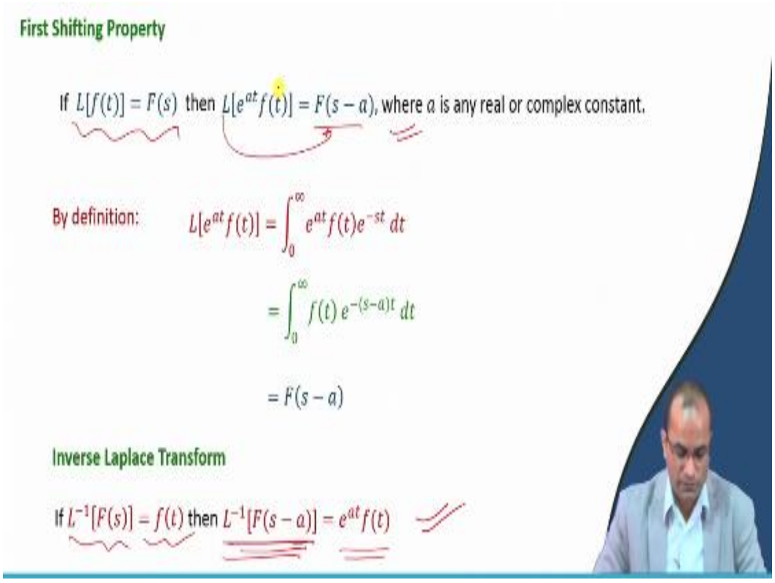
Engineering Mathematics-II
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Science – Kharagpur
Lecture -54
Properties of Laplace Transform

So welcome back to lectures on Engineering Mathematics II and this is lecture number 54 on Properties of Laplace Transform.

(Refer Slide Time: 00:20)

So today we will discuss some properties and using those properties we will realize that finding or evaluating this Laplace Transform or Inverse Laplace Transforms becomes easier. So there will be today in this lecture we will be discussing a shifting property and then we have the change of scale properties and multiplication properties. So these three properties will be discussed in this lecture.

(Refer Slide Time: 00:46)



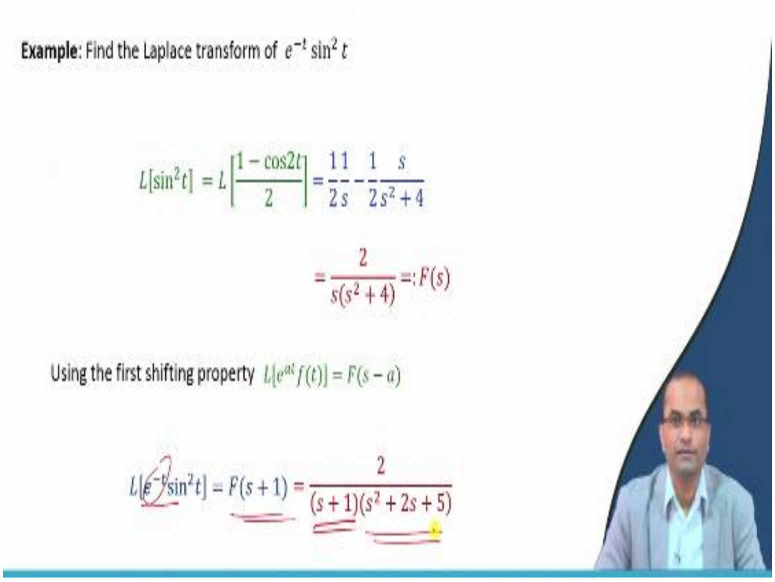
So coming to the first shifting property so if this Laplace of ft is Four Square, then this property says that the Laplace of exponentially at ft will be Fs minus a. So if we multiply this ft by this e power at then there will be a shift in the Laplace Transform. So s will be replaced by s minus a. So naturally once we have this property this will be much easier to find the Laplace Transform where we do the multiplication of some exponential function.

So by definition so we can go through the proof the proofs are very simple in these cases because finally we are dealing with this integrals only. So we have the Laplace of e power at ft e power at ft is now this function and then power minus st because of the Laplace Transform. So, then we have this ft and this e power at will be merged with this e power minus st.

That means we have e power minus s minus at dt and that is exactly the Fs minus a because our Fs is nothing but 0 to infinity e power minus st and then ft dt. So instead of this is s here we have s minus a therefore we have written this as F s minus a. The similar result we have for the inverse Laplace Transform. So together with the Laplace Transform we will be also listing here for Inverse Laplace Transform because these results are just the analog of the forward properties there.

So if we have the Laplace inverse of Fs this ft so we are just reversing this there then this says that L inverse of Fs minus a so this L goes to the right hand side, L inverse s s minus a will be e power at ft. So for finding the inverse we will be using this shifting property for finding the Laplace Transform we may use this shifting property.

(Refer Slide Time: 03:02)

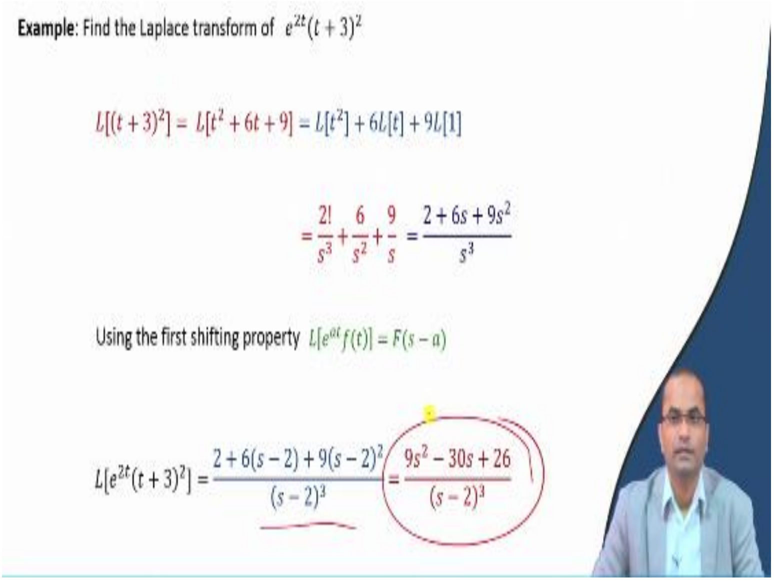


So coming to the examples, so we have first here we want to find the Laplace Transform of e power minus t sin square t for instance. So here with the function sin square t e power minus t is sitting. So that is exactly where we can apply this shifting property. So first we will evaluate the Laplace Transform of sin square t as 1 minus cos 2 t over 2 that means we have 1 by 2 the Laplace Transform of 1 that is 1 by s.

