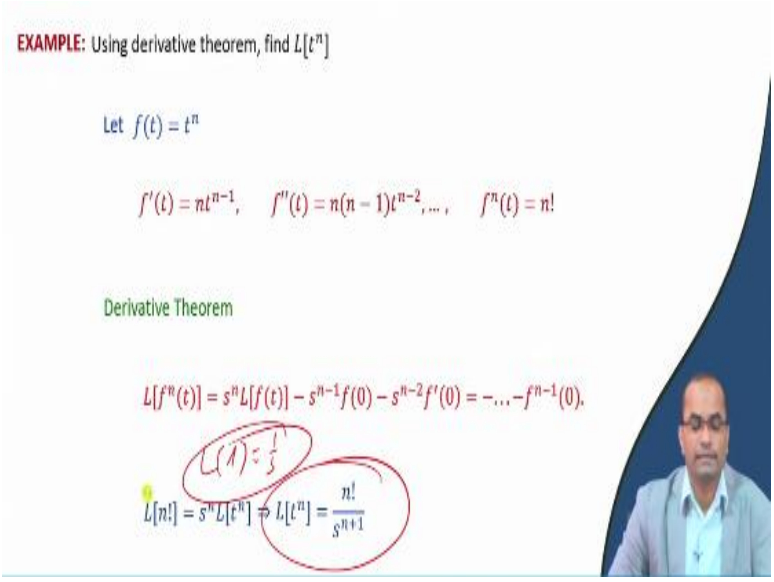

Well so here we are using this derivative theorem we will also find the Laplace of t power n which we have already evaluated, but this demonstrate using this derivative theorem how easy is to get this Laplace Transform of t power n. We know that if we have this ft is t power n then its first derivative is nt n minus 1 then n minus 1 t n n minus 1 t power n minus 2 and so on.

The nth derivative will have just the factorial n and if we use this derivative theorem that the Laplace of f the nth order derivative is given by this one what is interesting here that f0 is 0 the f prime is also 0 at 0 this is also 0. So all these derivatives up to n minus 1 they all will become 0. So this portion will be 0 for this function and then we have only this Laplace t power f n t is equal to s power n and Laplace of ft.

So applying this the left hand side we have the Laplace of this nth derivative which is factorial n. So Laplace of factorial n is equal to s power n and then the Laplace of t power n. The Laplace of this 1 because factorial n is constant so here Laplace of 1 we know it is 1 over s. So we have 1 over s from here with factorial n and then s power n will go to the denominator there.

So we have the result that the Laplace of t power n is factorial n over s power n plus 1. So we got this result just by using that the Laplace Transform of 1 is 1 over s.

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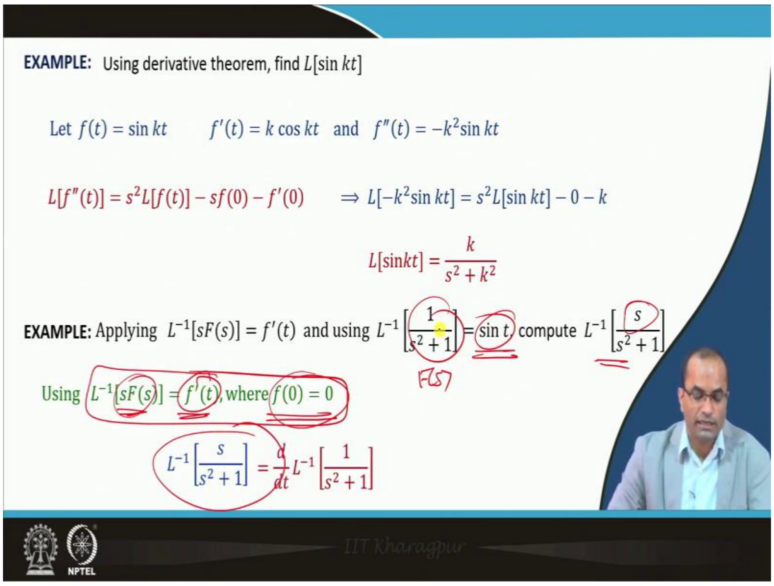

