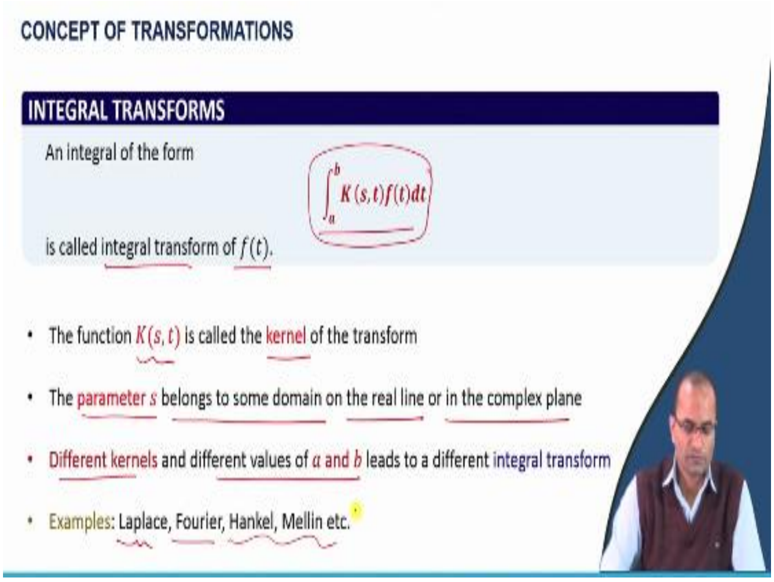
Engineering Mathematics-II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Science – Kharagpur
Lecture -51
Laplace Transform of Some Elementary Functions

Welcome back to lectures on Engineering Mathematics 2. So this is lecture number 51 on Laplace Transform and today we will be talking about the Laplace Transform some elementary functions.

(Refer Slide Time: 00:25)

So first after introducing the Laplace Transform which is similar to what we have done in the Fourier transform. So Fourier transform and its applications is already done in previous lectures. So today this is a new transformation where we call this Laplace Transform and in this lecture we will be talking about how to find the Laplace Transform of some elementary functions.

(Refer Slide Time: 00:51)



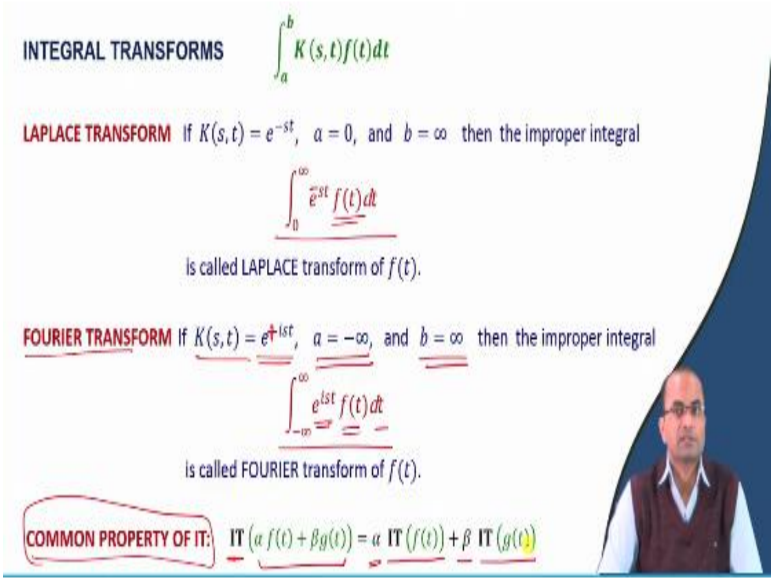
So in general the concept of such transformations, though we have already discussed the Laplace Transform. So just to review again this concept. So we have an integral of this form. So some a to b and then there will be a function K of s and t and then ft dt. So this function ft, when we integrate over this t with the help of this K so called kernel, we get thus the function was in t and now we will get something in terms of s.

So this is transform to other variable s and this is the general context of integral transform, one of the example we have already studies that was the Fourier transform. So in general when we have such transforms through this integral we call them as integral transform of ft. So here in this context, this K st is called the kernel of the transforms for different functions, for different kernels we have different transforms.

And the parameter this s which belongs to some domain on the real line or in the complex plane and different kernels and different values of a and b leads to a different integral transform and the examples of variate transforms are like Laplace Transform, which we will study now in this lecture.

We have the Fourier transform which we have already studied, but there we did not directly introduce the transform, the Fourier transform through such integrals. But there was a theoretical background where from the Fourier series it is integral representation and then we have introduced the Fourier transform. Then we have other transforms as well, the Hankel transform, Mellin transform, etcetera.

(Refer Slide Time: 02:55)

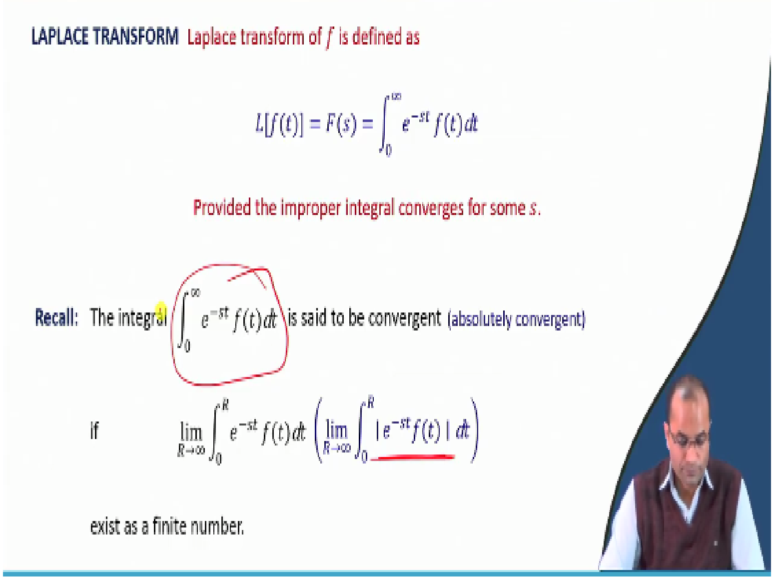


So these integral transform coming back to the Laplace Transform again that is the main topic of this lecture. So if we take here this K start, this kernel e power minus st and if we set a to 0 and b to infinity then this improper integral that means 0 to infinity e power start, e power minus st and ft dt, this is called the Laplace transform of ft and for the Fourier case since we have already done this we can relate now.