So it is a linearity property we are using of the Laplace Transform. Then we have 1 by 2 there with the minus sin and the Laplace Transform of cos 2t which is s square over this sorry s over s square plus 4 then we can just simplify this so we got this 2 over s s square plus 4 and then this is what we call as Fs the Laplace of this sin square t and then to get the Laplace of e power minus t sin square t we will apply this shifting property which is called the first shifting property.

So using this property we have e power minus t sin square t and then we have here Fs plus 1. So the s will be replaced by s plus 1 because of this e power minus t. So here now we have this 2 over s is replaced by s plus 1 s square plus 2 s plus 5.

(Refer Slide Time: 04:35)

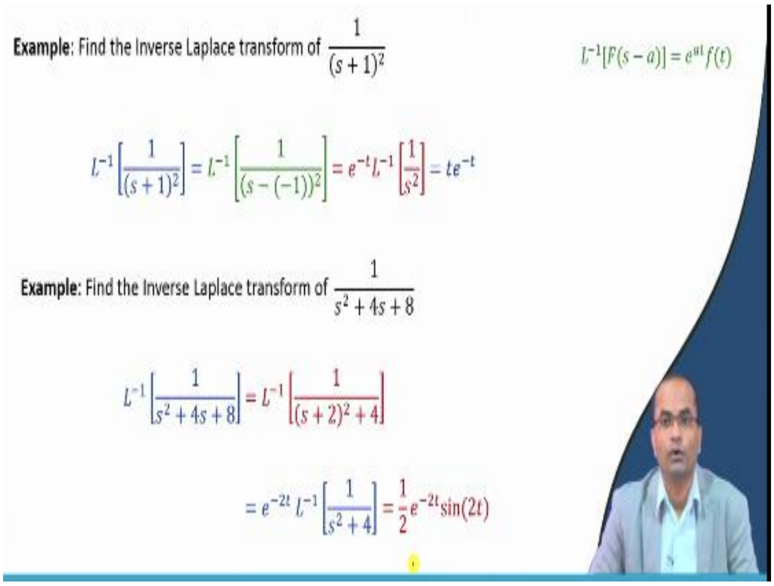


So we have now the next problem where we want to find the Laplace Transform of 2 power t and t plus 3 square. So again the similar situation we have the multiplication with this exponential function. So if we find the Laplace Transform of this t plus 3 square then we can get the Laplace Transform of this e power 2 t into t plus 3 whole square. So this t plus 3 whole square is the t square plus 6t plus 9 and then we can use the linearity property to find this Laplace Transform.

So we have the Laplace of t square 6 times Laplace of t plus this 9 times Laplace of 1. So here the Laplace of t square is 2 over factorial 2 over s cube then here Laplace of t is 1 over s square and this Laplace of 1 is 1 over s. So we have this 2 over s cube 6 over s square and then over s which can be simplified to get this 2 plus 6 s plus 9 s over s cube. Then using this first shifting property which says that the exponential e power at ft the Laplace Transform will be s F of s minus a.

Now this s in this expression here the s will be replaced by s minus a and a in our case is just 2 there. So that means this 2 plus 6s minus 2 and then here 9 s minus 2 whole square divided by this s minus 2 cube. So this is the Laplace Transform of e power 2t sin t plus 3 square which again can be simplified to finally give this expression here.

(Refer Slide Time: 06:29)

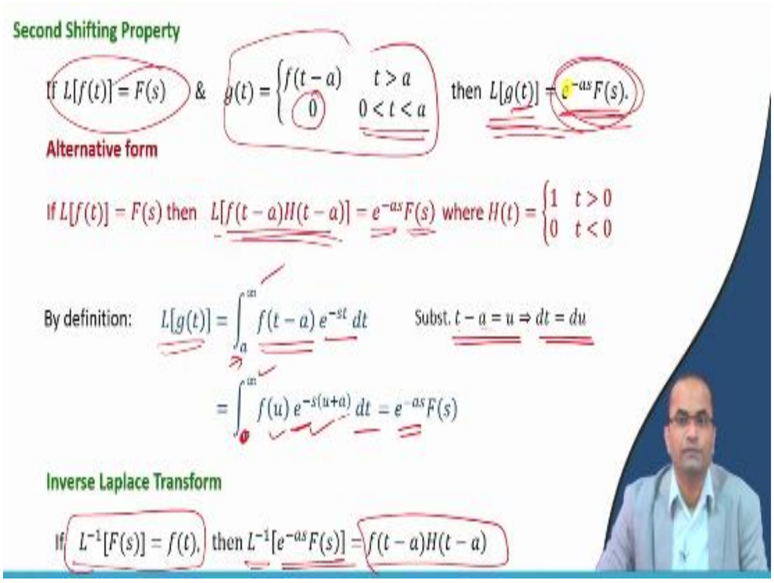


We want to find now the inverse Laplace Transform for instance of this function 1 over s plus 1 square. So in that case we will rewrite this s minus and this a which is minus 1 now whole square and then we can use this shifting property that the Laplace inverse of Fs minus a is e power at and ft. So here we have this s is just s minus a that means if we get if we use this property there so we can have the e power this minus t outside and then the Laplace of 1 over s square. So e power minus t and the Laplace inverse of this 1 over s square.