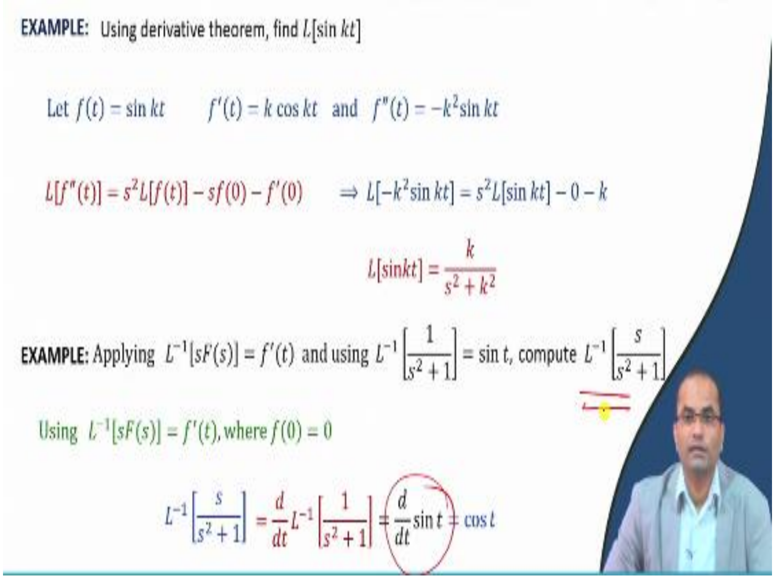

Now using this derivative theorem we will find the Laplace Transform of sin kt. So ft is sin kt if we assume. So here purely the application of the derivative theorem will be used to find out the sin kt. So the derivative of first derivative is k cos kt and we take another derivative once more we have minus k square and the sin kt. Now we will apply this derivative theorem for the second order.

So Laplace of this f double prime t is s square Laplace of ft minus sf0 which is 0 and f prime 0 which is just k there. So substituting these values there we have the Laplace of minus k square sin kt is equal to s square and the Laplace of sin kt and minus this 0 and we have minus f prime as f prime 0 as k. So we have this identity now which here we have Laplace kt here also we have Laplace kt so these can be merged now.

And there will be a factor k square plus s square and the other side we have this k so k over s square plus k square will be the value of the Laplace kt. Applying this L inverse s fs as f prime t and using this result that the Laplace inverse 1 over s square plus 1 is sin t. We want to find for instance the Laplace inverse just if we multiply here by s so that means s over s square plus 1 using naturally the idea of this derivative theorem.

So from the derivative theorem we know that L inverse s Fs is f prime t and this is valid when this f is 0 is 0. So in our case if we take this ft as sin t f0 is 0 and sFs. So this is exactly sFs if this is Fs then we have there sFs. So the Laplace Inverse of sFs will be f prime t that means the derivative of sin t. So the derivative of sin t is cos t. So this L inverse of s over s square plus 1 is nothing but cos t.

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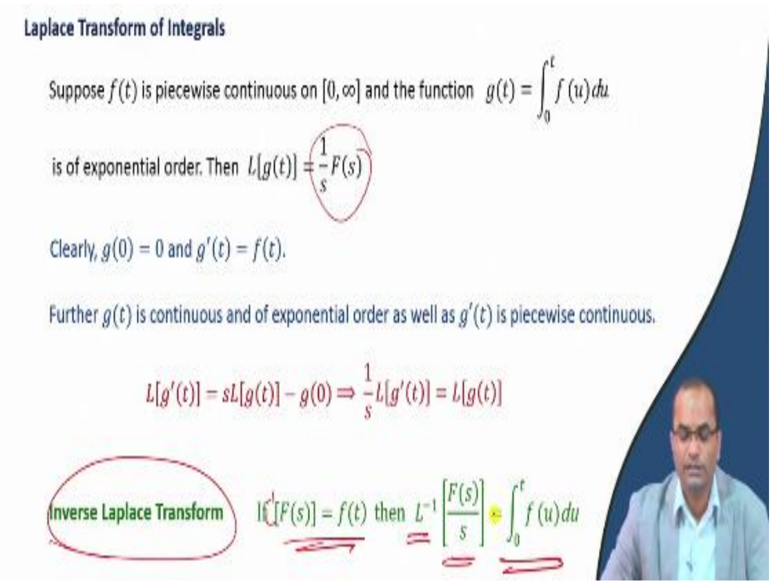

Now finally we want to discuss this Laplace Transform for integrals and here we assume that ft is piecewise continuous on this interval 0 to open infinity and this function gt as 0 to infinity fu du. Suppose this gt is the integral of this f from 0 to t then we assume that this function is of exponential order and then the Laplace Transform of gt we want to find as 1 over s Fs.

So clearly here g0 because when t is 0 here we will get this g0 so g0 is 0 and g prime again from there g prime is just ft. So now this gt is continuous because f is piecewise continuous function and this is integral gt is the integral there so gt will become continuous and is of exponential order that is given there in the problem and g prime t is piecewise continuous because g prime t is ft and ft is piecewise continuous.

So naturally g prime is piecewise continuous. So this gt is continuous of exponential order and g prime t is piecewise continuous. So we can use this derivative theorem just discussed before that Laplace of g prime t is s Laplace of gt minus this g0. So we have 1 over s Laplace of g prime t is equal to Laplace of gt because this g0 is 0. So we got the result already that the Laplace of gt is 1 over s.

