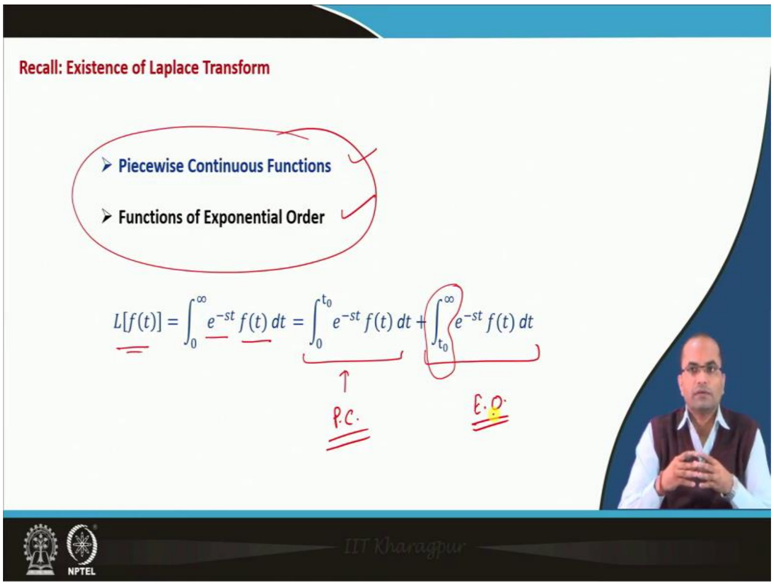
Engineering Mathematics-II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Science – Kharagpur
Lecture -53
Inverse Laplace Transform

So welcome back to lectures on Engineering Mathematics 2. So this lecture number 53 on Inverse Fourier Transform. So in this lecture we will be talking about the Inverse Fourier Transform and some of its property like linearity, etcetera.

(Refer Slide Time: 00:30)

So we will also discuss the uniqueness theorem which has to be discussed now in case of the Inverse Laplace Transform because Inverse Laplace Transform is not unique and then some worked problems will be demonstrated for Inverse Laplace Transform.

(Refer Slide Time: 00:53)

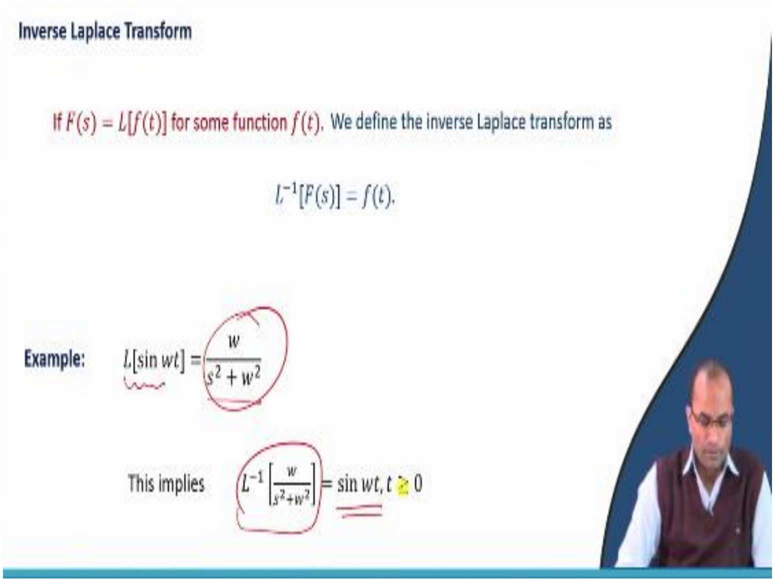


So coming to the existence of Laplace Transform which was the last lecture. We need a function which is piecewise continuous and of exponential order and these two conditions were sufficient for the existence of Laplace Transform. So the idea was we have the Laplace Transform e power minus st ft dt and if we just break into two parts 0 to t naught and then t naught there is some finite number t naught to infinity e minus st ft dt.

So the idea was for the existence of Laplace Transform that this piecewise continuity takes care for the existence of this portion where we have the limits finite and for this portion when we have t naught to infinity then this exponential order, the functions of exponential order takes care for the existence and as a whole we discuss that these two conditions are actually sufficient conditions for the existence of Laplace Transform.

But they are not necessary and we have shown some example where the functions were not piecewise continuous and then the Laplace Transform exist also the function was not of exponential order, but still the Laplace Transform exist.

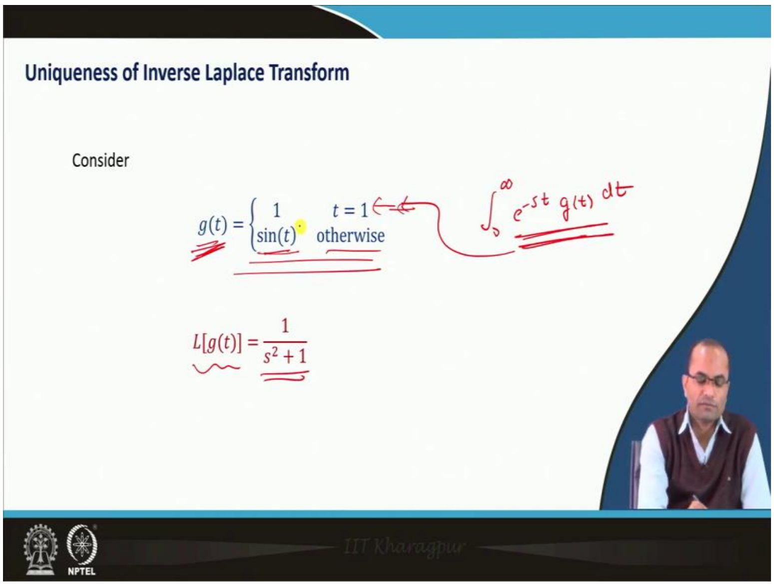
(Refer Slide Time: 02:24)

