Engineering Mathematics-II
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Lecture -52
Existence of Laplace Transform

So welcome back and this is lecture number 52 on existence of Laplace Transform. So in the previous lecture we have seen that some elementary functions, we can use in this integral which is a Laplace integral and a close form of the solution can be obtained or this integral can be evaluated.

So we have evaluated several Laplace Transforms or the Laplace Transform of several elementary functions which include the exponential function, the polynomial so t power n and the constant function one etcetera. So in this lecture we will be talking about in more general context that what are the conditions under which the Laplace Transform exist.

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So we will be talking about the piecewise continuity of the functions because that is required as in sufficient condition, we will be also talking about the functions of exponential order and then finally what are the sufficient conditions for the existence of Laplace Transform.

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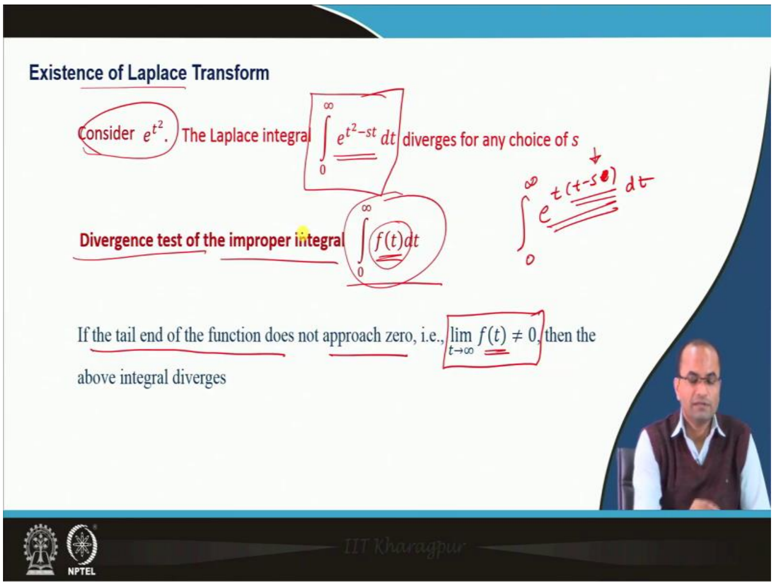

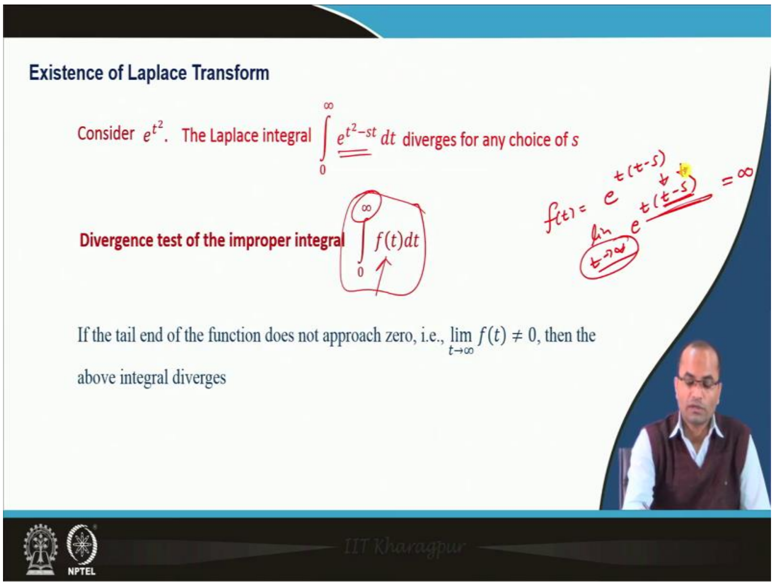

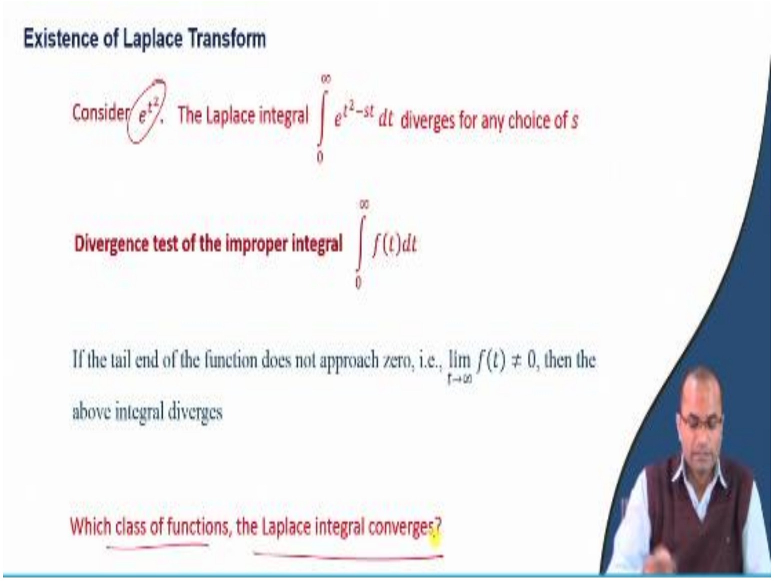

So the existence of Laplace Transform if we consider for instance e power t square such functions e power t square and try to compute this integral this improper integral 0 to infinity e power minus st and e power t square. So we are trying to compute this Laplace integral, but what is happening in this case now because exponential here is like t and t minus st and this integral is over 0 to infinity.

So this integral does not converge for any choice of this s, whatever s, how large s we take here, but so this is t minus s. So whatever value of this s we choose this will never converge. So just to recall what are the conditions, the necessary conditions rather than for the convergence of such integral. So the divergence test we have for such improper integral which says that if the tail end of the function does not approach to 0.

That means if we compute the limit as t approaches to infinity, this ft of the integrant and if this does not go to zero, then this above integral will diverge that is the basic test we have for the divergence of such improper integral so if this ft it does not go to zero as t approaches to this infinity then definitely this integer will diverge and this is exactly happening here. So if we look at this ft I mean the integrant of this e power t and t minus s.

So if we talk about this limit as t approaches to infinity of this e power t and t minus s so whatever value of s we choose when this t approaches to infinity this will go to infinity it will never go to 0 because of this t minus s factor here and t can be very-very large. So t is approaching to infinity. So whatever s we choose there again when this t approaches to infinity this can never go to 0.