And therefore this is e power t and then here we have this Laplace inverse of 1 over s square s t so this is t e power minus t using this first shifting property. For instance you want to find the Laplace inverse transform of this one 1 over s square plus 4 s plus 8. This also can be handled exactly in a similar fashion that we have this s square plus 4s plus 8 which is nothing but s plus 2 whole square plus 4.

So this s is just s plus 2 there and we will now applying this shifting property to get this e power minus 2t outside and then the Laplace inverse of 1 over s square so this will be becoming now s square so s square plus 4. So that means we have this half because of this we need 2 there to have the sin 2t. So e power minus t is there and then this is half sin 2t. So we got this inverse transform using this for shifting property.

(Refer Slide Time: 08:31)



There is another result which is called the second shifting property where we have the Laplace Transform of ft is Fs if this is given and also the gt is now given by a shift here when t greater than a it is f t minus a. So there is a shift in the function now and then from 0 to a where this function is shifted the value is set to 0. So we have this again second shifting theorem where the shift is done in the function itself.

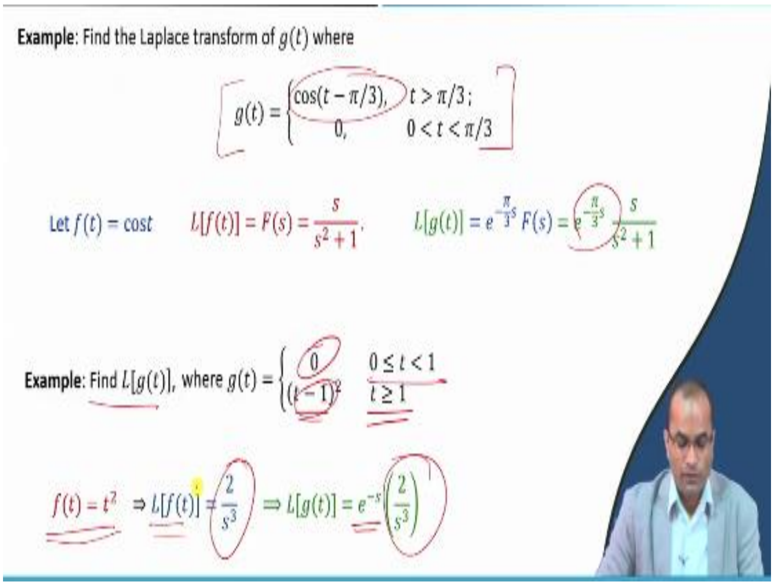
Now the question is what is the Laplace Transform of ft. So to Laplace Transform of gt is nothing but e power minus as and Fs. The alternate form which is also convenient to write this gt instead of gt writing in this way we can actually write in this compact form ft minus a Heaviside function t minus a. Recall this Heaviside function when t is greater than 0 it is 1t less than 0 it is 0.

And now this ft minus a Ht minus a is exactly this function gt because when t is greater than a this will become 1 here H so we have ft minus a and when t is less than a so this will become 0. So instead of writing this gt in this way we can also write this gt in this way ft minus a Ht minus a and we have e power minus as and then Fs. So by definition if we go the Laplace Transform of gt because is 0 to a the value of the function gt is 0.

So the integral will start from a to infinity and the function is ft minus a e power minus std t and if we substitute this t minus a as u that means dt is du so this will become now when t is a then u will become so this is 0 and then here we have t infinity, so u will be also infinity then we have fu and e power minus s and t is a plus u dt. So here e power minus as we can bring out and then the remaining part is just the Fs the Laplace Transform of ft.

For the Inverse Laplace Transform we have again a similar result if Laplace inverse Fs is ft so parallel to this one then the Laplace inverse of e power minus as so Laplace inverse here of e power minus as Fs will be the Laplace Transform or the gt gt is we have written here in compact form that is ft minus a and H t minus a. So we have the counterpart for the Inverse Laplace Transform which is exactly can be written from this second shifting property.

(Refer Slide Time: 11:41)



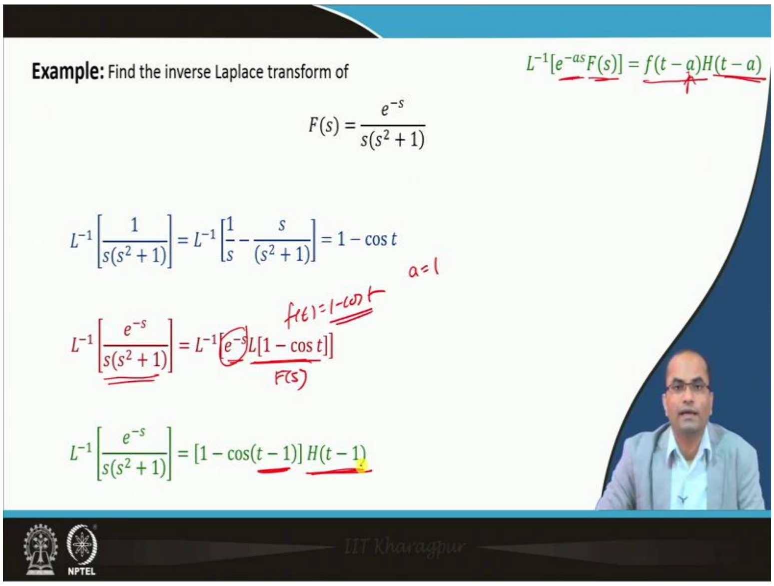
Coming to some examples so we have for instances here you find the Laplace Transform of this gt where gt is cos t minus pi by 3 when t is greater than pi by 3 and 0 in this range 0 to pi by 3. So the cos function is shifted for this t greater than pi by 3, but the value is exactly starting from 0 itself because cos t minus pi by 3 is there in the argument. So here if we take this ft as cos t we know the Laplace Transform of this ft we have already done this.