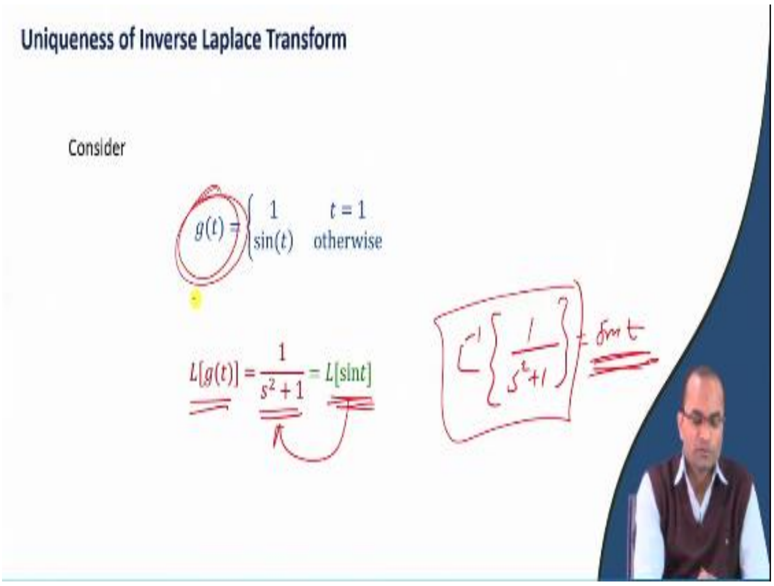


Now we will come to inverse Laplace Transform and the idea is simple that if we have for instance this Fs is the Laplace Transform of ft for some function this ft, then we define the inverse Laplace Transform as Laplace Inverse Transform of this Fs is equal to ft. If we know that the Fs is the Laplace Transform of ft then the L inverse, the Laplace inverse of Fs will be ft.

So that is the simple inversion, for instance we know that the Laplace Transform of sin wt is w over s square plus this w square. In that case we can say that the L inverse of w over this s square plus w square is equal to sin wt for t positive.

(Refer Slide Time: 03:21)





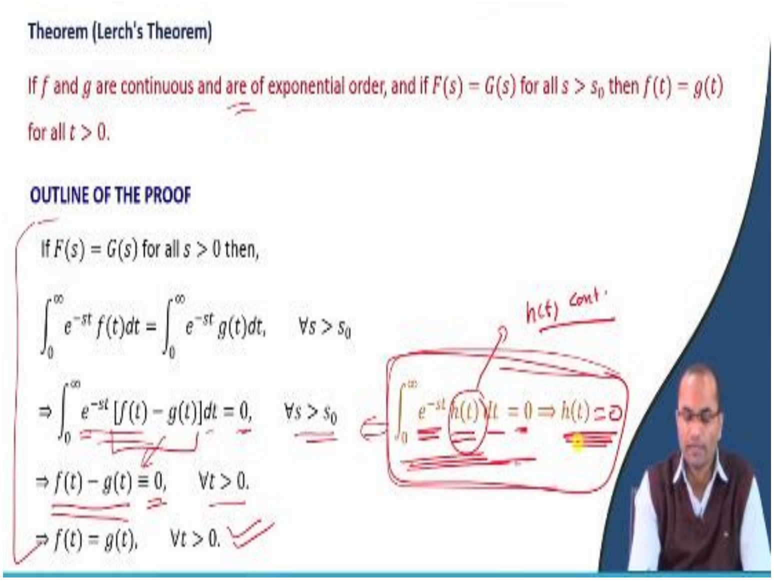
But there is a problem here for instance if we consider the function gt we define as that t is equal to 1 the value is 1 and otherwise it is sin t. So at t is equal to 1 it is not sin 1, but we have defined different value as 1 and other than this t is equal to 1 we have sin t. Actually we can define many such finite points where the value defer for this function gt and otherwise it is sin t.

So if we take the Laplace Transform of this gt which is defined here as t equal to 1 and otherwise sin t in that case we will end up with 1 over s square plus 1 and it is clear because the integral which is 0 to infinity e power minus st and for instance this gt dt, it does not read the value at discrete points. So eventually this is the Laplace Transform of sin t only. So we will get here the Laplace Transform of this gt he same as the Laplace Transform of sin t.

Now having the earlier definition which we have given that if the Laplace Transform of sin t, for example, is s square 1 over s square plus 1, then the L inverse of this 1 over s square plus 1 will be sin t, but here the problem is now we have a gt function which is actually different than sin t. So the Laplace Transform of this 1 over 1 plus s square.

Whether we should say it as sin t or we should say it as gt or we can define several other functions whose Laplace Transform will be 1 over s square plus 1. So in that sense the inverse Laplace is not unique because we have the Laplace inverse of 1 over s square plus 1 sin t we have also the Laplace inverse of 1 over 1 plus s square as gt.

(Refer Slide Time: 05:35)



So there is a theorem here the nice result by this Lerch's theorem. If f and g are continuous and are of exponential order and then if we have this Fs is equal to Gs, if the Laplace Transform are equal for this s greater than s naught then ft would be also equal to gt for all t. So at least we have this uniqueness when we are talking about the continuous function.