So therefore because of this divergence test we know that for any value, any choice of this s this integral, the Laplace integral diverges that means the conclusion is that for this function e power t square we cannot compute the Laplace Transform and that is the point of discussion here that which class of functions we can actually find the Laplace Transform. So which class of functions the Laplace integral converges.

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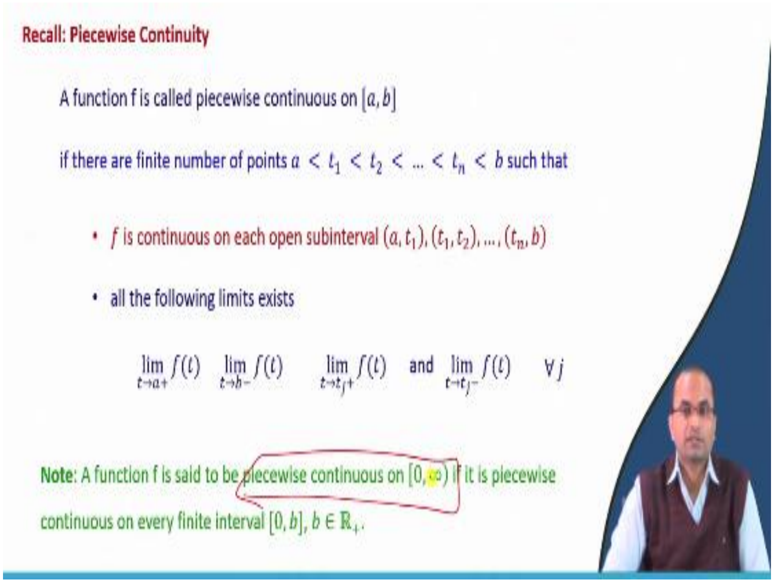

For that we need to just recall the piecewise continuity which was already discussed in earlier lectures on Fourier transform. So a function f is called piecewise continuous where if there are finite number of points t1, t2, tn between this a and b such that f is continuous in each open interval a to t1, t1 to t2 etcetera and moreover the following limits when t approaches to a plus so from the right hand because here we are starting with a only.

So this limit should exist at the other end from the negative side that limit from the left hand side this limit should exist and at all other points both from the right hand from the left hand both the limits should exist for all such points and then we call that the function is piecewise continuous. For the piecewise continuity there are two conditions, one that the function must be continuous in each these open subintervals.

And at all these points where we have either the function is not defined or the function is not continuous, in those cases at least this limit should exist and then we call that the function is piecewise continuous. So a function f said to be a piecewise continuous on this whole interval 0 to infinity if it is piecewise continuous on every finite interval 0 to b and if we can do this for any b from this real positive, then we call that the function f is piecewise continuous on this interval 0 to infinity.

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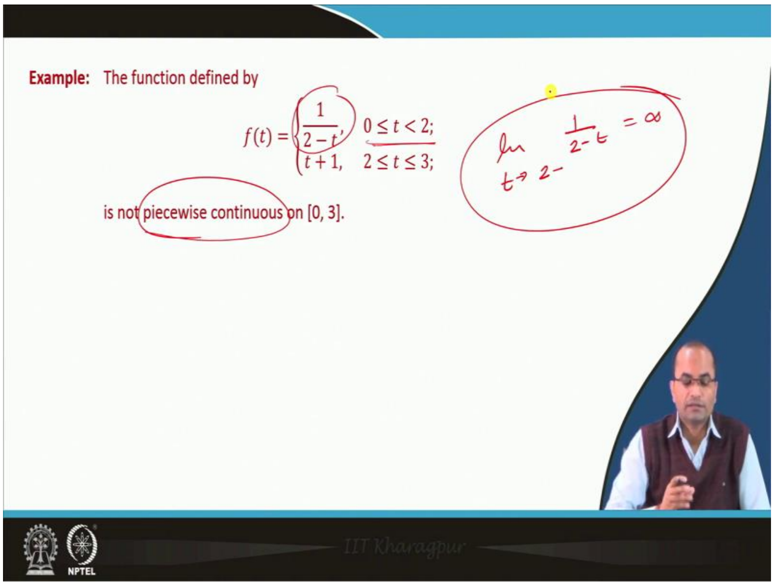

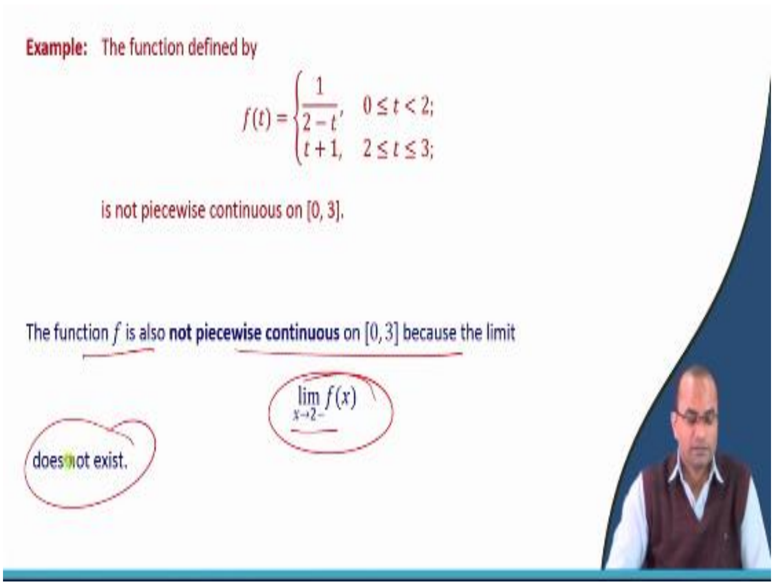

Well so we have this example here the function defined by this ft as 1 over 2 t and the t plus 1 in this interval 2 to 3 and 1 over and 1 over 2 minus t in the interval 0 to 2. So this is not piecewise continuous and the reason why it is not piecewise continuous because if we look at this limit here, when t approaches to 2 from the left hand this is approaching to infinity 1 over 2 minus t, this is approaching to infinity.

So that is the reason we do not have finite limit at some point here at 2, so this function is not piecewise continuous in this interval 0 to 3. Hence the function is not piecewise continuous because this limit here fx x from the left hand does not exist.