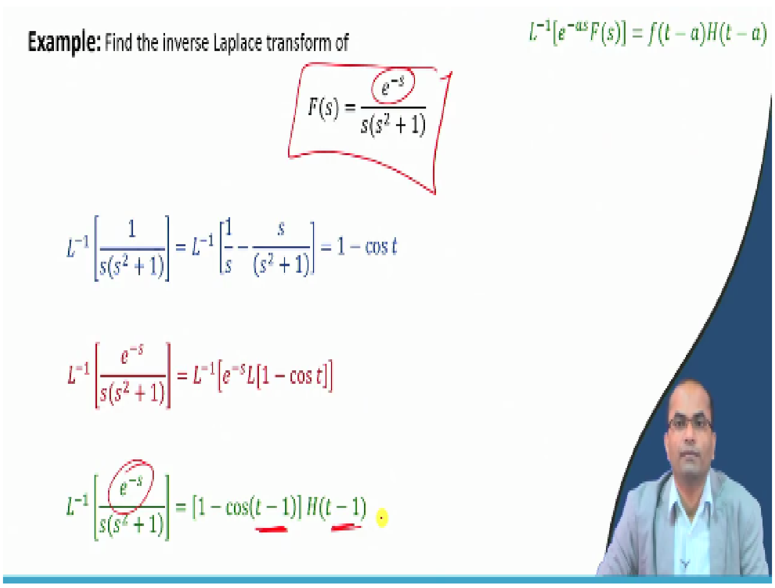
So s over s square plus 1 and then the Laplace Transform of gt where we can apply this first shifting theorem which says e power this minus pi by 3 s will go out and then the Fourier Transform or the Laplace Transform of ft that is just the cos t so the Laplace Transform is s over s square plus 1. So we are done with this second shifting property and the application is very easy.

Now with the help of this we can directly write down the result just by multiplying this e power minus pi by 3 s. If you want to find this Laplace of this function which is again a shift here so 0 to 1 the value is 0 and t greater than the value is t minus 1 square. So the function is t square and there is a shift to this by 1 so again we can use the second shifting theorem property. So ft is t square and the Laplace of ft means t square is 2 over this s cube.

So the Laplace of this gt will be e power minus s e power minus s is coming because of that shift theorem because here there is a shift by minus 1 so e power minus 1 s and then the Laplace Transform of this ft which is 2 over s cube.

(Refer Slide Time: 13:53)





Well so we can also look at the Inverse Laplace Transform how to apply this property of the Laplace Transform for the inverse case. So we have for instance here Fs as e power minus s and s square plus 1 into s. So just to recall that this e power minus s can be adjust with this inverse property with this second shifting property. So we need to get the Laplace inverse of 1 over s and s square plus 1 where we can do the partial fractions we have already discussed yesterday that this is one of the techniques to find the partial fractions.

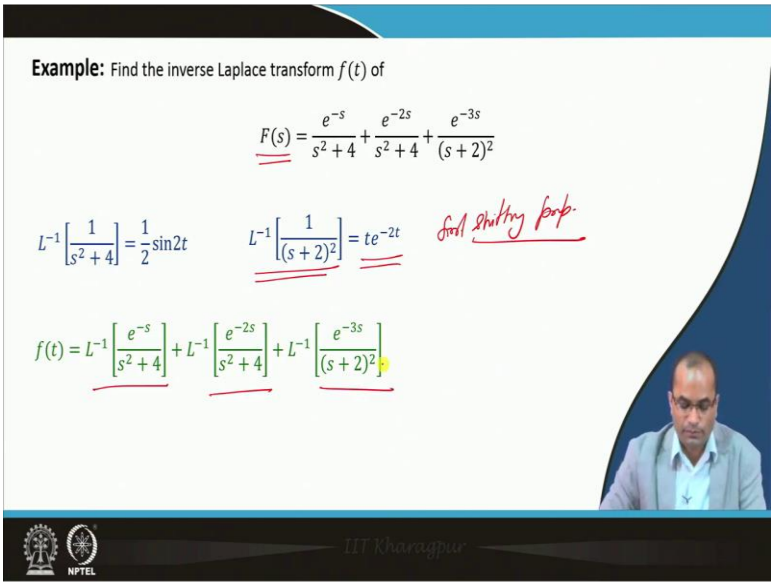
So to find Inverse Laplace Transform so here 1 over s and minus s over s square plus 1 will come where we can apply this linearity property on this 1 over s. So L inverse 1 over s and then we have their L inverse s over s square plus 1. So the L inverse of 1 over s is 1 and the L inverse of this s over s square plus 1 this is cos t. So we have this 1 over cos t the result of the Laplace inverse of 1 over s and s square plus 1.

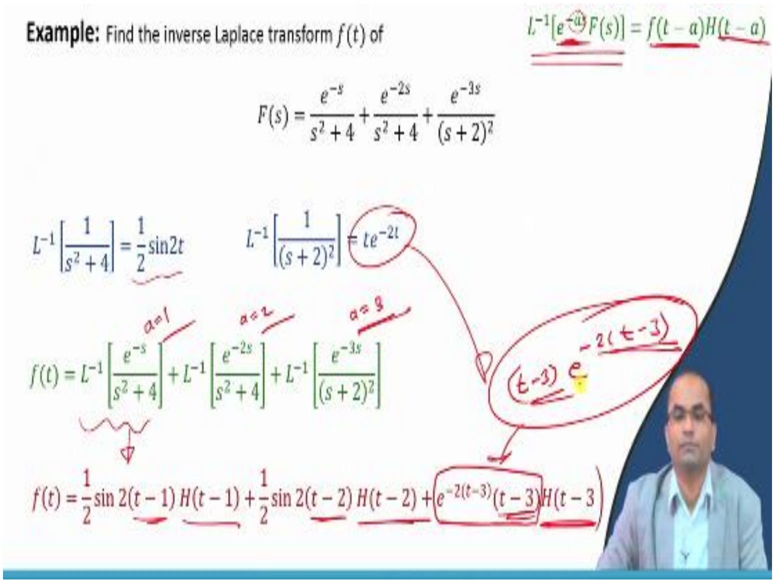
Now getting to the Laplace inverse of e power minus s with this s into s square plus 1 so we will apply this shifting theorem that says that L inverse e power this minus s and then the Laplace of 1 minus cos t because this is the Laplace inverse of this is 1 minus cos t that means a Laplace of this so this is what given here that e power minus s and some kind of Fs is given here.