So if f and g are continuous and their Laplace Transform is same then we can say that actually these functions are same, that Laplace transform cannot be, the functions cannot be different if their Laplace Transform is same, but this is only true when we are talking about the continuous function. So let us first go through this theorem and then we will have little more discussion on this uniqueness part. So just we will present here outline of the proof.

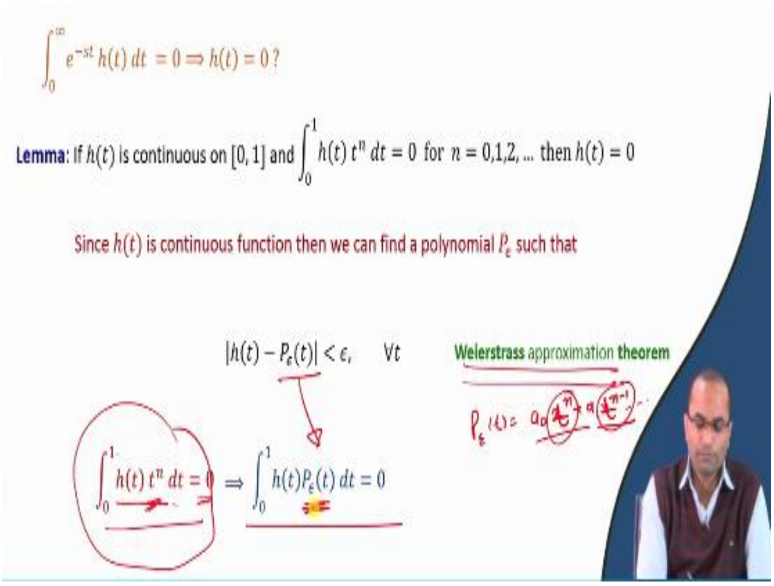
So suppose this Fs is equal to Gs for all s greater than 0 in that case we have these two integrals equal because this is the Laplace Transform of this ft and here we have the Laplace Transform of gt which are denoted by Fs and Gs respectively. So having this equation equality we can bring everything to the left hand side.

Then we have the integral e power minus st ft minus gt and integrated over t from 0 to infinity and that is equal to 0 for all s greater than s naught. Having this now we have a nice result which we will proof in the next slide which says that if we have this 0 to infinity e power minus st ht dt equal to 0 this implies that ht is equal to 0, but we need to proof this result let us first use this result.

And proof this Lerch's theorem later on we will show that indeed this is the case when we have e power minus st ht dt equal to 0 this will imply that ht equal to 0. So here having this result with us we have e power minus st and there is a function here ft minus gt dt is equal to 0 and this holds actually this ht is continuous. So in this case also we have this continuity because f and g are continuous.

So in that case we can say that ft minus gt is identically 0 for all t that means this ft equal to gt for all t. So this is the outline of the proof, but we have to show that e power minus st ht dt equal to 0 implies that ht equal to 0.

(Refer Slide Time: 08:39)



So to prove this result we will consider here this Lemma and using this Lemma we can show that this results holds. So here this Lemma says if ht is continuous on this 0 to 1 in this interval 0 to 1 and when we have this integral now 0 to 1 ht t power t power n dt equal to 0 for all this n 0, 1, 2, 3 and so on that implies that ht equal to 0. So first we will proof this Lemma and with the help of this Lemma we will go, get back to this result.

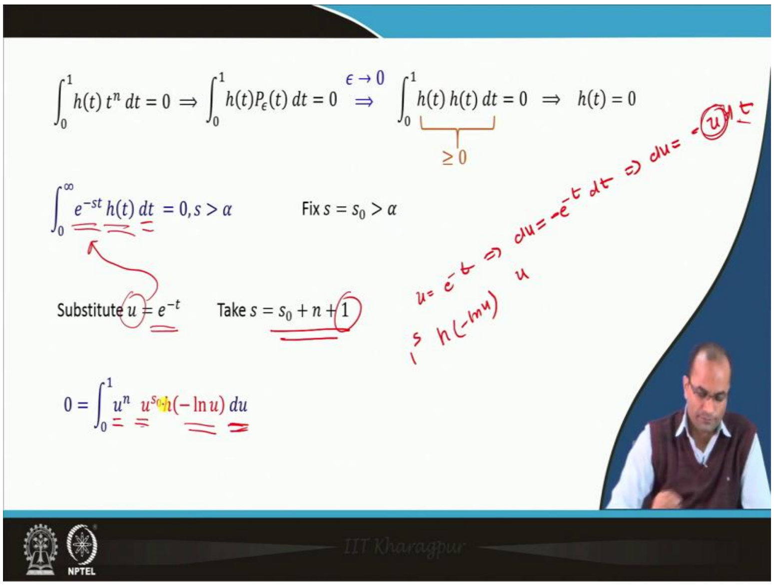
So since here the ht is assumed to be continuous, so ht is continuous function then we can find a polynomial P epsilon such that ht minus P epsilon is less than equal to epsilon. So this is the result which we call the Weierstrass approximation theorem and it says that if we have this continuous function defined on this close interval like here 0 to 1, then we can find a polynomial of as good accuracy as we want.