So this K st is a function here e power minus st where a was taken as minus infinity and b was taken as plus infinity and then improper integral again, I mean depends sometimes we take the plus sign there or sometimes we take the minus sign and accordingly the inverse was adjusted. So here let us go with the positive sign, so e power st ft dt and this integral over t from minus infinity to plus infinity is called Fourier transform.

And the common property which we have already discussed for Fourier transform and all these integral transform have this common property that is the linearity. So if we apply this integral transform to such a linear combination alpha ft plus beta Gillette, then we will get alpha, the integral transform of ft plus beta and integral transform of the function gt. So this is the general context of these integral transforms.

(Refer Slide Time: 04:41)



And now we will focus on Laplace Transform again. So the Laplace Transform of f as discussed before is defined as e power minus st ft dt and this function because this is a function of s, we usually denote this by Fs and this L symbol is used for the Laplace. So the Laplace of ft e power minus st ft dt and this integral as a function of s we name it as Fs.

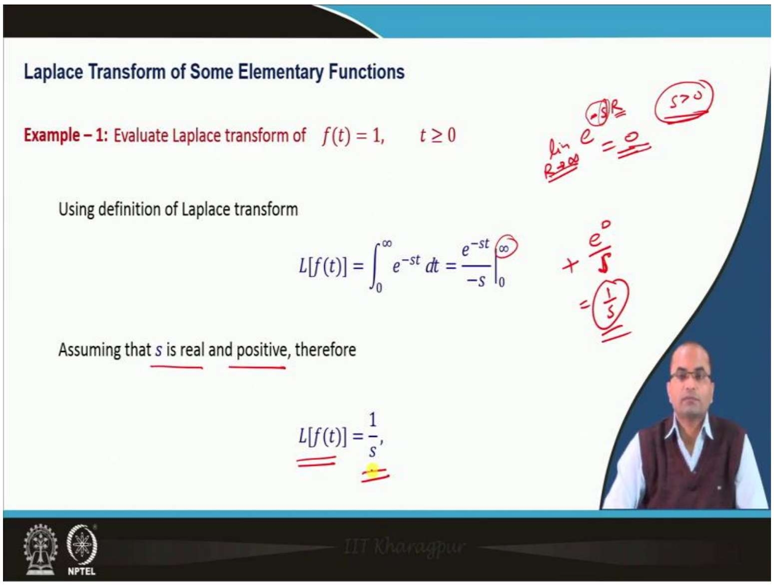
So naturally the point is that since we are talking here this improper integral so there is a issue of convergence. So this makes sense when the improper integral converges for at least some interval of s, this parameter s because this depends on this s not for all values of s this integral make this, but for some s this integral may exist and if it exist for some s there is no problem.

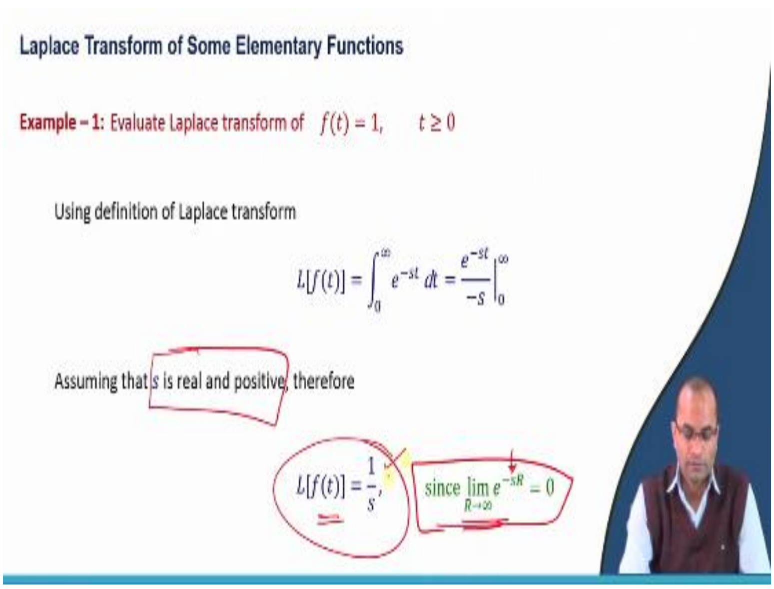
So, still we can define this Laplace Transform and then we have to mention that the range of s together with this Laplace Transform. So there is a always issues about this because improper integral and the convergence has to be discuss there and just to recall that an integral of this type, an improper integral of this e power minus st ft dt is said to be convergent if this limit, so if we replace this infinity by R there.

And then compute this integral and later on we can set this limit R to infinity and if this limit exist then we call that the improper integral is convergent if this limit exist as a finite number. So there is another definition which we use for absolute convergent. So this integral converges absolutely when we have this limit R to infinity with the absolute value of the integrant.

So what is the difference between a simple convergence and the absolute convergence? In absolute convergence we are talking about the absolute value of the integrant. So if this converges which is more stringent than this one, because now we are talking about the only positive values, so here if this converges with the absolute value of the integrant then we call that this integral, this improper integral converges absolutely.

(Refer Slide Time: 07:30)





Well so having this we can, because we know the definitions. So let us go with some elementary functions which can be integrated easily and then later on in the next lecture possibly we will be talking about more on convergence and the existence of the Laplace Transform. So now we will continue with some example, which are trivial, which are simple.

But once we know the Laplace Transform of these elementary functions, then with the help of these elementary functions we can actually derive Laplace Transform of more complicated functions so that we will see later. So here in this example we want to evaluate for instant this ft is equal to 1, the constant function 1 which is defined for all t positive.

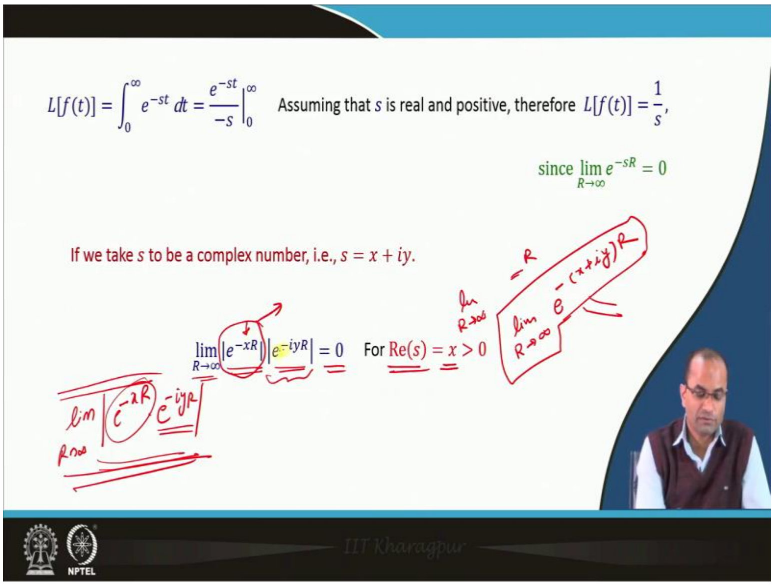
So using the definition of the Laplace Transform what we have here the Laplace Transform of ft is e power minus st and the function which is 1 there. So naturally this can easily be integrated and we have e power minus st over this minus s and then the limit from 0 to infinity. So now the question is that what will happen when we take this infinity that means we are interested here.