The second shifting theorem says that e power minus as Fs. So here we have e power minus s and this is our Fs here which is 1 over s square plus 1 and the result is ft minus a Ht minus a. So the ft here is simply 1 minus this cos t and then we have to shift this by this a. So a is 1 in our case and then Ht minus a will also come together. So we have 1 minus this cos t the t will be shifted by 1 so t minus 1 and then this Heaviside function H t minus 1.

So using this second shifting property we can now deal such functions where e power minus s is appearing in the multiplication. So this e power s is handled here by this shift t minus 1 and multiplied by Ht minus 1.

(Refer Slide Time: 16:52)





One more example to find this to apply this inverse, to apply this second shifting property on this Inverse Laplace Transform. So we have Fs e power minus s s square plus 4, e power minus 2s s square plus 4 and e power minus 3s s plus 2 whole square. So again here because this e power minus s or minus 2s or minus 3s appears in the multiplication, so we will just get the Laplace Inverse in this first case 1 over s square plus 4 here we will get 1 over s square plus 4.

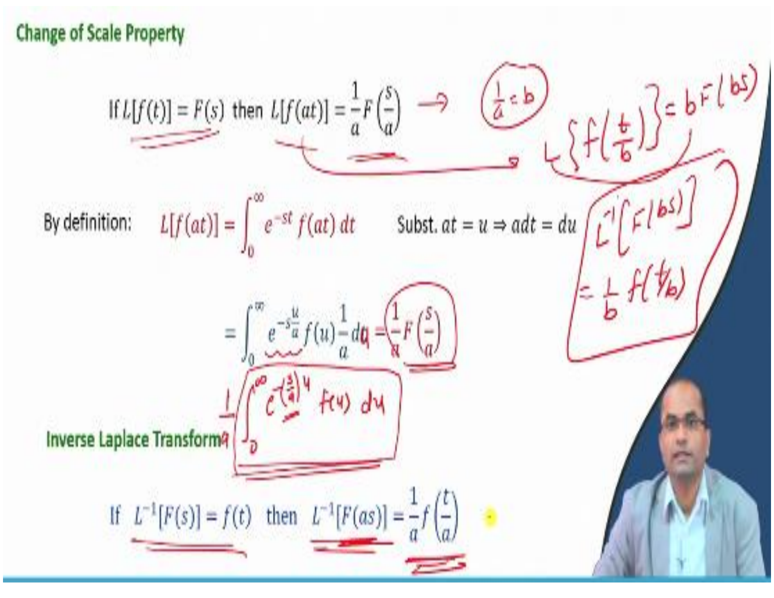
So same thing but here the shift is different and here we have s plus 2 square. So getting the Laplace Inverse of this we can apply the shifting property to get this desired inverse Laplace Transform. So this Laplace Inverse of 1 over s square plus 4 is half because 1 can multiply 2 there and divide by 2 there. So this is exactly the sin 2t and then this half has to be there. The second we need to get this Laplace Inverse of 1 over s plus 2 whole square.

So s plus 2 whole square is we know already because of this 2 we can have this shift property which says it is e power minus 2t and then 1 over s square that will be just t. So with Laplace inverse of this 1 over s plus 2 square is te power minus 2t using the first shifting property. So we have use here the first shifting property and now we can get this inverse. So Laplace inverse e power minus s s square plus 4 e power minus 2s s square plus 4.

And in this third place we have e power minus 3 s s plus 2 whole square. So here the shifting property is that e power minus as so this a in our case here it is 1 and here a is 2 and here a is 3. So we can have a direct shift there by a in f and then multiplied by Ht minus a. So in this case we know that this is half sin 2t, but now t will be replaced by t minus 1. So the first situation here is dealt with 2t minus 1 Ht minus 1, in the second case it will be Ht minus 2.

And this sin 2t will be sin 2 times t minus 2. In the third place we have e power minus s so there will be factor here H t minus 3 and everything will be replaced by t minus 3. So here the functions was this t, so the t power minus 2t so t will be t minus 3 and exponential minus 2t minus 3. So at every places we have this minus 2t minus 3 and here also t also t is replaced with t minus 3.

(Refer Slide Time: 20:08)



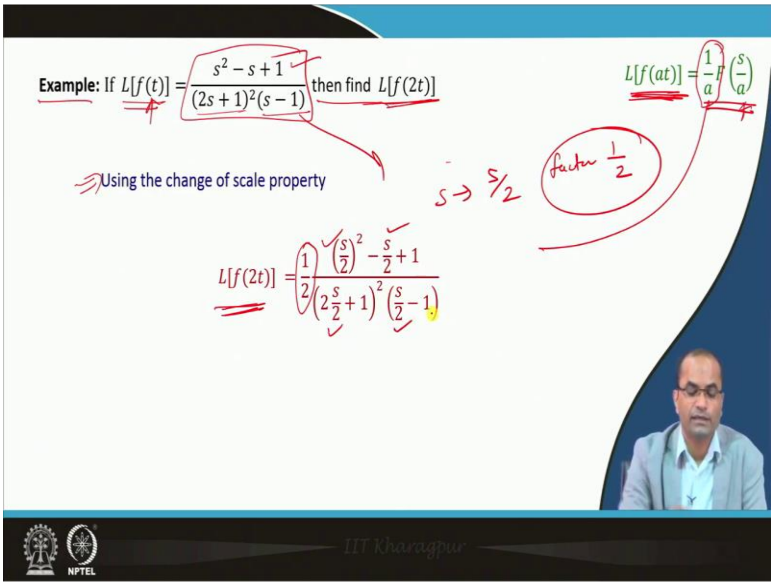
Well so we will come to another property which is the change of scale property. So the change of scale property says that if Laplace of ft is Fs in that case the Laplace of f at. So instead of t now if we have now at there then we can just adjust here 1 over s and the Laplace Transform this Fs where s is replaced by also s divided by a. So here also we can use the definition to prove to get the idea how this property is coming up.