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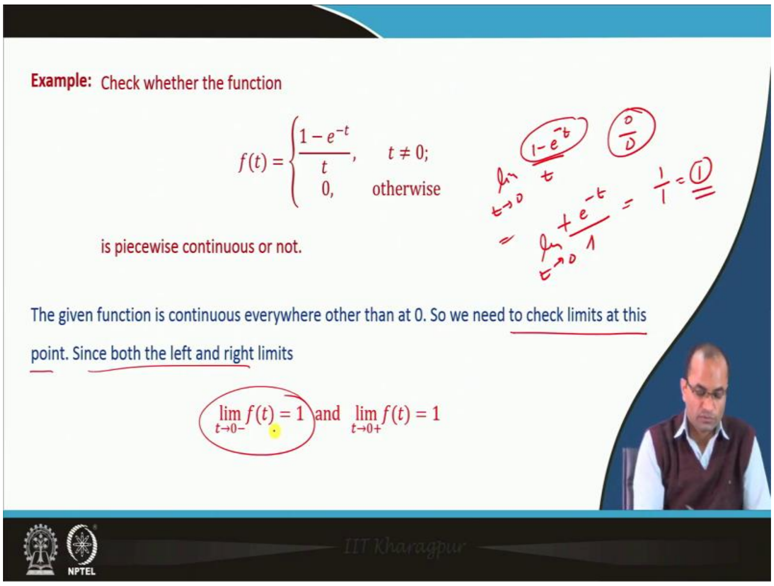

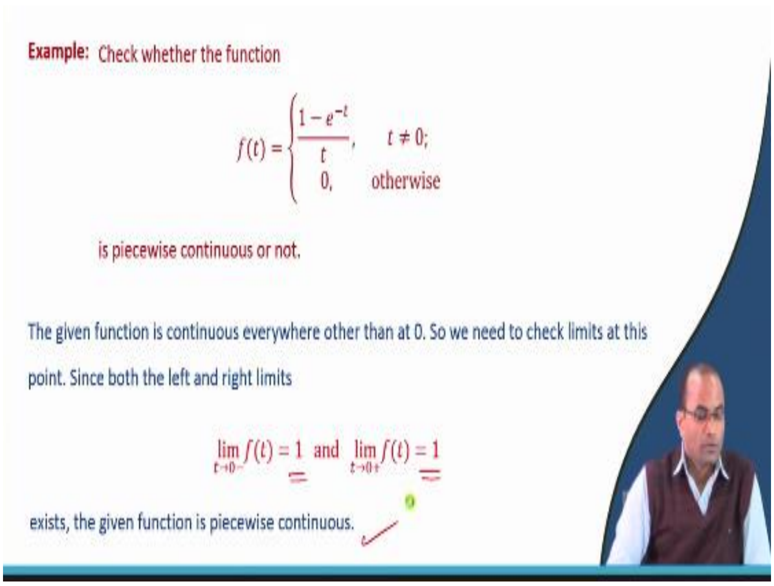

And check now whether this function is piecewise continuous or not because here we have at t not equal to 0 the function is defined as this one and otherwise it is so that t is equal to 0 basically this is 0. So here we may have problem at t is equal to 0. So just to check the given function is continuous everywhere rather than 0 because we have this quotient of two nice functions so it is continuous other than t equal to 0.

So we do not have to worry about the piecewise continuity of the function there, but we need to check this limit point that means at 0 we have to check the limit from both the ends that means but if we do so because this 1 minus et over this tn if we take the limit t approaches to 0 from any end. So this is 0 by 0 form. So we can use the LHopital rule which says that this is minus with plus so e power minus t and we have 1 and then t goes to 0.

So this is e01 so 1 over 1 then we have this 1. So the limit from both the ends it is 1 there so finite limit and at all other points there is no problem as such. So this function is obviously piecewise continuous then in this whole real axis.

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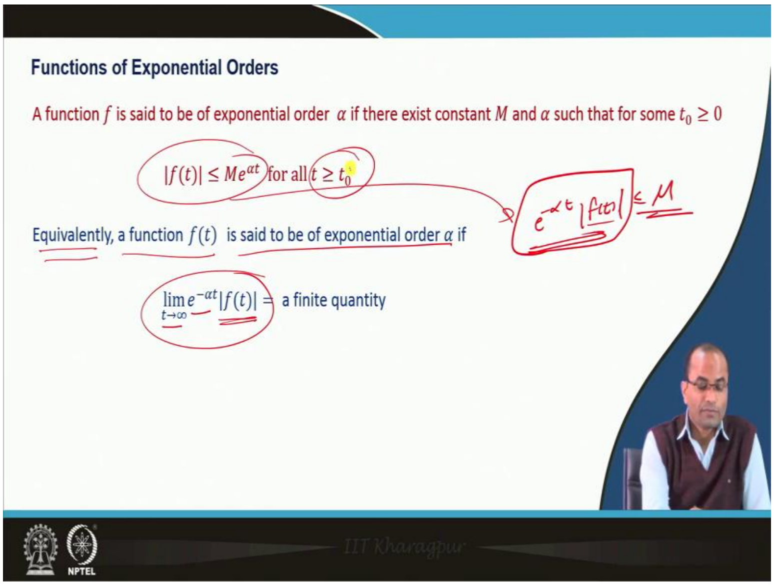