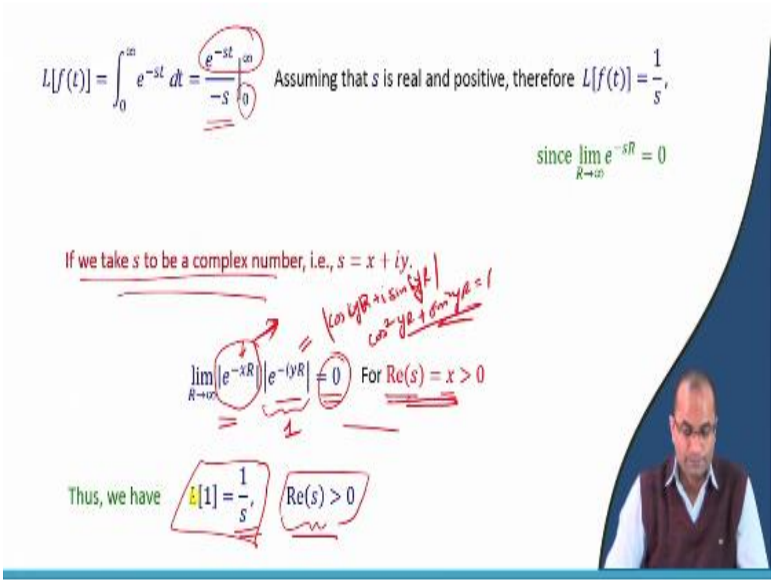
Now this sR can limit R to infinity that what is this number here. So naturally depends on this s not all values of s this is a finite number, but when s is positive. So if we take this s positive, then this minus sign and sR goes to infinity, we know that this will be 0. But if s is negative this minus s will make it positive with this R and then R approaches to infinity this will go to infinity then.

So we make an assumption that s is real and positive and in that case this integral will converge to this Lt is equal to 1 over s because this infinity will make this 0. And then we have for 0 e power minus 0 that is 1, and so 1 over this s and this minus will be also plus because you are talking about the minus so we have 1 over S.

So the Laplace Transform of this ft is equal to 1 over S and now this is also important to notice that this limit when R approaches to infinity was 0 for positive value of s. So together with this Laplace Transform we should actually mention that the range of s that means s is real and positive then the value of this Laplace Transform of ft is 1 over s.

(Refer Slide Time: 10:31)





Now since this s can be real and positive which situation we have seen and the Laplace Transform is 1 over s and the reason was because this limit was getting to 0, but now what will happen if we take s to be a complex number because we are free to take s a complex number as well. So if we take this s to be a complex number that means this s is equal to x plus iy of this form, then what will happen to this limit because we had the limit, they were e power minus this st or sR and then R we led to infinity.

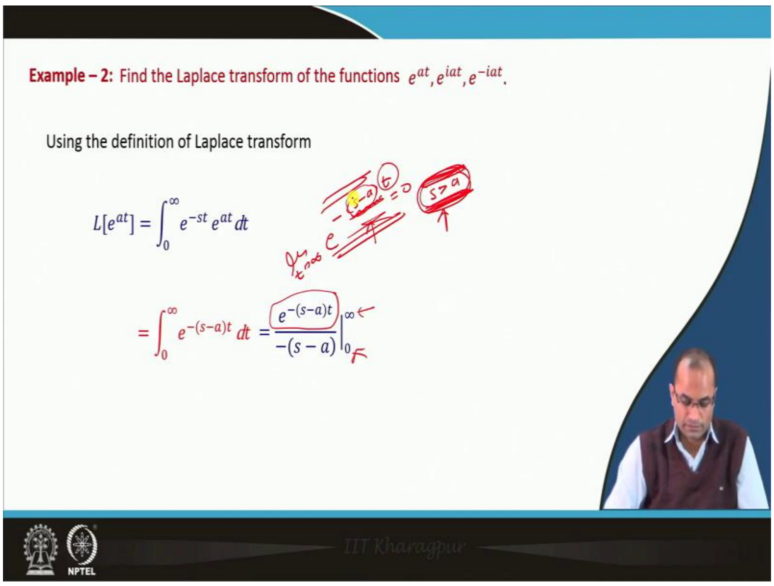
So this was the point of discussion now. So here this s if we replace, if take this not only just real and positive if we take here e power minus this x plus iy a complex number R and then we are interested now that what will happen to this limit. So instead of this we are considering now this limit here R to infinity and the absolute value of e power minus one is the xR term there and then we have minus iyR term.

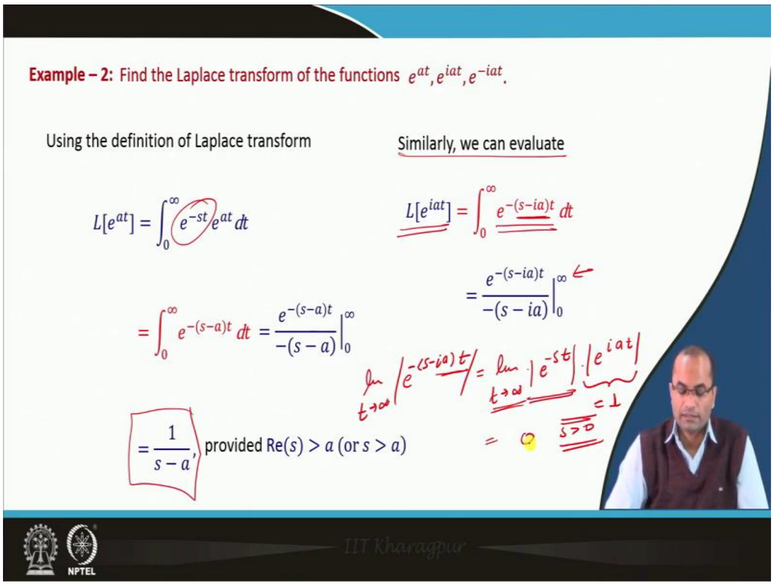
So we are taking a look on the absolute value of this number and if this absolute value goes to suppose 0 then naturally this will also go to 0. So let us see what happens to now this limit here limit R to infinity, then we have e power minus xR and then the absolute value of e power iyR. So getting back to this absolute value here of iyR, this will go to a 0 if this real s is greater than s especially because of this term.