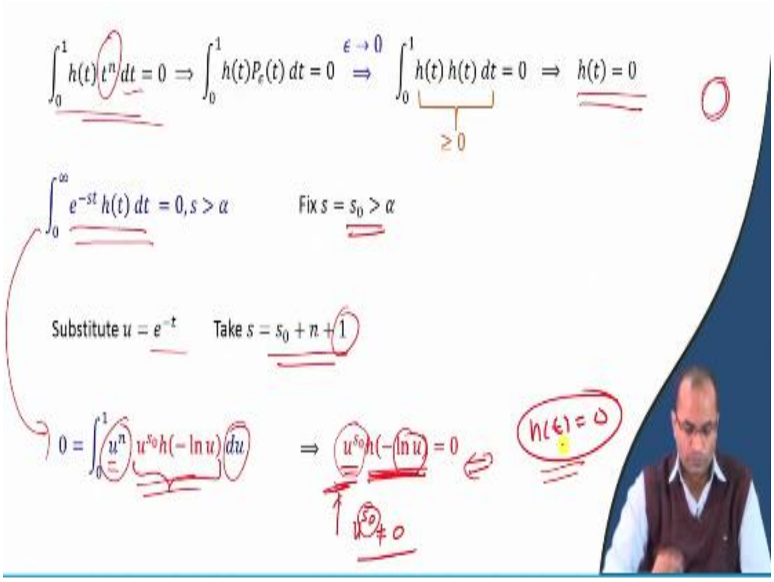
So we can basically approximate given continuous function by the polynomial and the accuracy we can set here as written here epsilon. So for any epsilon we can find the function the polynomial P epsilon which actually the difference from this function is less than the epsilon. So this is so called the Weierstrass approximation theorem and using this now we will move further.

So we have this information that 0 to 1 ht t power n dt is 0 for all n, 0, 1, 2, 3 and so on. So having this result now what we can say that 0 to ht this P epsilon t dt will always be 0 because this is a polynomial and the polynomial will have some form of like a0 then xn then a1 x power n minus 1 or t rather we are talking about t here. The t n minus 1 and so on and then individually we know that this ht t power n is 0.

So we have basically here t power n t power n minus 1 and so on. So this polynomial by this result because for any value of n this is 0 then here we are keeping the polynomial and polynomial will have this linear combination of such terms t power n. So definitely then this integral will be also 0.

(Refer Slide Time: 11:26)





So having this integral equal to 0 what we can now say if we take the limit epsilon goes to 0 so having said that epsilon goes to 0 meaning that P epsilon the polynomial which we have taking for the approximation of the function ht it will become actually ht it will go very close to ht as we take the smaller value of epsilon. So in the limiting case when epsilon goes to 0 this polynomial goes exactly to this ht it will meet with this graph of ht.

That means in the limiting case here when epsilon goes to 0 we have ht and this polynomial will also go to will converge to ht that is the Weierstrass theorem we have seen that this polynomial epsilon goes to when epsilon goes to 0 this actually goes to this continuous function. So we have this ht square dt and which means we have this positive or ht square in the integrant and that will imply that this has to be identically 0 to have this integral equal to 0.

So what we now need to proof that when we have e power minus st ht dt is equal to 0 for all s is greater than alpha then this implies that ht is equal to 0 that we will prove now. So we will use indeed this result that means ht t power n dt equal to 0 implies ht equal to 0. So here we will fix this s we will take s naught which is greater than this alpha. So this integral will be valid for all these s.

And now we substitute u equal to e power t that means we have ln u is equal to minus t or we have basically du also minus e power minus t dt and then so du and e power minus t is u. So this du can be replaced by this minus u dt. So having this and we will take this s as s naught plus this n plus 1. So here we are introducing n we are going to this direction now. So here this result says that 0 equal to this 0 to 1 then the limits here which was 0 to infinity when we have this t naught this will go to 1 and when t is infinity this will go to 0.

So we have this result and this minus will be adjusted to reverse the order of the integration. So then we have here e power minus st and this e power minus t so this will be written in terms of this u then h and t is this minus ln u and then we have this du and there will be factor u over u power s naught when we substitute this in this. So we have this u e power minus t so when we substitute this e power minus t this will be u power this s u power this s.

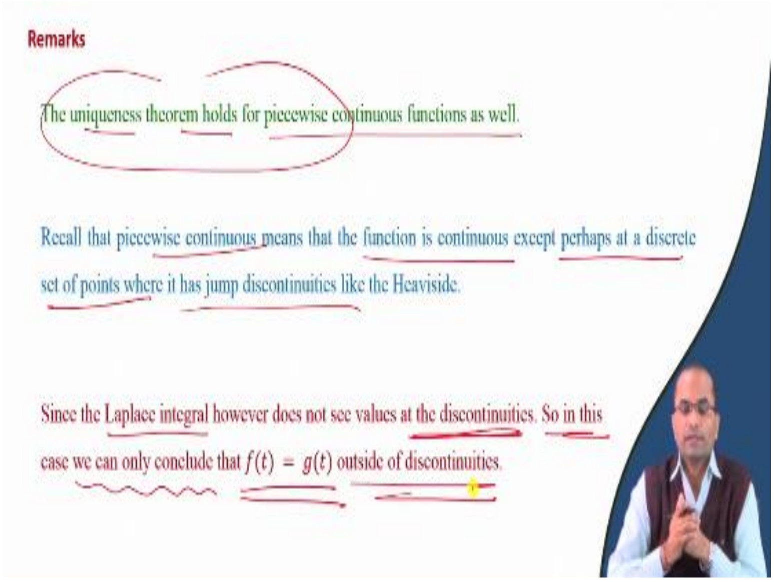
And then we have h and t is minus ln u in the integrant and this du so one u will be there which is coming from this du because the du will be e power minus t dt and then this implies that du is equal to minus sign with u and dt. So this when we replace this dt there will be a u term there so 1 over u, but when we are keeping this u power n and n is taken as s naught plus n plus 1.