And the Laplace of this g prime t which is ft so the Laplace of ft which is Fs there. For inverse counterpart so we have this as follows that the L inverse of this Fs is ft in that case L inverse of this 1 over sFs will be gt that means 0 to t fu du.

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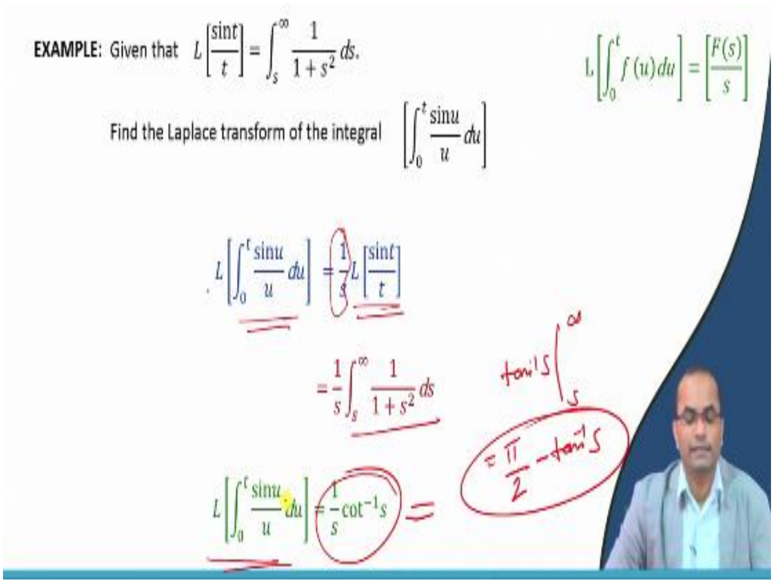

Coming to the examples so we discuss here that it is given that sin t over t the Laplace of this is s to infinity 1 over 1 plus s square ds and we want to find the Laplace Transform of this integral 0 to t and sin u over u. So if sin over t is given then what is the Laplace of the integral of this sin t over t. We know the result that the integral of the Laplace of the integral is Fs over s.

So basically we have to get just Fs the Fs is the Laplace Transform of this integrant here that is sin u over u and that Laplace is given already in the problem. So the Laplace of this integral will be 1 over s the Laplace of sin t over t which is already there s to infinity 1 over 1 plus s square and this 1 over 1 plus s square is tan inverse s and then the limit s to infinity which is tan inverse infinity is pi by 2 minus this tan inverse s. And this pi by 2 minus tan inverse s is 1 by s cot inverse s. So this is done for the integral.

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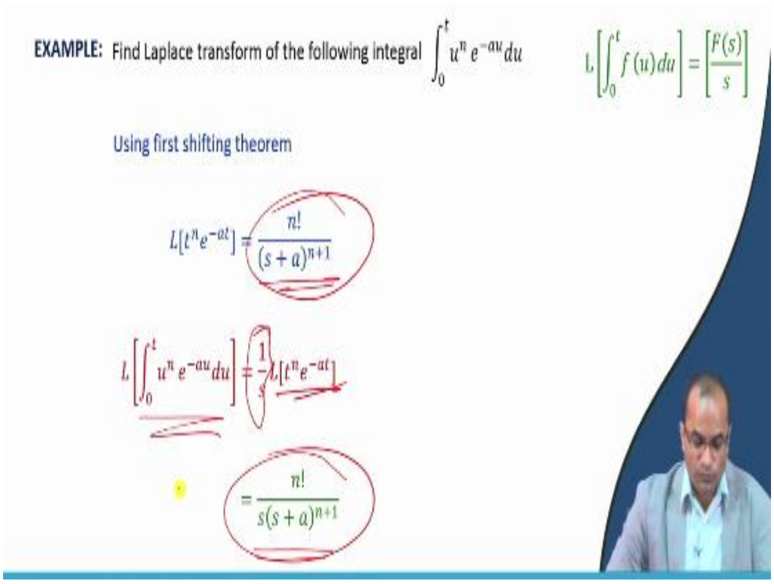

The next is find the Laplace Transform of this integral 0 to t u power n e power minus this au. So again we will apply this that the integral here of this fu du so here fu is u power n e minus au and so we have to get just the Laplace Transform of this and then we can just divide by s. So the Laplace Transform of t power n e power minus at by this shift theorem.

We know already that is a factorial n over s plus a this plus a is coming because of this shift there. So we have factorial n s plus a power n and then this integral will be just 1 over s and this factorial n over s into this s plus 1 the power n plus 1. So that is also done.