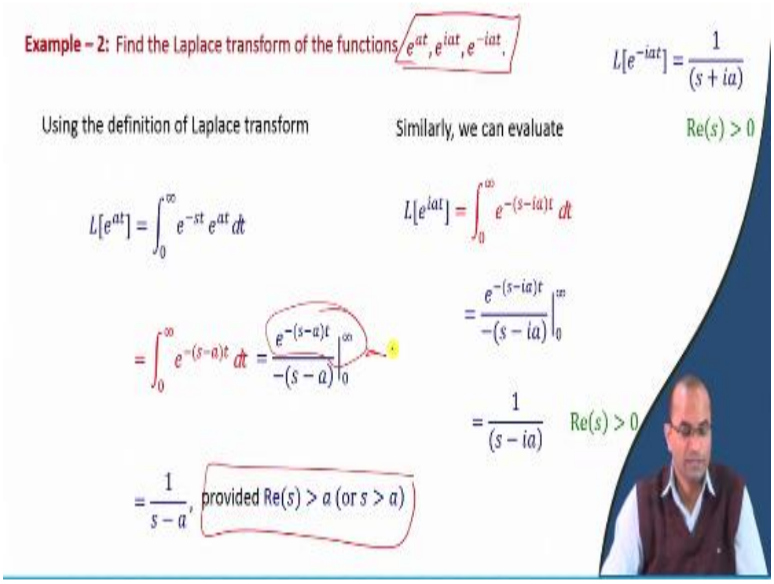
If this real s which is x if x is positive here this term will go to 0 and the second term is nothing but it is cos iyR plus i sin iyR and then the absolute value. So there we can say this cos square. So cos yR plus i sin yR then we have cos square yR plus sin square yR so that is 1. So this quantity here is 1 and this goes to 0 when R approaches to infinity for this real s greater than 0.

So having this now we conclude that again we have the same situation, if we take s to be a complex number again when we approach to this infinity this will be 0 and then if this t approaches to 0 we will have this as a 1 over s. So in this case also we have the Laplace Transform of 1 as 1 over s, but the condition now together with this we have that R, the real value of this s must be positive, then only this is possible that the real, the Laplace Transform of 1 is 1 over s.

(Refer Slide Time: 14:01)







Okay, getting to the new problem now we need to find now the Laplace Transform of the functions, the exponential type, then that means we have e power at e power iat and e power minus iat or we can have also e power minus at. So this kind of functions they are actually similar so we will just focus on one the other can be computed analogously. So we have the Laplace of e power at that means exponential minus st and then we have e power at and then dt.

So having this 0 to infinity, e power minus s minus at so we have combined the two, the two exponent here with dt and that will be that can be integrated now. So we have e power minus s minus at divided by this minus s minus a where this t goes from 0 to infinity. So again the same problem here that will happen when this t approaches to infinity to this exponential which is s minus at.

And recalling from the previous slide so if this number here is positive then when t approaches to infinity this can go to 0 and the integral will exist. So here to have this positive basically we make an assumption s is a real number and greater than a, then naturally this will go to 0 when limit t approaches to infinity. So this is a one condition that for s greater than a this will exist and when we put this t equal to 0 naturally this will be 1 over s plus a.

So under this range of s that s is greater than a, this limit will go to 0 and we have the finite value of the integral. The other situation again which can be handled similar to the previous one we have the extension that what will happen when s is complex number. So in that case the condition because of this again it will come out that if real s is greater than a again then this will satisfy and we will have when limit t approaches to infinity the value will be 0 again.

So we have finally this 1 over s minus a the value of this integral this improper integral provided either we call that s is greater than a or we call this real s greater than a. So this is more general domain for this s and this is if we assume that s is a real number then s greater than a but we have this Laplace Transform as 1 over s minus a. Similarly, we can evaluate for instance e power iat and in this case we have again by combining with e power minus st.

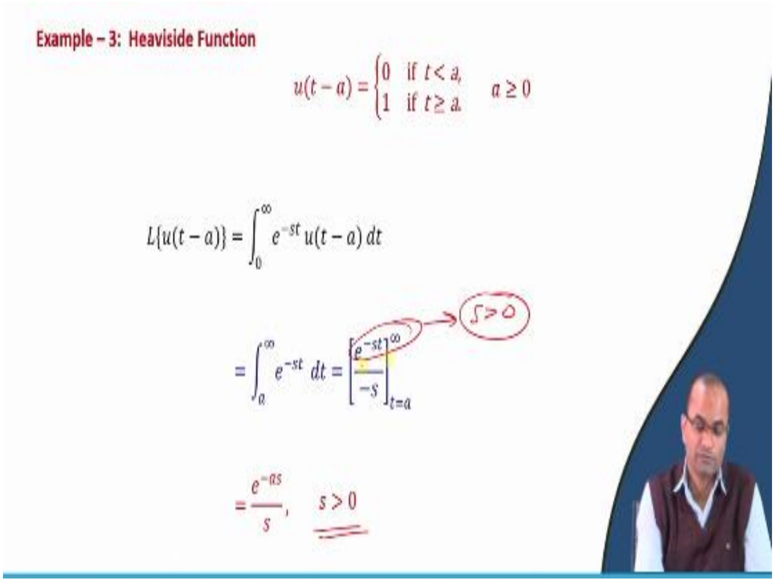
We have e power minus and s minus ait so this time we have this ia and now if we look at again that what will happen to this when t goes to infinity. So in that case we have e power minus s minus this ia and t if we take the absolute value and then try to compute that what will happen when t approaches to infinity. So in this case the situation is little different than the earlier one.

Here we have this e power minus st and one term and the other term is with i so e power iat the other is e power iat which the value is for which the value is 1 so we have just e power minus st and to get to this 0 with this limit we have just the condition that s must be just greater than 0 or real s must be greater than 0. So under this condition the infinity can be handled and then we have the value here when this t approaches to 0.