So this one power will be cancelled out with this u and then we have u power n and then u power s naught. So having this now this integral which is just after substitution we can get now we look at this form here we have u power n similar to what we have there t power n and then we have some function here some continuous function sitting and then we have this du which was dt there naturally.

So if this result implies this ht equal to 0 identically here also we are getting the same result that means this u power s naught and the h the function of this minus ln u will be equal to 0. So we got and this is naturally here ln u is jut a t there minus ln u is nothing but t. So this ht and this u power s naught u power s naught is not 0 u power s naught is not equal to 0 because u varies from this 0 to 1 and s naught is also a number here greater than this alpha.

So this cannot be 0 only possibility is that this ht this has to be 0 now to make that integral 0. So that is what we have proved now that e power minus std t equal to 0 implies that ht equal to 0.

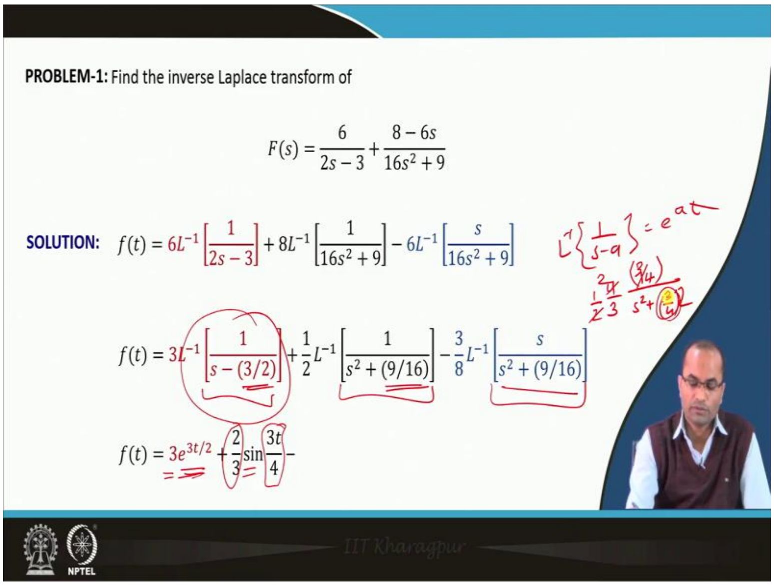
(Refer Slide Time: 17:02)

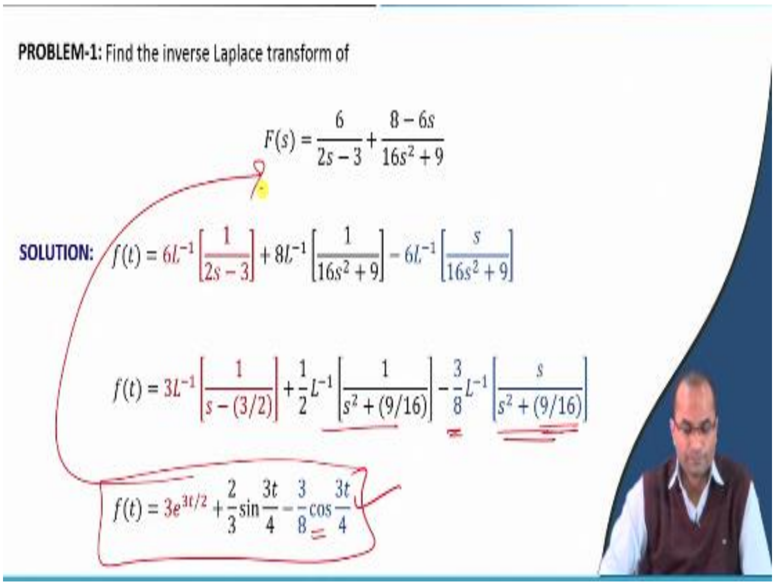


So we have this now some remarks the uniqueness theorem holds for piecewise continuous function as well, but they will be a difference that for piecewise continuous means that the function is continuous except perhaps at discrete set of points where it has kind of jumped discontinuities like this Heaviside function we have seen. So since this Laplace integral does not read the values at the discontinuities.

So in this case what we have? In this case means the case when we have the piecewise continuous function, so in this case we can only conclude that ft will be equal to gt outside the point of discontinuities. So what the result can be also proved formally though we understood this intuitively that when we have two Laplace Transform is equal then the functions whose Laplace are this functions, those functions will be equal other than or outside the points of discontinuity.

(Refer Slide Time: 18:15)





Well, so let us go through some problems based on the inverse Laplace Transform. So if you want to find the inverse Laplace Transform for instance of this function which is 6 over 2s minus 3 and 8 minus 6 s divided by 16 s square plus 9. So looking at the structure here of this Laplace Transform it seems that here we can have some kind of the exponential functions and then here we have this square plus something so cosine or sin type of functions.