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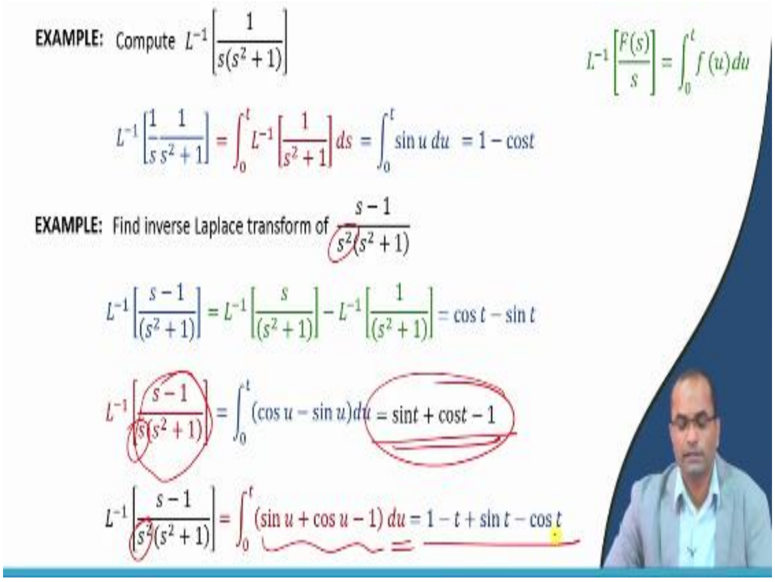

And now we want to compute for instance this L inverse 1 over s s square plus 1 and we will use this analogous definition of this integral for the inverse. So we have L inverse Fs over s so if we know for instance the L inverse of 1 over s square plus 1 then we can get for Fs when we divide by s just by integrating that function. So here we need to get this L inverse of 1 over s square plus 1 which we know it is a sin function.

So we have sin u then du which we have to integrate now here that will give 1 minus cos t because we have integrated here 0 to t. The last example we want to find the sin inverse find this Inverse Laplace Transform of s minus 1 over s square into this s square plus 1. So here we have this s there which can be handled with this division theorem. So we need to find L inverse of s minus 1 over s square plus 1 which again the linearity says that this is L inverse of s over s square plus 1 minus L inverse 1 over s square plus 1.

And then here we have cos t and then this is sin t so if we just consider this 1 over s there that means we have to integrate this 0 to t that means 0 to t this cos u minus sin u over this du. So here we have the cos will be sin this t and sin will be cos u. So we have the cos t and then cos 0, 1 will be also coming there. So we have now this here the L inverse of this s minus 1 over s s square plus 1.

But in the question it is asked for this s square. So we have to apply this once again to have to accommodate this one more s there that means we have to integrate this sin t plus cos t minus 1 from 0 to t over this du. So after this integration we get 1 minus t plus sin t minus this cos t.

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Well, so these are the references we have used for preparing this lecture.

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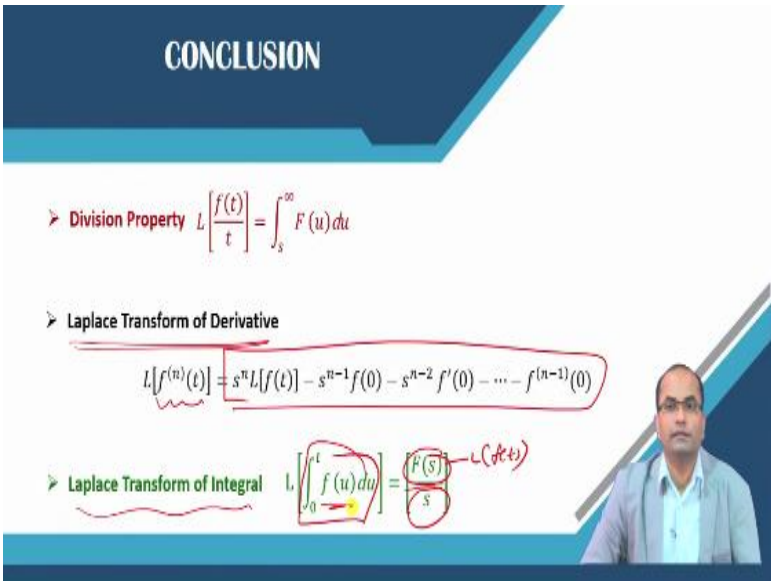

And just to conclude so today we have discuss this division property which says that if the Laplace Transform of this ft is Fs then we can get this Laplace of ft over t as s to infinity fu du. So we have to integrate there if you want to accommodate this extra t in this denominator there. The Laplace of the derivative one of the important most important result in this Laplace Transform.

So we have the Laplace of this f the nth derivative can be computed with this help of this formula and this Laplace of the integral that means if we have this ft and we want to get the

Laplace of this integral then this is just the Fs which is the Laplace Transform of that ft and we have to divide here by s to accommodate this integral there.

So with the help of all these properties we have observed that the evaluation of Laplace as well as the Inverse Laplace Transform of much more complex functions become easier. So we will continue with some more properties in the last lecture and that is all for this lecture and I thank you for your attention.