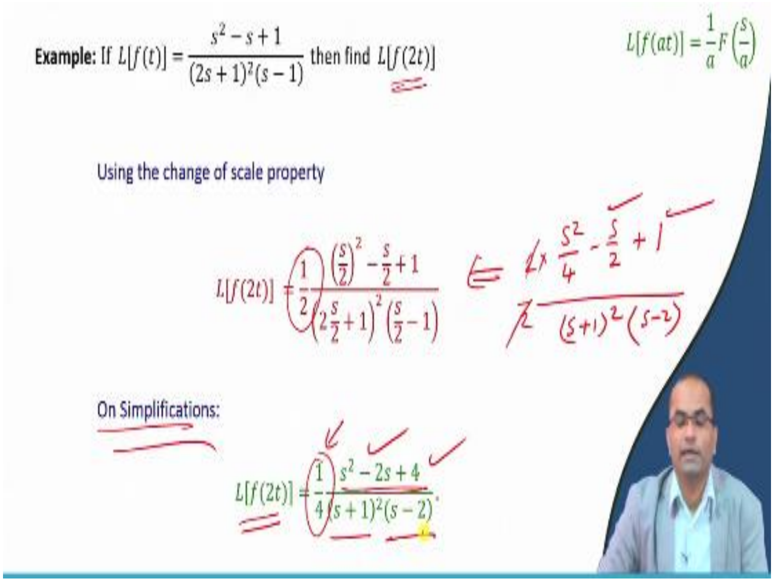
So we have Laplace of f at, the Laplace of f at then by definition we have e power minus st and f at dt. So if we substitute this at is equal to u that means we have adt is equal to du. So now we have to now replace this t by u by a. So here we have this t there that is replaced by u by a then we have fu and then du will be du by a. So dt will be du by a so here it is u. So we have now if we think as e power minus s over a and then u.

And then we have here Fu letting this 1 over a outside we have 0 to infinity and then we have du. So looking at this Fs now instead of s we have this s over a here otherwise this is Fs over a. So here 1 over a Fs minus a is coming as result. Looking at this counterpart for the Inverse Laplace Transform so we have similar result that if this L inverse Fs is ft then L inverse F as so here also s is now as then the result will be 1 over a and ft over a.

Because here if we set in this relation let us say 1 over a we take as b so what will happen now here the Laplace of f t over b will be a b and the F bs or if we take the other side now this Laplace so the Laplace inverse of this F bs will be 1 over b and the f of t over b. So we have this property for the inverse that the Laplace inverse for this F as is 1 over a f t over a.

(Refer Slide Time: 23:10)



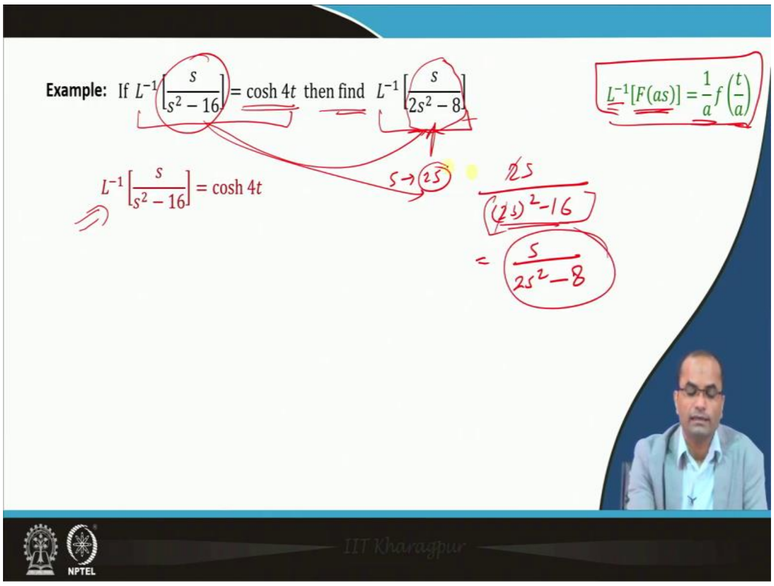


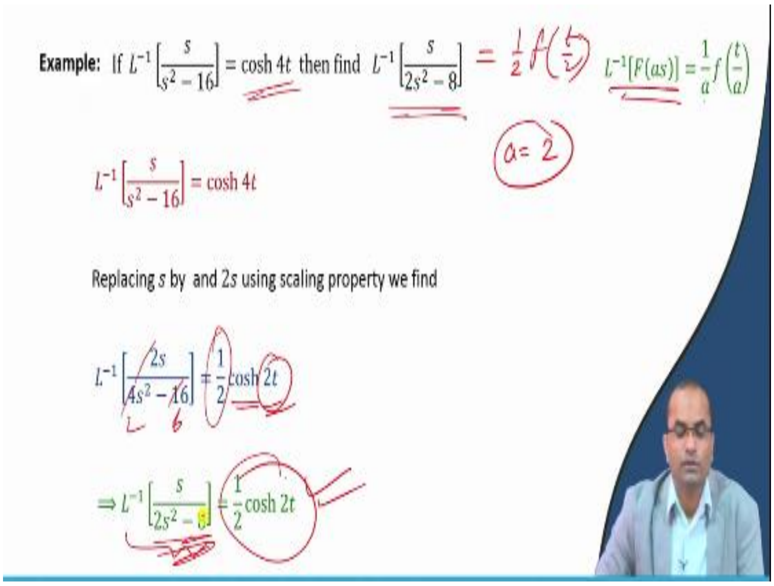
Coming to the examples so we have the Laplace of ft is given as s square minus s plus 1 2 s plus 1 whole square and s minus 1 and then we want to find the Laplace of f 2t. So we will use naturally the scale property which says that the Laplace of f at is nothing but 1 over a and the Laplace Transform of this ft we have to replace s by s over a. So here we have to now get for Laplace of f of 2t so just applying directly this result we have to replace this s by s by 2 so s will be replace there by s by 2 and there will be a factor sitting outside as 1 by 2.