So if we take this Laplace we can use the linearity property that means this ft taking the inverse Laplace Transform both the sides. We have the Laplace of 1 over 2 s minus 3 then here Laplace inverse of 1 over 16 s square and then Laplace inverse of s over 16 s square plus

9. So what we will do now in the first case we will take this common so that we have s minus a form.

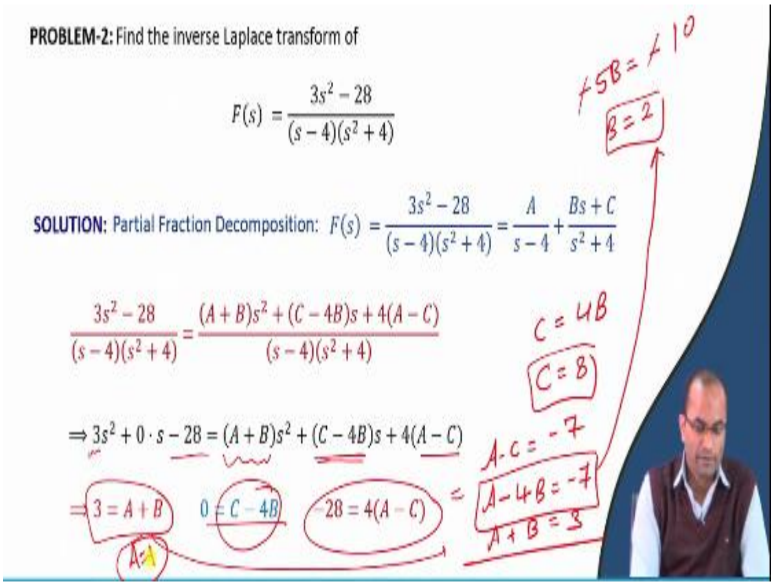
So taking this common here 2 that means we have now 3 there L inverse 1 over s minus 3 by 2. In this case we will take the 16 outside this bracket. So we have 1 by 2 and then L inverse 1 over s square plus this 9 by 16 and in the third case we will take this 16 out so we have 3 by 8 and L inverse s over s square plus this 9 over 16. So we have now the structure which we know from the basic functions that whose Laplace Transform is 1 over s minus s whose Laplace Transform is this s square plus a square form.

And s over s square plus a square form. So having that knowledge we can say now we can conclude that 3 this is the Laplace Transform of e power 3 by 2 t so 1 over s minus a the Laplace inverse it is e power at. So we are writing here e power this a was 3 t by 2 and then in the second case we have here 3 by 4. So we have the situation that the s square plus 3 by 4 whole square.

So we need 3 by 4 also there to have the sin function that means 4 by 3 we have to multiply and 1 by 2 was already sitting there, so that is a 2 by 3 now. So here we have 2 by 3 and then sin of 3 t by because 3 by 4 is there so 354 t. So we are done with this term also here concerning this one we have direct result because s over s square plus a square and the a square is this 9 by 16 that means s 3 by 4 again.

So this L inverse of this that is cos at so we have 3 by 8 and L inverse 3 by 4 t. So that is the function whose Laplace Transform is given as this 6 over 2s minus 3 and 8 minus 6 s over 16 s square plus 9.

(Refer Slide Time: 21:35)



The second problem we will consider the inverse Laplace Transform of this function which is Fs equal to 3 s square minus 28 then we have s minus 4 and then we have s square plus 4. So in this case what we have to do and that is one trick, which we always most of the times it works it is the partial fraction, the decomposition to the partial fractions because we have this s minus 4 and s square plus 4.

So if we can decompose using this idea of partial fraction then we will get something of this 1 by 4 and then s square plus 4 and then we can convert into the known functions the Laplace Transform of known function then we can get back to the original function. So using this partial fraction decomposition what we have that this can be written now A over this s minus 4 this factor and then here we have this quadric terms s square plus 4.

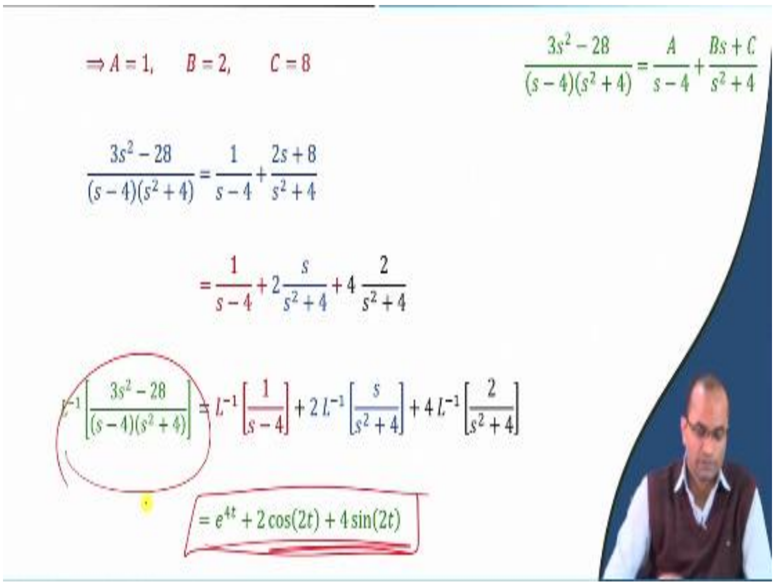
So there will be a linear term there Bs plus C. So A, B, C are constant which have to be evaluated so that this relation holds. Once we are done with the calculation of this A, B and C we can just apply the inverse Laplace both the sides and we can get the desired result. Okay so going to this partial fractions we have this 3 s square minus 28 so here also we make this denominator equal.

So then in the numerator what we will get we have A s square and then will be Bs square term. So with s square we will have A plus B with the s we will here minus 4s and there will be Cs. So C minus 4 Bs that is the term with s and the third one we have the constant term. So there will be 4A and minus 4C, so 4A minus 4C. So we have now the numerator here which has to be equal to the left hand side which is 3 s square minus 28.