So that means s minus a will be the value of this e power iat. The similar result but the difference here is now that this, the range of s here is real s positive whereas here the range was real s greater than a and a is this exponent there. Similarly when we have e power minus at so that result will be just s plus ia instead of this s minus ia under the same restriction that real value of s is greater than 0.

So that was another elementary function the exponential functions which we can handle easily and we can get this range of s as well by noting down that what will happen when here this t approaches to infinity.

(Refer Slide Time: 19:27)



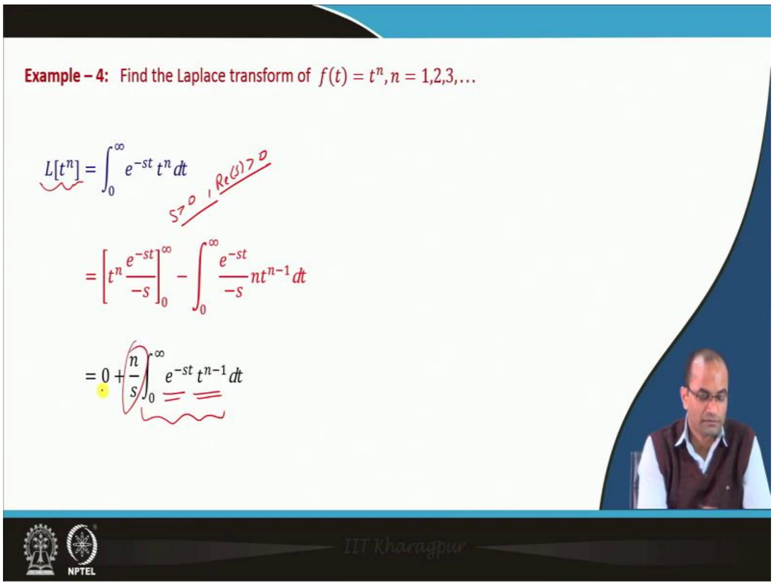
Another simple function that is the Heaviside function and we can compute its Laplace Transform. So here we are defining this Heaviside function u t minus a or sometimes it is also defined as H t minus a so this is the notation and here it says that t less than a, the value is 0 and t greater than or equal to a the value of this function is 1. So it is again a simple function to integrate, it is similar to what we have done as taken as a constant function 1.

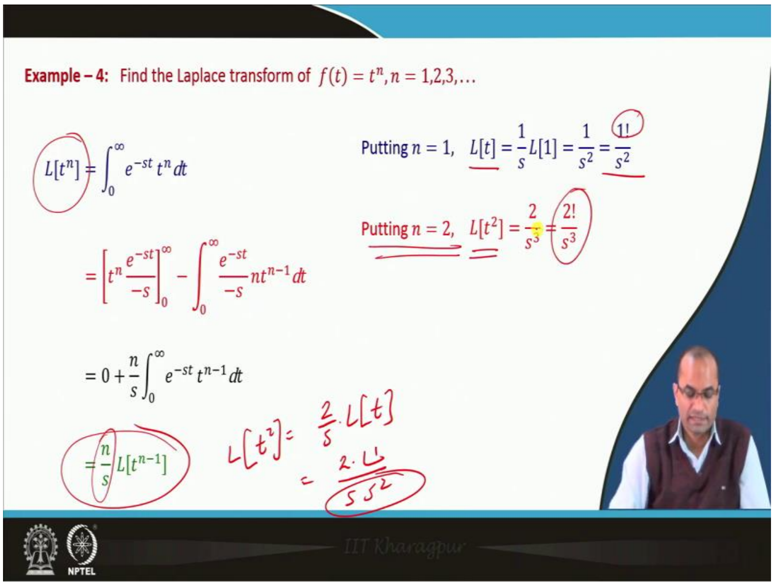
But here the range is from a to infinity not from 0 to infinity because 0 to a the value will be 0. So computing this integral now if we substitute this u minus a there in this definition. So what will happen we have a to infinity now and e power minus st and since the function is 1. So after this integration we have e power minus st over s limit t to infinity and again the same situation.

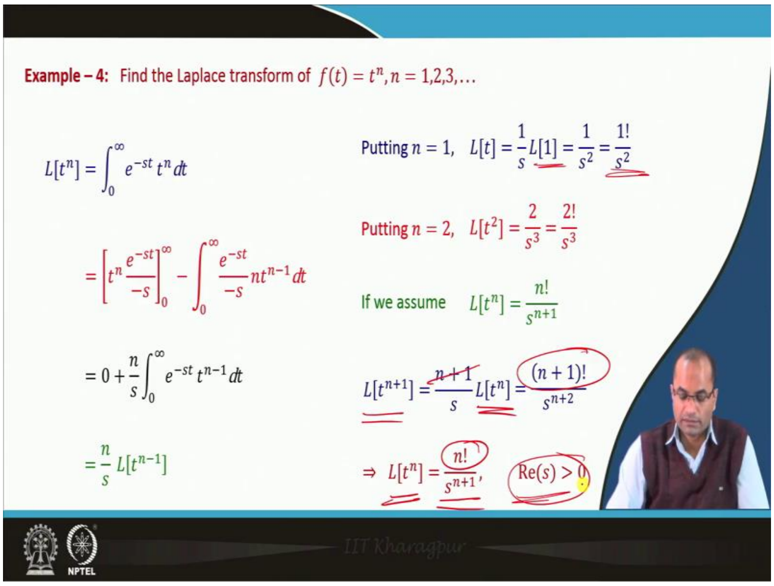
That if this s, the real s is positive or this s is positive naturally this t approaches to infinity that term will vanish and when t is equal to a, now we can say that it is e power minus as over s. So when a is for instance 0 so this is exactly the case when we have the Laplace of 1 because when a is 0 so we have this e power 0 that means the Laplace will be 1 over s because in the range 0 to infinity the function is 1.