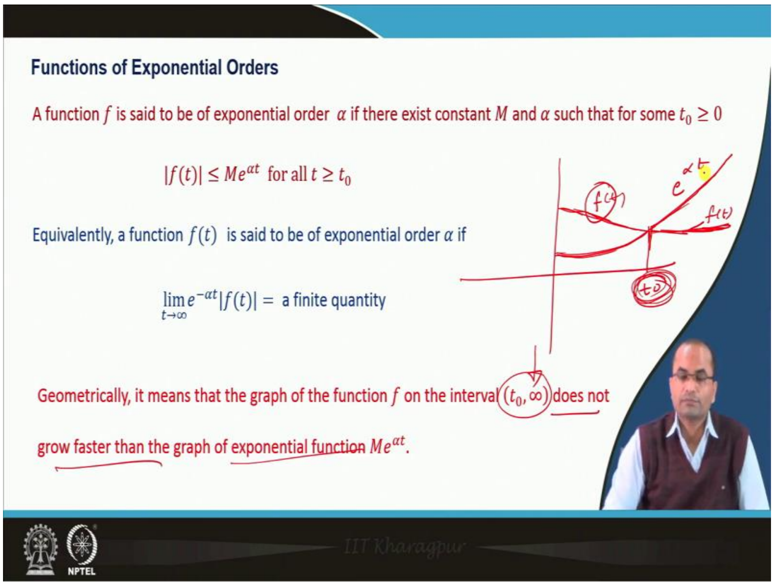

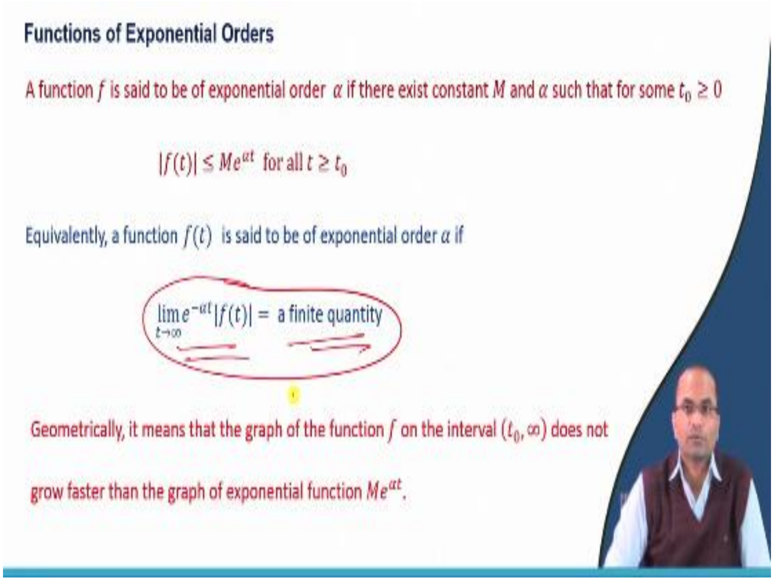

Well the other class of functions you will consider before we go for the final theorem, it is a functions of exponential order so what are these functions. A function f is said to be of exponential order alpha if there exist constant M and alpha such that for any t naught greater than 0 we have this condition that ft, the ft the absolute value is bounded by M, and e power alpha t so the function is bounded by exponential function actually.

So M times the exponential function alpha t for some t greater than t naught, I mean for all, sorry for all t greater than t naught. So if this happens for all t greater than some t naught in that case we call that this function is of exponential order. So if there exist a constant M and this alpha such that for some t, the t naught could be a large number there is no problem, but there should exist a t naught also so that for all t now from that point onwards this ft is bounded by the exponential function.

So in other words or which is a equivalent definition which is rather used for knowing whether it is exponential order or not or testing the function whether it is of exponential order or not. So a function ft is said to be of exponential order if we compute this limit here t to infinity e power minus alpha t ft, if this comes to be a finite number the finite quantity then we call that the function is of exponential order which is exactly equivalent to this one.

Because if we look and e power minus alpha t if you multiply both the sides so we are getting there that this number here e power minus alpha t ft is bounded by this constant M and for all t it should happen for all t greater than t naught. The main issue is when t approaches to infinity because this is where it can actually go above this exponential function, so we want to bound the function, behavior as t approaches to infinity by the exponential function.

And for that we are talking about these exponential order. So our concern is now that what is happening at t approaches to infinity because here we stated here that for all t but the main problem will be just when t approaches to infinity. So actually we are looking of the function behavior when t approaches to infinity. When t is finite there is no issues as such later on we will realize in this sufficient condition where we need the functions of exponential order.

And actually we need to control the behavior of this function as t approaches to infinity and for that we are talking about this exponential order. So if this limit with this exponential minus alpha t, the product with this ft is a finite quantity that means this function ft is of the same order like e power minus alpha t and then we call that the function is of exponential order.

Geometrically it means that the graph of the function f is this interval t naught to infinity. So again the important point is this infinity. So from t naught to infinity does not grow faster than the graph of the exponential function. So if we have this function f which is suppose growing like this and we have the exponential function. So this is exponential function alpha t and this is our function f there.

So what is important that after this t naught, t naught could be very large, but after this t naught this e power alpha t is growing faster than our function which is the function ft the given function ft. So in this interval it does not grow faster than the graph of this one. So after here in this range from 0 to t does not matter the exponential function was below the function ft, but after some point here t naught, it could be very large number, but still if we, as I said before we are talking about its behavior at infinity.

So if we can have a large number t naught also where we have this condition that from that t naught onwards the exponential function is the graph of the exponential function is above the graph of ft. So that is what we have the functions of exponential order, but for testing the functions of exponential order this definition will be a very useful to get this limit and if it is finite quantity we say that the function is of exponential order.

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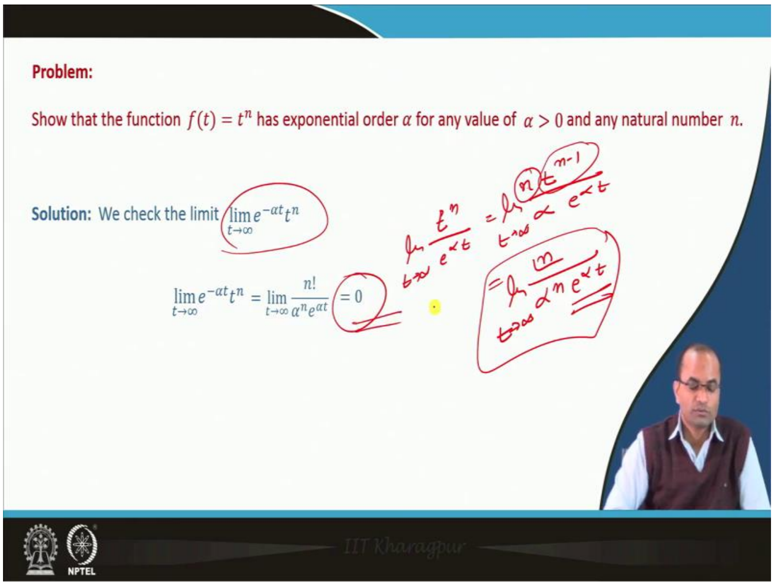

So now for instance we can show here that t power n these functions for any value of n has exponential order alpha and alpha can be anything positive and this holds for any natural number n. So this t power n is an exponential order alpha, alpha positive which we can show with this help of this limit. So we will take this limit t to infinity e power minus alpha t with this t power n.

And now here we see that if we write this t power n and e power this alpha t so this is going to infinity that means and this is also going to infinity. So we have the situation infinity divided by infinity. So we can use LHopital to compute this limit and it turns out to be because we have here the t power n and then e power alpha t and we keep on applying the LHopital rule. So in the first occurrence we will get n and then tn minus 1 and here we will get alpha.

And e power alpha t and then we have to again apply the LHopital rule because the same situation infinity over infinity is happening there. So when we keep on doing this here we will get n n minus 1 and so on at the end the factorial n and then this t will disappear there and here we will get this alpha power n and again this e power alpha t and we have to consider again t approaches to infinity.