So we can compare here because here 3 s square and the coefficient of s is 0 left hand side and then minus we have 28 the right hand side we have A plus B s square then we have C minus Bs and then we have this constant term 4 into A minus C. So by comparing this we realize that this A plus B equal to 3 and here C minus 4B equal to 0 and this A minus C is equal to minus 28 or from here we can say that this A minus C is equal to minus 7.

And then here we have this result that the C is equal to 4 B so we can substitute here indeed 1 minus this 4 B is equal to minus 7 this is one equation in AB and then we have the other equation in AB as A plus B is equal to 3. So solving these equations what we get now so if we substrate this the A will get cancel and we have minus 5 B is equal to minus 10. So here we will get as B is equal to 2. So B is 2 then C is 8 and then from here we can get if this B is 2 then A has to be 1. So we have A1 then B2 and C8.

(Refer Slide Time: 25:22)

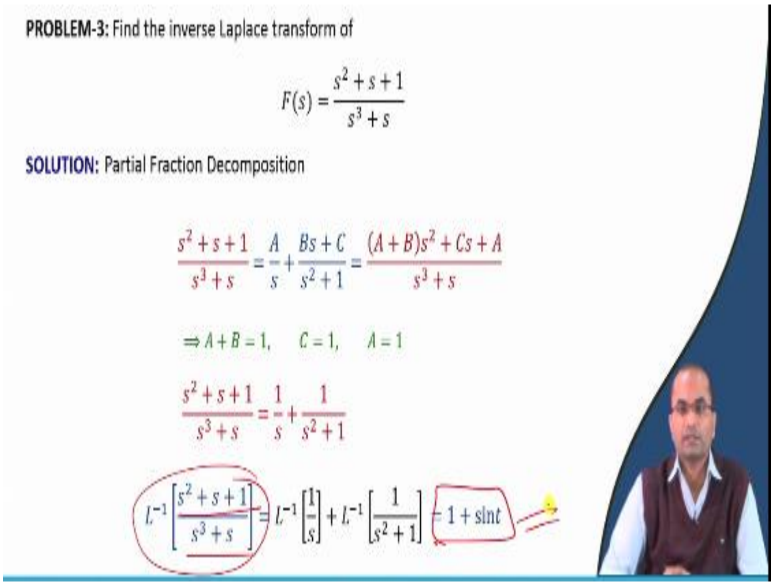


A1 B2 and C8 and this was the idea of the partial fraction that A over s minus 4 here we have Bs plus C by s square plus 4. So having this equal to now we have A so 1 over s minus 4 then B is 2 so 2s plus 8 by 4 and now we are done. So the main work here is to get this partial fraction once we have done the partial fractions it is just a matter of inverting because we know now whose Laplace Transform is 1 over s minus 4.

And here also we can further write 2s over s square plus 4 and then 8 over s square plus 4. So now we can get the Laplace inverse that means the Laplace inverse of 1 over s minus 4 Laplace inverse of s over s square plus 4 and Laplace inverse here 4 into 2 so here we have 4 L inverse 2 over s square plus 4.

So we have the direct results now that e power 4 t will be the function whose Laplace Transform was 1 over s minus 4. Here s over s square plus 4 again non function that is cos 2t. Here we have 2 over s square plus 4 so we have sin 2t. So this e power 4t 2 cos t and plus 4 sin 2 t that is the Laplace inverse of this given function.

(Refer Slide Time: 27:01)

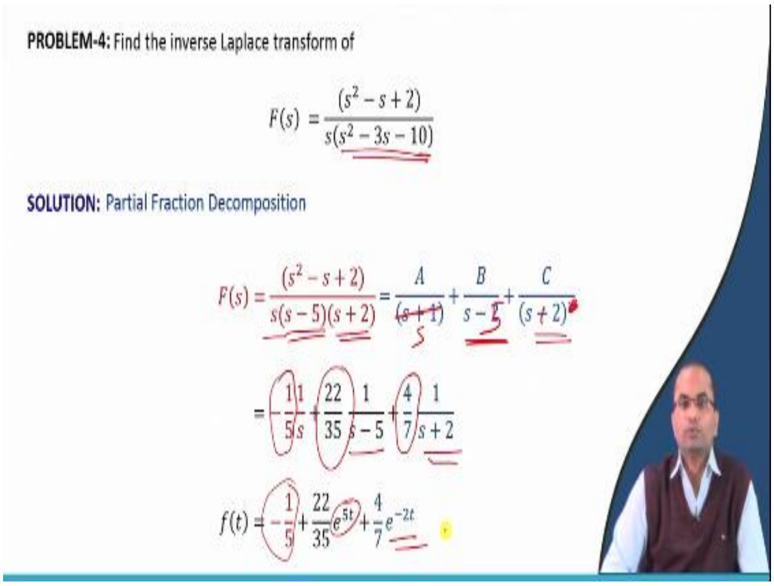


Now we want to find the Laplace inverse of this function s square plus s plus 1 over s cube plus s. Now the idea is same as I said before that we have to do this partial fraction once the partial fractions are done we can simply use the existing result to get the functions back to the t domain. So we have the partial fraction decomposition now again because here we have this s and s square plus 1.

So we can write down then A over s and 4 s square plus 1 again this Bs plus C will come. So here this s into s square plus 1 and then numerator will be A plus B s square plus Cs is equal to A. So again we get the equations here that A plus B is equal to 1 then C1 and A1 and once A is 1 then B is 0 from the first equation.