So this Laplace of f 2t will be 1 by 2 this factor 1 by a is coming and then s is replaced by s by 2 here also s by 2 then 2s then s by 2 plus 1 and s by 2 minus 1. So we have directly applied this result to get this f 2t and on simplification of this what we will get the Laplace f 2t will be 1 by 4 there and this will become s square minus 2s and plus 4 because this is s square by the numerator is s square by 4 and minus we have s by 2 and then we have 1 there.

And here we have this s plus 1 whole square and then s minus 2 and this 2 will go up there and we have 1 by 2. So this 2 gets cancelled and this 1 by 4 we take common so we get s square minus 2s and then plus 4 will come and we have s plus 1 square into s minus 2. So it was a very simple application of this scaling property.

(Refer Slide Time: 25:26)





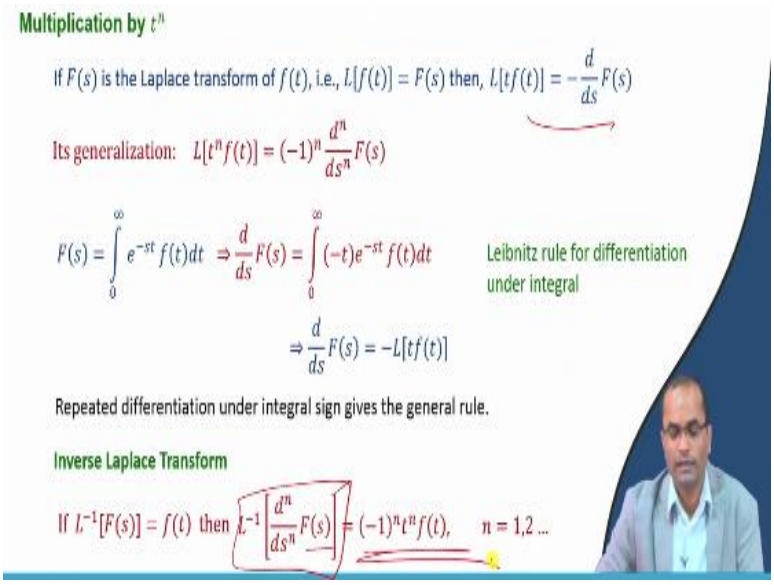
If we have now the L inverse s over s square minus 16 given as cos hyperbolic 4t then we need to find L inverse of s over 2s square minus 8 so we have to again see that what kind of scaling is done in this Laplace Transform and then we can apply this scaling property of the inverse transform. So what we notice that this is given already s over s square minus 16 as cos hyperbolic 2t and we know this scaling property that L inverse of F as is 1 over a f t over a.

So we have to just see that what is the scaling term from this s over s square minus 16 to get this 2 s square minus 8 over s over 2s square minus 8. So it is clear here that if s is replaced by 2s that what will happen we have 2s there and we have 2s square minus 16 which is s over

4 we take common or we take 2 common from there. So we have 2s square and that 2 will get cancel then with this when we take the common here 2 so we get still 2 s square and minus 8.

So this is exactly the scaling done here s is replaced by 2s that means our a is 2 now so a is 2 here. So the result will be the Laplace inverse of this will be just 1 over 2 and the f which is already there t by 2. So using this property we will get this half and cos hyperbolic and this t will be replaced by t by 2 that means we will have instead of 4t we will have just 2t. So the Laplace inverse of this s over this 2s square minus 8 the desired one. So this 2 gets cancelled there and we have exactly 2 square minus 8. So this is half cos hyperbolic 2t.

(Refer Slide Time: 27:41)



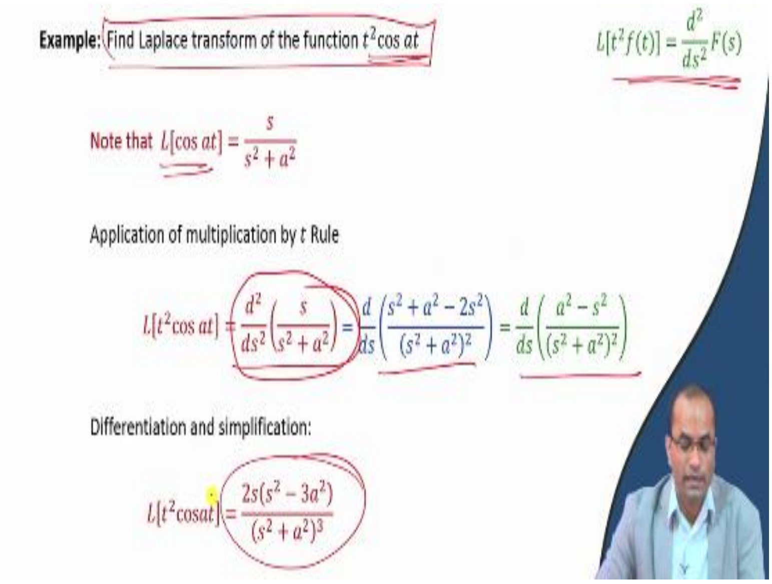
The last result or the last property here for today's lecture when we see this multiplication by t power n. So we have this Laplace Transform of if Fs is the Laplace Transform of ft that we have the Laplace Transform of t ft, so we can get t ft as minus d over ds Fs so that is another nice property and its generalization is that because if we apply repeatedly we have the t power n ft just minus 1 power n will come out and we have these derivatives there.