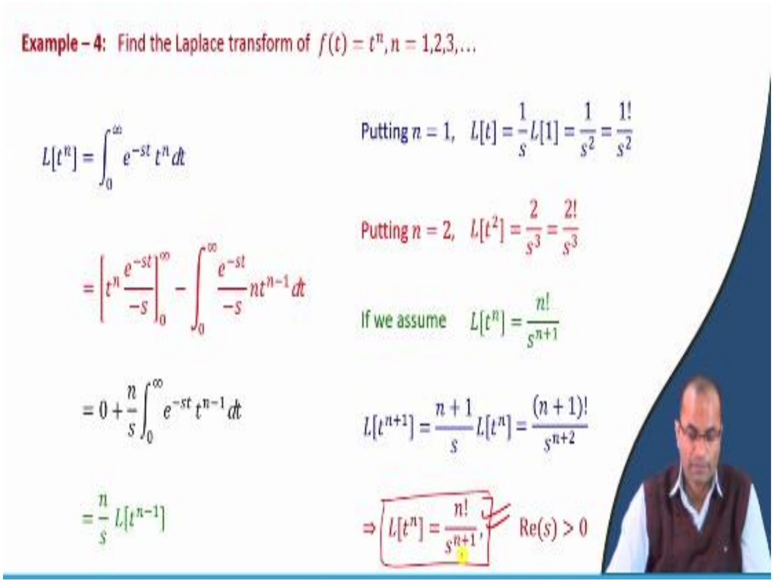
The only thing this function is defined in the whole range over the real axis, but it is 0 from minus infinity to this a. Okay that was the Heaviside function whose Laplace Transform is e power minus as over s. So this is valid when s is positive because that condition we got from here that when s positive then only this term will go to 0.

(Refer Slide Time: 21:44)









Okay so another important function and very elementary and this will be used again several times to compute more complicated functions that is the Laplace Transform of t power n. So once we know the Laplace Transform of t power n we can compute the Laplace Transform of any polynomial. So here we are taking now the Laplace Transform of t power n where n is 1, 2, 3 etcetera.

So the Laplace t power n is given by this e power minus st and we have this t power n so t power n and e power minus st over this minus s. Here we have e power minus st over minus s and then the differentiation of this t power n can be taken as nt power n minus 1. So we have differentiated this by parts treating this as second function this is first function. So this will remain as it is then the integration of this will come and similarly here.

So now the first part when this t approaches to infinity so assuming that this s is positive or the real s is positive in that case this will go to 0 and when t approaches to 0 again because of this t power n this will go to 0. So the first term is actually vanishing with the assumption that s is positive or this real s is positive under these two conditions this will vanish. So we have then n over s and then e power minus st.

And we have this reduction in the power t power n minus 1 that means we can get kind of this recurrence relation that Laplace of t power n is the Laplace of t power n minus 1 with the factor n over s. So let us formulae so we have t power n the value is n over s in the Laplace t power n minus 1. So with this relation, with this recurrence relation we can actually get the close form.

So here we will use now the induction principle, so when we take n is equal to 1 that means Laplace of t is 1 over s and this Laplace of 1 so this Laplace of 1 we know the result so that was 1 over s so we got this 1 over s square or which can be written as factorial 1 which is the value is 1 so factorial 1 over s square. Why we have written factorial that will be clear in the next step.

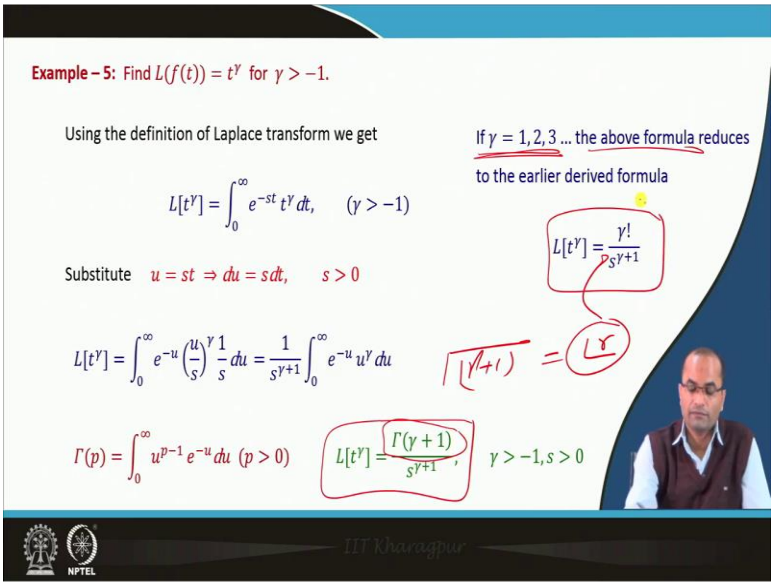
So when we put this n is equal to 2 what will happen from this relation that the Laplace of t square will be 2 over this s from here. So with this relation if we go here Laplace of t square will be 2 over s and the Laplace of t. So the Laplace of t we got already the factorial 1 and over this s and we have s square. So this will become s cube so we have factorial 2 over s cube in this case.

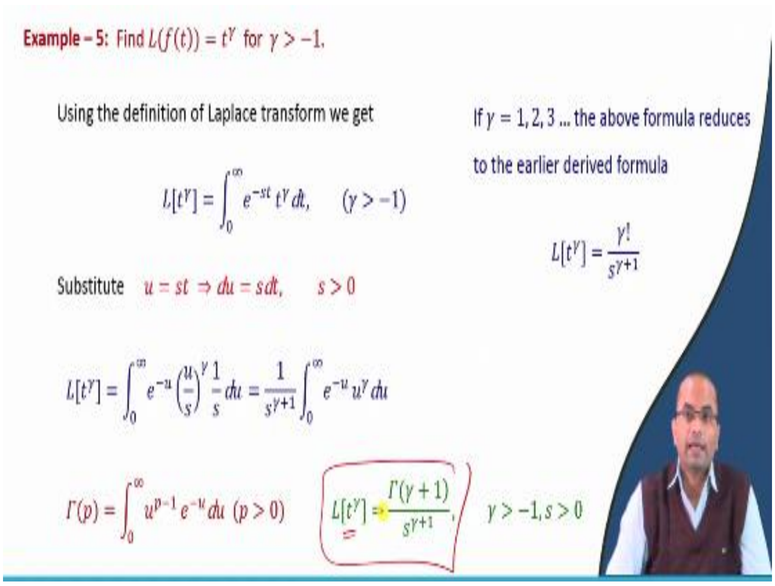
And then if we assume now so n is equal to 1 this is true n is equal to 2 this is factorial 2 over s3. So we know now what is happening now here L power, t power n is nothing but the factorial n over s power n plus 1 and having this formula, now we can try to compute for t power n again with this relation so we have n plus 1 over s and Laplace of t power n which is known here.