So having this now we have 1 over s because A was 1 and the B is 0 so there will be s term C is 1 so we have 1 over 1 plus s square. So it is very simple fractions we got here whose transform we know already L inverse 1 over s is 1 and the L inverse of 1 over s square plus 1 will be sin t. So we have the result here that this inverse Laplace Transform of this s square plus s plus 1 over s is cube plus s is 1 plus sin t.

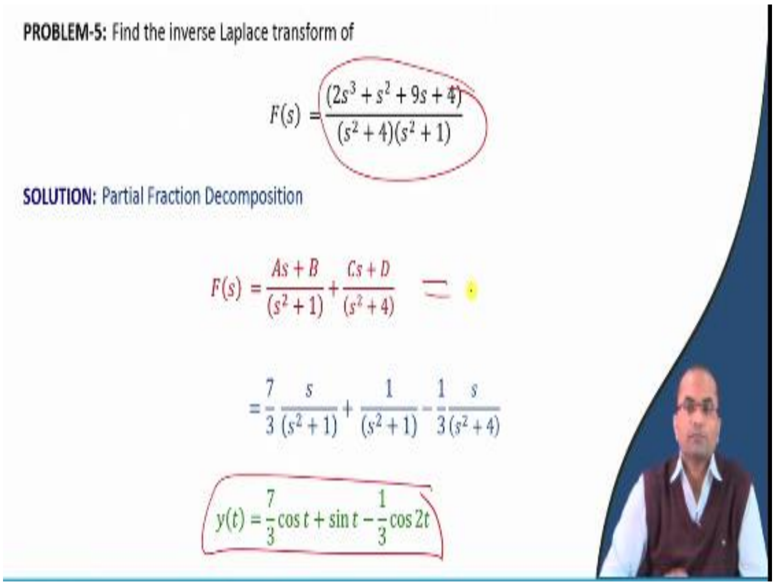
(Refer Slide Time: 28:44)



You want to find now the Laplace Transform of this function so s square minus s plus 2 where s s square minus 3s minus 10. So here also we have to just do the fractions so here s and this is s minus so we can have here, so s minus 5 and s plus 2. So this is the product of s minus 5 and s plus 2 we can factorize that and again we have to go for the partial fractions that means A over this s plus A over s plus 2.

So we have here A over s then B will be over s minus 5 and then C will be over this s plus 2. So by doing this partial fractions we will observe that this A factor is coming as minus 1 by 5 and then B with s minus 5 here it is coming 22 by 35 and with this s plus 2 the C is coming is 4 by 7 and then we can apply the idea of the inverse transform. So we get here 1 minus 1 by 5 and this is 1 only then s minus 5 will give e power 5 t and here we will get e power minus 2t.

(Refer Slide Time: 30:09)



The last problem where we will get this inverse transform of this and here now we have s square plus 4 and s square plus 1 again we need to go for the partial fractions, but this time we will have here also this linear term they are also linear term As plus B and Cs plus D. So in this case we have to compute these four coefficients A, B, C, D. So it could be difficult sometimes to get these partial fractions, but that we have to do to get this inverse.

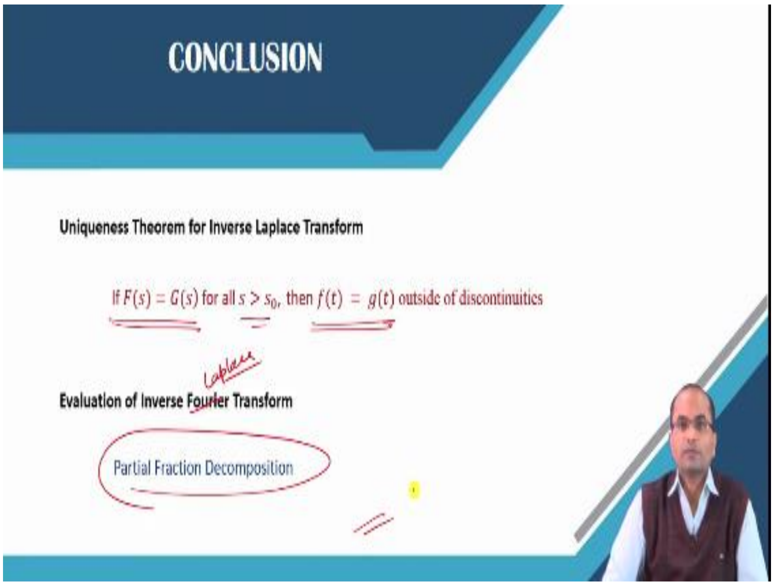
Because directly we do not know how to get it so we do the fractions into the known inverse transform, the known Laplace Transform and then we will take the inverse Laplace Transform. So here after doing the partial fractions it comes out to be that 7 by 3 is with s here over s square plus 1 and then this 1 over s square plus 1 for this B that is B is 1 and here the D is coming to be 0 and C is coming as minus 1 by 3.

And then we can go for the inverse transform because this is just for cos t this is for sin t and then here we have cos 2t. So using the idea of this partial fractions we can easily compute the inverse of some complicated expressions by this partial fractions.

(Refer Slide Time: 31:46)

So these are the references we have used for preparing this lecture.

(Refer Slide Time: 31:49)



And just to conclude we have discussed the uniqueness theorem for inverse Laplace Transform. So which says that if Fs is equal to Gs for all this s greater than s naught in that case at least this ft is equal to gt outside the points of discontinuities and the evaluation of the Inverse Laplace Transform is done based on this idea of the partial fractions. So this is one of the techniques to get the inverse of the Laplace Transform. Well so that is all for this lecture and I thank you for your attention.