So in this as well when t approaches to infinity this will go to 0. So we have this limit here is equal to 0 as t approaches to infinity. So hence this function is of exponential order that is clear now from here.

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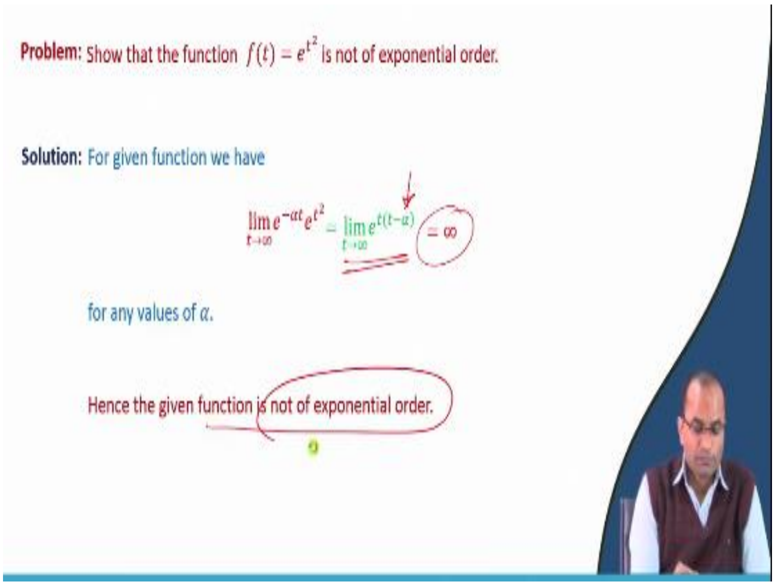

And for instance this e power t square which we have also discussed before that the Laplace integral does not exist if we take this function e power t square. So in this case we will show that actually this function is not of exponential order. So for this given function if we try to compute e power minus alpha t with this e power t square, then what is happening here e power t and this t minus alpha and we are letting this t to infinity.

So whatever alpha we take, however large alpha we take this will go to infinity only. So it is not going to any positive, any finite number it is going to infinity for any value of alpha and therefore this function is not of exponential order. So the given function is not of exponential order.

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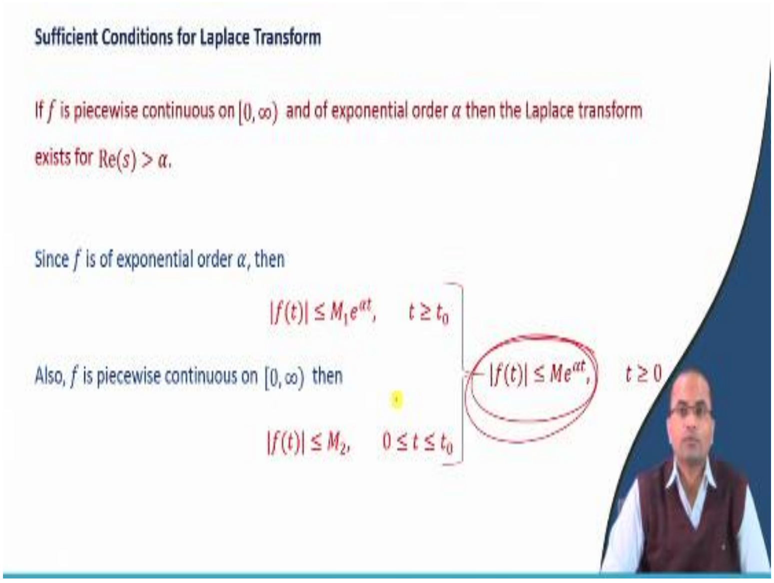

Now we are coming to the main result of this lecture and that is the sufficient conditions for Laplace Transform. So what we have if f is piecewise continuous that is a first condition here the piecewise continuity and the second condition is the exponential order alpha. So if we have these two conditions or a function satisfy these two conditions piecewise continuity and exponential order alpha.

Then the Laplace Transform exist for this all real s greater than exactly this number, this order alpha. So that is the most important theorem which tells about the sufficient conditions, these are not necessary conditions. We will be discussing later on what do we mean by necessary sufficient. So here under these two conditions or these two conditions are sufficient to find the Laplace Transform of a function.

And we can look at the short proof also. So since this f is exponential order alpha and we know as per the definition what do we mean by the exponential order alpha that means this ft can be bounded by M1 e power alpha t for t greater than some t0 naught, this was the definition of the exponential order and the function is also piecewise continuous.

So in this case if the function is piecewise continuous on the interval 0 to infinity then we know that we can bound this function ft by some number in this interval 0 to t naught because that is the beauty of this piecewise continuous function all these limits exist at all the points where the function is discontinuous.

So we can actually bound the function whatever t naught we are talking about there the function value we can bound by M2. So that is coming because of this piecewise continuity. And then we have because of the exponential order we have t greater than t naught this M e power exponential. So it is bounded there when t is greater than t naught by the exponential function and in the range 0 to t naught it is bounded by this M2.

So we can combine these two inequalities because one says that the function is bounded by this exponential function when t is greater than t naught, the other one says that the function is bounded in these intervals 0 to t naught. So combining the two we got at the absolute value of this ft is bounded by M and exponential alpha t because here it was anyway bounded by, it is a bounded function for 0 to t naught.

So that is not the concern, but for t greater than t naught it is bounded by the exponential function for all t actually this is bounded by this M e power alpha t because of this two reasons. We can find this M accordingly based on this M1 and M2. We can choose for instance the maximum of the two then this M will serve the purpose that it will be satisfying this inequality as well and naturally this inequality.

So we have the function f which is bounded by M e power alpha t in the whole range t greater than 0 and in the range 0 to t naught actually this helped the piecewise continuity was responsible to have such an inequality and in the range from t greater than t naughr this exponential order helped us to get this inequity that ft is bounded for all t greater than 0 by M e power alpha t. So having this result having this bounded result on this ft.