So that is factorial n plus 1 over s power n plus 2 and then we have the Laplace t power n is equal to the power one term will get cancel, we have factorial n there. So factorial n over s power n plus 1 and the condition naturally we have here that the real s is greater than 0, so which is the Laplace transform of one was use there which is valid for real s greater than 0.

So we got this relation that the Laplace of t power n is factorial n over s power n. So this is a very useful result which can be used for evaluation of a polynomials and many other functions wherever t power n appears.

(Refer Slide Time: 26:40)





So indeed we have more general result here that a Laplace of ft is t power gamma where gamma is greater than 1. So now we are not restricting here for the integers, but we have t power gamma, gamma is a real number greater than minus 1. So in this case also we can get the Laplace Transform again applying the definition of the Laplace transform. Why this restriction that will be clear in a minute?

And if we substitute here, in this integral that this u if we take, we replace by this st that means the du will be s time dt so the t variable is replaced with u variable now, assuming that s is positive. So when s is positive the limits will remain as it is 0 to infinity. We have e power minus u because st is u and then this t is u over s power gamma and then dt is du over s.

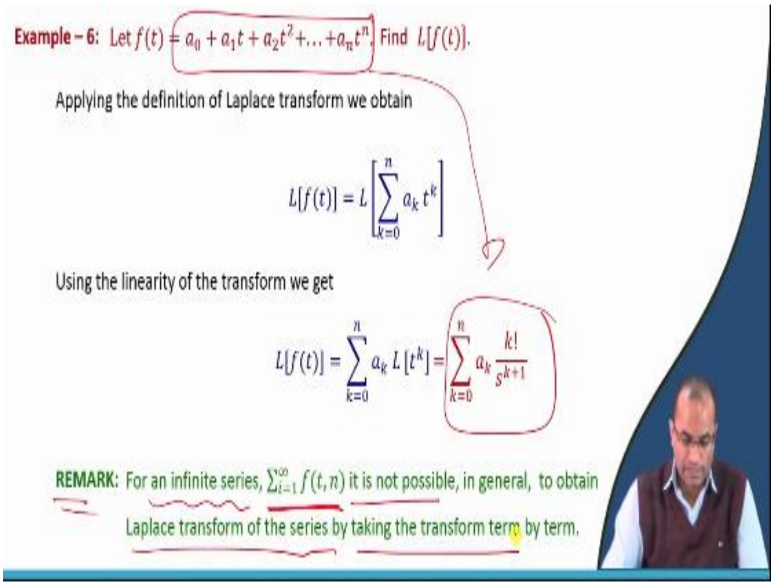
So having this integral now the idea is to get into this gamma integral which is just appearing now next to this. So we have 1 over s power gamma plus 1 which is coming from here and we have e power minus u u power gamma and that is exactly the definition we have for the gamma function. Just to recall so we have the gamma p is equal to 0 to infinity u power p minus 1 e power u du for p positive this was valid for p positive and this gamma p is defined as u power this p minus 1.

So here we have u power, e power minus u and u power gamma. So according to this definition the value of this will be so Laplace of t power gamma and this integral now here it is gamma, this gamma function of gamma plus 1 and then s power gamma plus 1 was already there and as per this restriction p greater than 0 that means gamma plus 1 must be greater than 0 that means gamma must be greater than minus 1 for the convergence of this gamma integral.

So we got the Laplace of this t power this gamma gamma here is a real number and then there is a gamma function, so both are named as gamma. So here the gamma function of this gamma plus 1 divided by s power gamma plus 1 that is the formula which we have for t power any real number greater than minus 1. Indeed if we take here gamma is equal to 1, 2, 3 that means the integer if we replace this gamma with the integers.

Then what will happen this we know the relation that this gamma are plus 1 will be nothing but the factorial gamma. So we can replace here with factorial and we get back to the original result we have for integers. So indeed this is a more general result which can be used for computing t power any real number greater than minus 1.

(Refer Slide Time: 29:55)

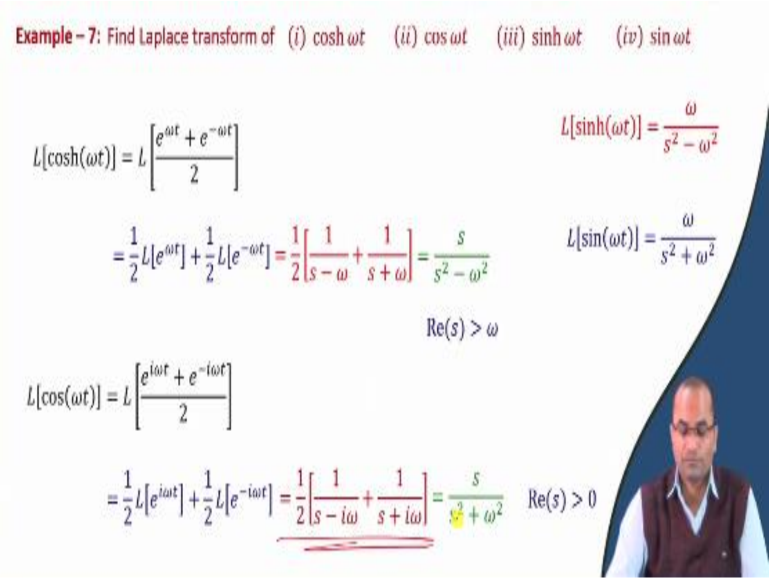


Then next example we have the function here a naught t and so on. So this is just a linear combination of different powers of t and naturally we know that the Laplace enjoys the linearity property. So we can apply the Laplace Transform here to each term and then we can get. So that means the Laplace of this sum will be a sum of the Laplace of t power k and we know this t power k is factorial k over s k.

So we basically got the result simply using the linearity property, but just a remark which can be discussed in some other lectures that in general for infinite series so when i 1 to infinity here we had finite terms so there is no issue we can use the linearity, but in general it is not possible to obtain Laplace Transform of the series by taking the Laplace Transform term by term.

Because after taking the Laplace Transform, the new series may not be convergent series so we have to again discuss the convergence issues and so it is in general not possible to take the Laplace term by term.