Looking at this result we can see that Fs is this e power minus st ft dt and if we use the Leibnitz rule of differentiations, so if we differentiate here df over ds that means here also we have to get this d over ds. So as a result of this d over ds of this minus t will come out and then we have e power minus st ft dt. So this is the Laplace Transform of this t ft and this minus sign because of this.

And now the function is t ft instead of ft. So we have this right hand side with minus Laplace of t ft and that is exactly the result we want to prove and this repeated differentiation here can give us this general rule, which says the t power n is power minus 1 d power n over d sn because if we differentiate again this here. So we will get d2 over ds2 and then here we have again d over ds.

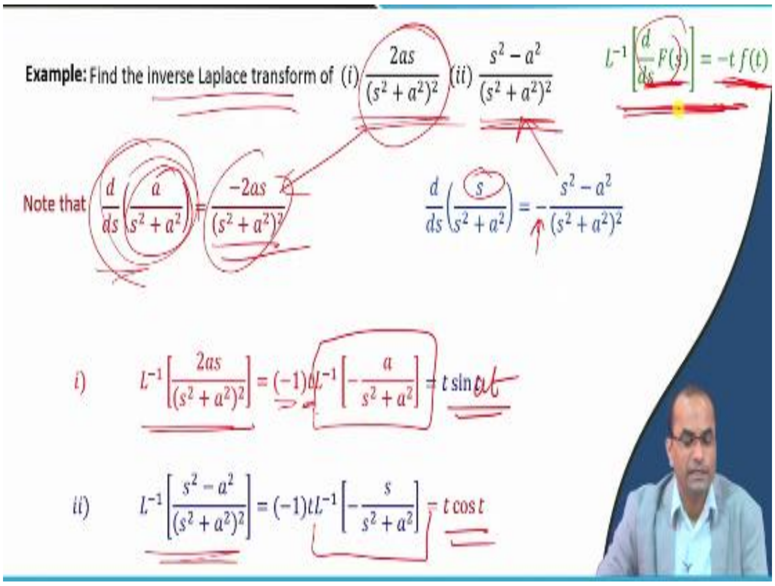
So this minus t will be out that means minus t square will be coming so this minus t will come out of this so it will be t square. So this Laplace Transform of t square ft will be just d22 over ds t Fs. So we can get all these general rules and getting to the Inverse Laplace Transform it is other way round that the Laplace of dn over ds Fs is minus 1 power n fn and ft.

(Refer Slide Time: 30:00)



So here we want to find the Laplace Transform for instance of t square cos at we know the Laplace Transform of cos at and then applying this rule two times or directly this d square Fs so we have to just get this derivative of s over s square plus a square which is simple and we can get this in 2 times. So on simplification we will get it is a 2s s square minus 3 a square divided by s square plus a square power 3.

(Refer Slide Time: 30:34)



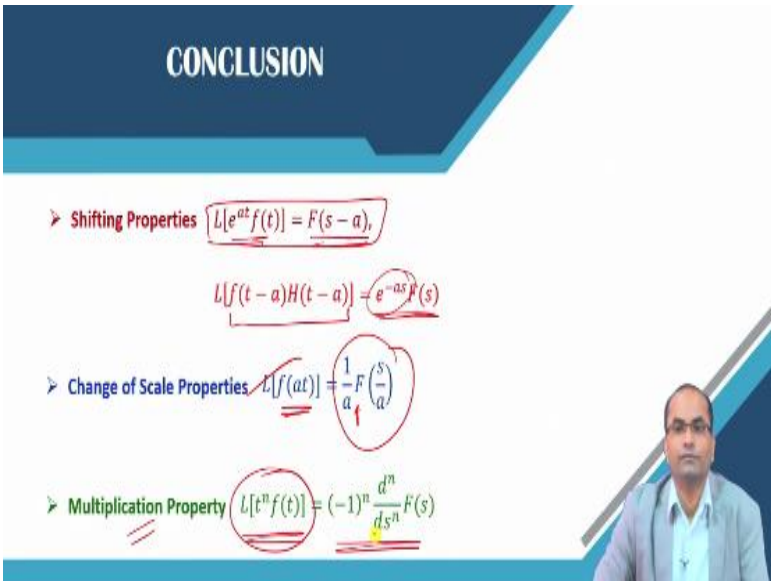
If you want to find the Inverse Laplace Transform of these functions for instance and again you will apply this property. So we have d over ds we should notice that d over ds of a over s square plus a square is this function here with minus sign and if we take their s this is actually this function with again minus sign. So we know the result that d over ds of this Fs is minus t ft. So we can apply this result there to get the Inverse Laplace of these given functions.

So if we apply here the Laplace inverse of this 2as over s square plus a square which is the derivative of this Fs. So we have the minus sign there and t and then we have this ft that means Laplace Inverse of a over s square plus a square which is sin at. So here we have sin at and t sin at. Similarly for the second case we have just the inverse of this minus s over s square plus a square which is t cos t. So here also we have applied this result of the derivative of this Fs is equal to minus t ft.

(Refer Slide Time: 31:55)

So these are the references we have used for preparing this lecture.

(Refer Slide Time: 31:59)



And just to conclude we have discussed here the shifting properties which where there are two shifting properties. The first shifting property e power minus at ft then there will be a shift in the Laplace Transform, and if we have a shift here in the function then there will be a component here e power minus as outside of Fs.

Change of scale properties, though if we know the Laplace of ft we can get the Laplace of f at by 1 over a and F s over a, this F is the Laplace Transform of ft. This multiplication property where t power n ft the Laplace Transform of this is nothing but minus 1 power n and nth derivative of this Fs with respect to s. So that is all for this lecture and I thank you for your attention.