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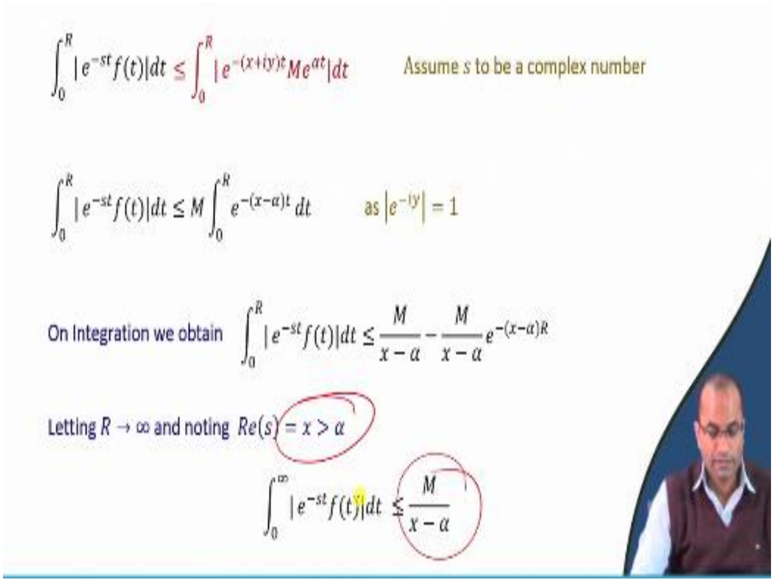

Now we can discuss with the absolute value this Laplace integral and assuming that s is a complex number, so we can consider more general case. So we have e power minus x plus yt and then here this ft is a place by M e power alpha t because in the whole range 0 to R this is bounded by M e power alpha t. So we can bring this M outside and then we have a positive there the exponential functions, we have e power minus x minus alpha t.

And the absolute value of e power minus iyt this will be 1. So here we have this result now and now on integration of this because we can integrate this we are getting here M over this x minus alpha and M over x minus alpha with this e power minus x minus alpha R. So and now we will let this R to infinity to get exactly the Laplace integral. So we let R to infinity and we note that the real s is greater than this alpha.

So assuming that this x is greater than alpha we have this positive number here with this R and if R is approaching to infinity this will go to 0 and hence we will get only this M over x minus alpha. So this absolute value of this Laplace integral with the absolute value of the integral what we have this M over x minus alpha. So this is bounded and that is what we want to show that this Laplace integral exist because this exist now with the absolute value indeed.

So naturally without absolute value this will exist where that value will be smaller than this value. So here what we have shown that under these two conditions that means the piecewise continuity and the functions of exponential orders. If we have these two conditions this such integral can be bounded by this M over x minus alpha for this real s greater than alpha real s greater than alpha under these conditions. So we have this M over x minus alpha the integral is bounded meaning the integral, the Laplace Transform of such functions exist.

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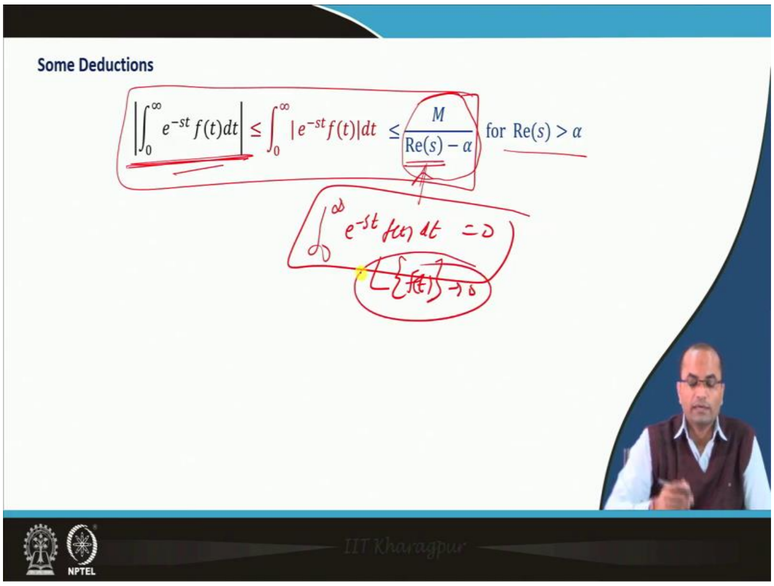

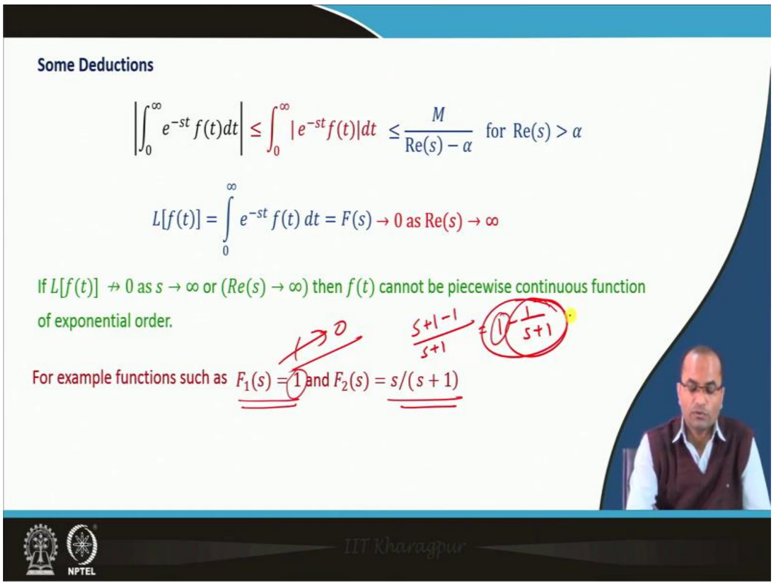

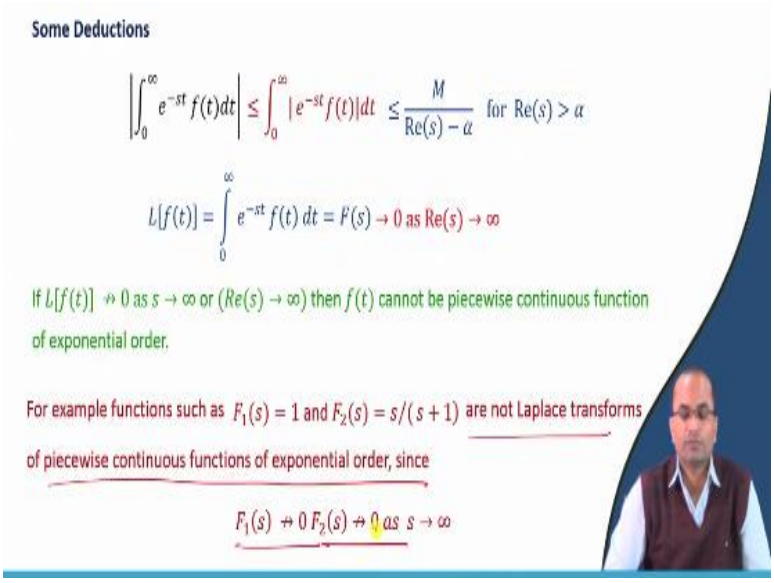

Well so some deduction from this derivation, so we have seen that this 0 to infinity power minus st ft dt which can be bounded by we can bring this absolute value inside and we have seen that this is bounded by M over real s minus alpha that was the last conclusion where we have concluded that the Laplace Transform exist. So here with this inequality what we can deduce now that when this real s goes to infinity for instance this is M real s minus alpha.