(Refer Slide Time: 31:12)



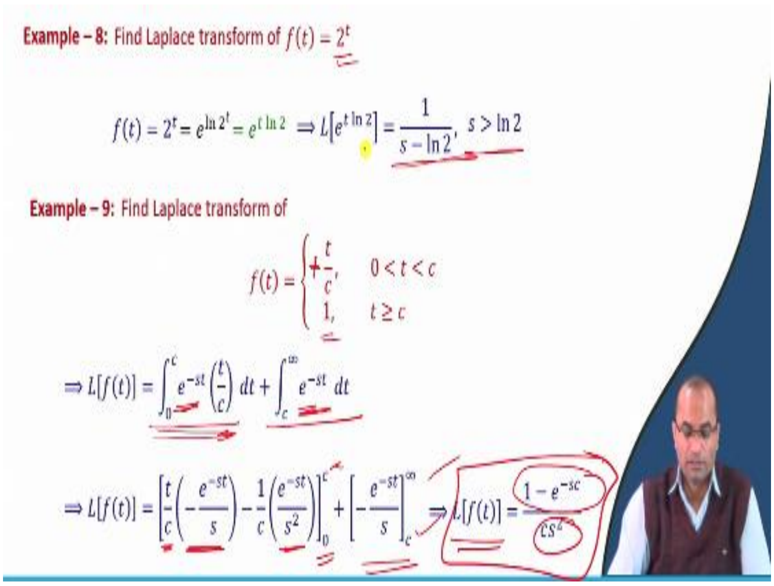
Then next example here we have the to find the Laplace of the cos hyperbolic cos omega t sin hyperbolic omega t and sin omega t. So these are also elementary functions, but we will realize here that these functions indeed can be written in terms of the exponential function and we know already the result of the exponential function. So, for example, this cos hyperbolic omega t is e power omega t plus e power minus omega t divided by 2.

And then we can use the linearity property of the Laplace Transform and here we know the results of the Laplace e power omega t. So indeed now we have the Laplace here 1 over s minus omega n1 and 1 over s plus omega so we will get s over s square minus omega square where this real s greater than omega because remember that this result e power omega t, the Laplace was 1 over s minus omega for real s greater than omega.

And here we have the real s positive or real s greater than minus omega. So in either case when we combine the two this is valid now for real s greater than omega. So this is a cos hyperbolic omega t and then cos omega t so we have again the exponential function e power i omega t plus e power minus i omega t by 2. Again the linearity says that we have to just get this Laplace of i omega t and minus i omega t exponential functions which we have already evaluated.

So we have 1 over s minus omega s plus omega and this is again valid for real s positive. So just simplifying this we got the result s over s square plus omega. So that is the Laplace Transform of this cos omega t. Now the s over s square plus omega square and this is valid for real s positive. Coming to this sin hyperbolic omega t again with the similar splitting we can obtain omega over s square minus omega square. And the Laplace of the sin omega t instead of this s there omega will appear and we have the same idea absolutely to get these Laplace Transforms.

(Refer Slide Time: 33:40)



So we have for instance here if you want to find the Laplace Transform of 2 power t, again this function can be converted into this known transforms. So ft is equal to 2 power t can be written as exponential ln 2 power t. So exponential and ln so we have again this 2 power t and this can be written as t ln 2. So what form we have it is an exponential function of at where a is here now as ln 2.

So this can be again used to get this Laplace Transform, which says 1 over s minus this a, which is 2 where s must be greater than now ln2. The last example where we will find the Laplace Transform of such a function which is defined 0 to c by t over c and t greater than c it is defined as 1. So this is also not a complicated function, we can apply now the definition of the Fourier transform that e power minus st and then t over c is there and for a second case we have this e power minus st so this is t over c.

And this we have this 1 there and now we can differentiate this, we can integrate by parts to have again this t over c as it is, here the integration is coming minus e power minus st over minus s with the minus sign and then again two times we have to integrate so this s square will be coming with e power minus st and this 0 to c and then here again we have this e power minus st integrated as e power minus st over s and c to infinity.

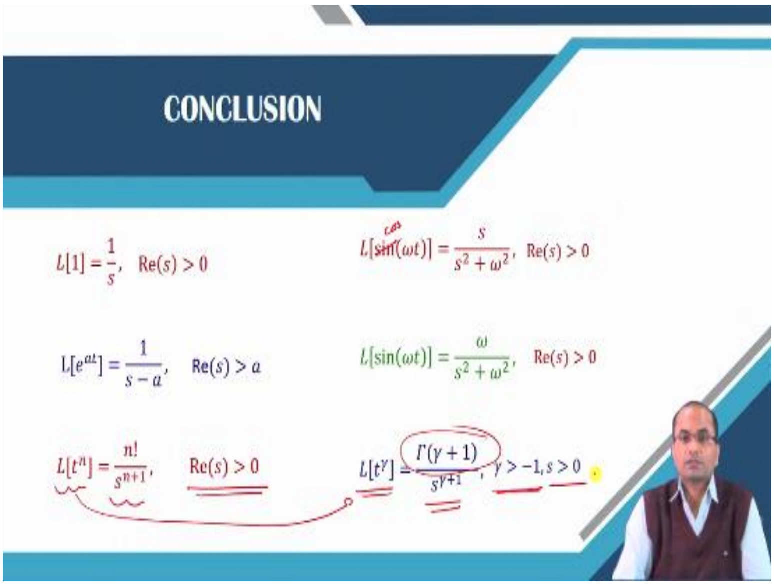
So this can be simplified to give that the Laplace Transform of ft is e power minus sc over this cs square. So what we have observed that once we know the Laplace Transform of very simple functions that to function 1 for instance the function t power n or t power gamma and the exponential functions.

So these are the most basic one and then many other functions which we have seen the cos sin, cos hyperbolic etcetera or 2 power t, they can be directly computed using simple linearity property of the Laplace Transform and we can get the Laplace Transform of these linear combinations.

(Refer Slide Time: 36:22)

These are the references we have used for preparing this lecture.

(Refer Slide Time: 36:26)



And just to conclude the most important the elementary functions we have covered in today's lecture the function 1 whose Laplace Transform is 1 over s for this real s greater than 0 it is valid e power at we have s minus a and the restriction on s was real s must be greater than a Laplace of sin omega t was s over s square plus omega square and for sin this was for cos omega t and then for sin omega t we have the omega over s square plus omega square.

So there is a correction here this is cos omega t and then we have the Laplace of t power n that is factorial n over this s power n plus 1 and this is valid for real s positive and this was generalized later on to have the Laplace Transform of t power gamma as this gamma of gamma plus 1 and s power gamma plus 1 and gamma greater than minus 1 this is valid and for the s positive this Laplace Transform exist.

So these are the most basic elementary functions which can be used as a direct result for getting some more or the combinations of these or some more complicated functions that we will see in some other lectures that is all for this lecture and I thank you for your attention.