So if we let this s to infinity this Laplace integral is bounded by 0 that means the integral 0 to infinity e power minus st ft and dt is equal to 0. That means the Laplace Transform of ft will go to 0 when this real s or if we are talking about only the real number that means s tends to infinity. So that is a nice deduction here that if we have the functions of exponential order and piecewise continuous there Laplace Transform will go to 0 as real s goes to infinity or s goes to infinity in case of real s.

So since we have this Laplace Transform is Fs what we notice from here that it has to go to 0 if this real s goes to infinity that is the deduction coming from this above inequality. So if we find that Laplace Transform of ft does not go to 0 for instance as s goes to infinity or real s goes to infinity in general case then the ft cannot be piecewise continuous function of exponential order.

So we should understand this because we have proved here that if f is exponential function of exponential order if f is in piecewise continuous function of exponential order. In that case we have deduce here that the Laplace Transform will go to 0 as real s or s goes to infinity, but if we notice in some problem that the Laplace Transform for ft does not go to 0 as s goes to infinity.

Then we are sure that ft cannot be the piecewise continuous function, I mean the function whose Laplace Transform is not going to 0 that ft cannot be piecewise continuous function of exponential order that is what we can definitely deduce if we find that the Laplace Transform is not going to 0 as s approaches to infinity. So, for example, the functions like Fs is equal to 1, we will see later on that.

We have the Laplace, we have some functions whose Laplace Transform is 1 for instance or s over s plus 1. In either case so this is also not going to 0 as s approaches to infinity because this is a constant value 1 and here also it is s over s plus 1. So we can write s plus 1 minus 1 over s plus 1 so this is 1 minus 1 over s plus 1. So if this s goes to infinity here also this is not going to be 0 it is going to 1.

So in either case here this is given 1 and this is going to 1 not going to 0. So these two functions the functions whose Laplace Transform is given as F1 s and F2 s they cannot be piecewise continuous of exponential order this is what we can deduce and later on we will see that indeed those functions whose Laplace Transform are given by these functions are not piecewise continuous of exponential order.

So if such as this are not Laplace Transform piecewise continuous function of exponential order because in either case these F1 s and F2 s they are not tending to 0 as s approaching to infinity.

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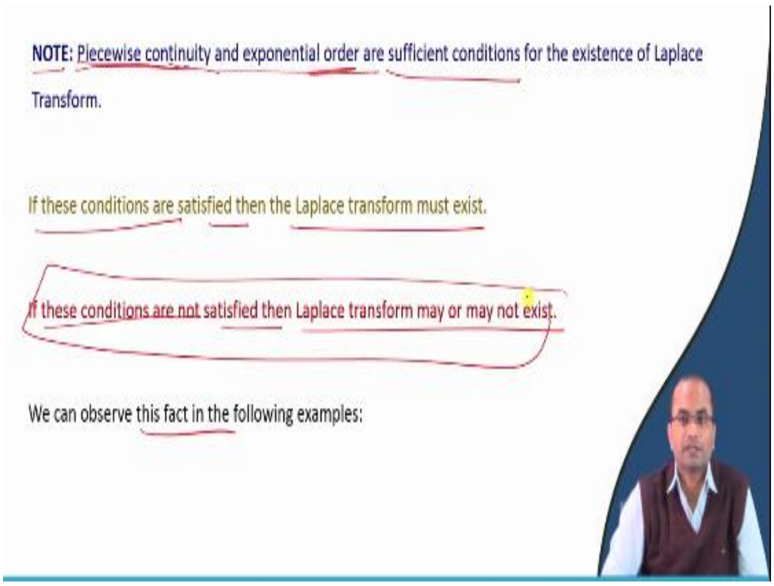

Now just a note here that this piecewise continuity and this exponential order, these two conditions, they are sufficient conditions for the existence of Laplace Transform, they are not necessary conditions for the existence of Laplace Transform. So what do we mean by this. If one of or both the conditions are not fulfilled by a function, it can still has a Laplace Transform that is what we mean by sufficient conditions.

So under these conditions definitely the Laplace Transform will exist so these are more than what we actually need, but we do not know the necessary and sufficient conditions we know only the sufficient conditions under which we are sure that the Laplace Transform will exist, but if there are functions if they are not piecewise continuous or they are not of exponential order is still the Laplace Transform may exist.

So if these conditions are satisfied then the Laplace Transform must exist, if these conditions are not satisfied then Laplace Transform may or may not exist. We can observe this fact with the following examples, so we have two examples now which can tell little more about the sufficient conditions and the remark we have made here.

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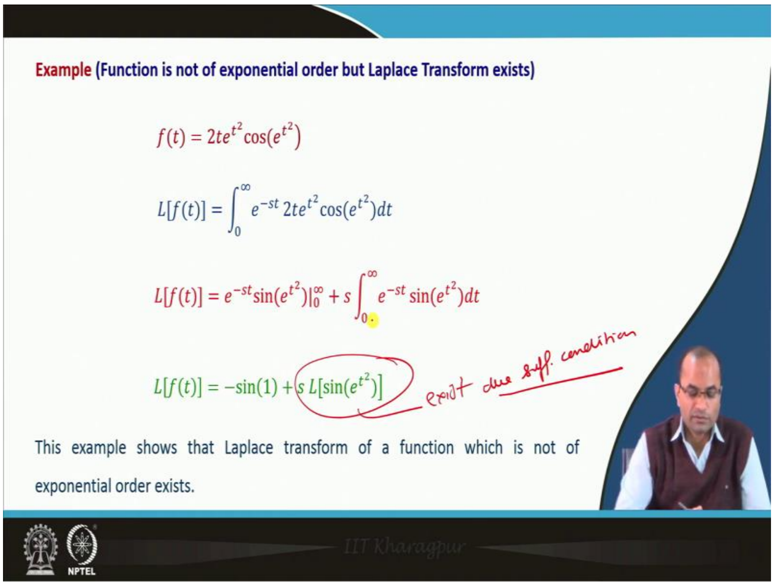

So first example we will take where the function is not of exponential order, but Laplace Transform exist. So here in this case for instance if we take ft as 2t power t square and we have here cos e power t square. So we have already seen the cos e power t square is coming here and this cannot be bounded by the exponential function of order alpha. So here this because of the presence of e power t square this function is not of exponential order, not of exponential order.

So we have this example here which is not of exponential order and now if we compute this Laplace Transform that means e power minus st 2t et square and cos e t square dt. So we can integrate this by parts so e power minus st and then here this cos will become the sin and this e power t square is sitting, the derivative is sitting there. So we can integrate this easily to have the sin e power t square.

The limits 0 to infinity and then we have this s there because this has to be differentiated so e power minus s and e power minus st and then minus s so that minus s and this minus will become plus. So we have e power minus st and then sin e power t square. So what we note now, so we have Laplace of ft this is when t approaches to infinity this will go to 0 and when this t goes to 0 we will get just minus sin 1.

So we have minus sin 1 because of this first term, the second term s the second term and you notice that this is the Laplace Transform of this function e power t square because e power minus t minus st and then we have here a sin et square. So this is the Laplace Transform of this sin et square where product with this e power minus st and then we have s there. So Laplace of ft is minus sin 1 s the Laplace of sin e t square.

So what this example shows? The Laplace of this et square exist because of our sufficient conditions, this sin e power t square is a continuous function and bounded function so it is a piecewise continuous and of exponential order function and the sufficient condition says that it is Laplace Transform will exist. So this exist because of due to the sufficient conditions the function is indeed continuous and has piecewise continuous.

And the function is bounded so hence it is of exponential order. So the second term has the existence and first is minus sin 1 that means the Laplace Transform of ft exists. So this example shows that the Laplace Transform of a function which is not of exponential order this exist.

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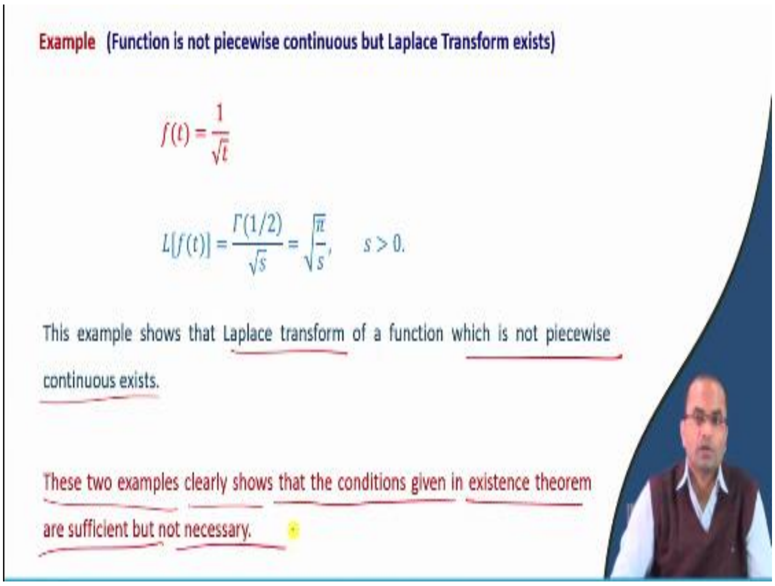

The second example we will show that this is not piecewise continuous, but the Laplace Transform exist. For example if we consider ft 1 over square root t and as t approaches to 0 we know that this will go to infinity. So this ft is equal to 1 over t this limit does not exist hence it is not a piecewise continuous function, but the Laplace Transform for ft which we have already derived this formula it is t power minus half and that says the gamma half over this square root s.

So we have this square pi and the gamma half is square root pi over this square root s for s positive. So we have the Laplace Transform for this function which is not piecewise continuous. So again this piecewise continuity and the exponential order these are the sufficient conditions, these are not necessary condition for the existence of Laplace Transform.

So this example again shows that the Laplace Transform of a function which is not piecewise continuous and it is Laplace Transform exist. So these two examples we have or clearly shows that the conditions given in exercise in this existence theorem that means piecewise continuity and exponential order they are sufficient, but not necessary conditions.

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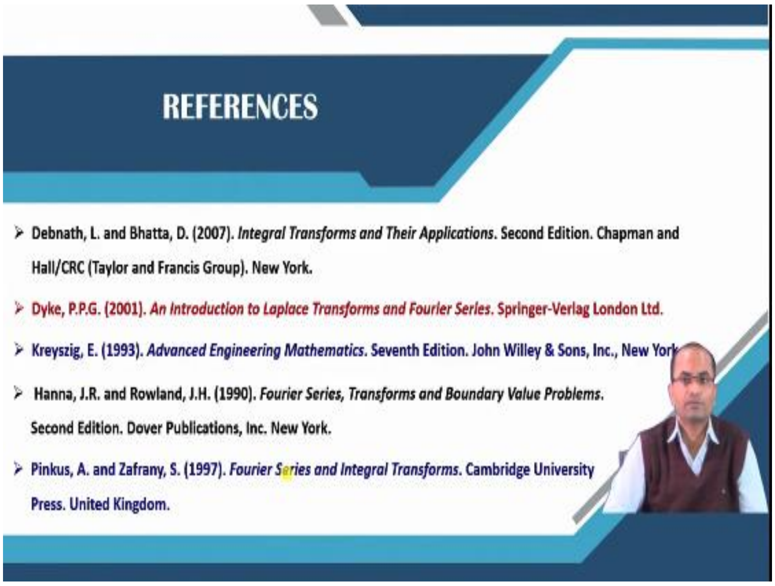

Well so these are the references we have used for preparing this lecture.

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And now coming to conclusion. So we have discussed in this lecture sufficient conditions for the existence of Laplace Transform and in particular the piecewise continuity of the function and the functions of exponential order they come to this class where we can find the Laplace Transform, but just remember that these are the sufficient conditions, the piecewise continuity and this exponential order, these two are sufficient conditions for the existence of Laplace Transform but they are not necessary.

If a function fails to have one of these conditions whether the function is not continuous or the function is not of exponential order is still we can find its Laplace Transform and we have seen indeed two example the one function which was not piecewise continuous and we are able to find the Laplace Transform. In the other case the function was not of exponential order and we have seen the existence of Laplace Transform. So that is all for this lecture and I thank you for your attention.