Engineering Mathematics II
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Lecture 50 - Applications of Fourier Transform to PDEs (Part II)

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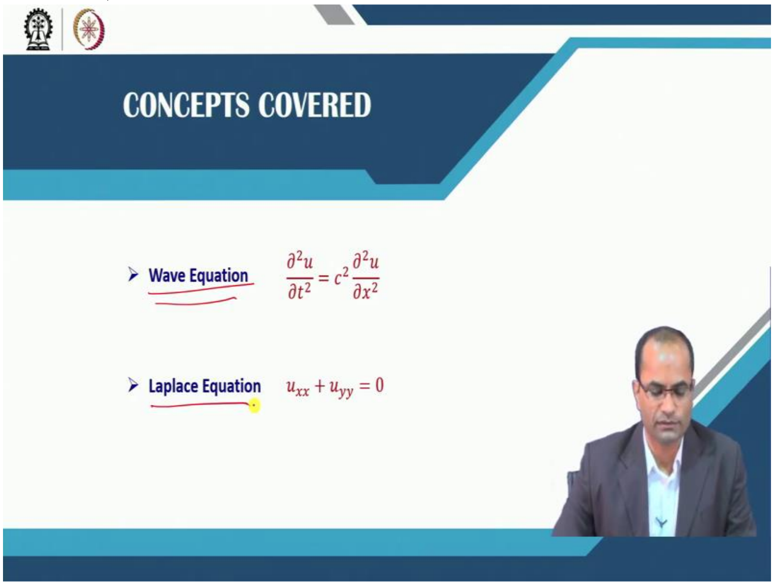

So welcome back to lectures on Engineering Mathematics II. This is lecture number 50 on Applications of Fourier Transform to PDEs and this is part II. We have already discussed in part I how to solve the heat equation and in this lecture, we will continue with the same idea applied to the wave equation and then to the Laplace equation.

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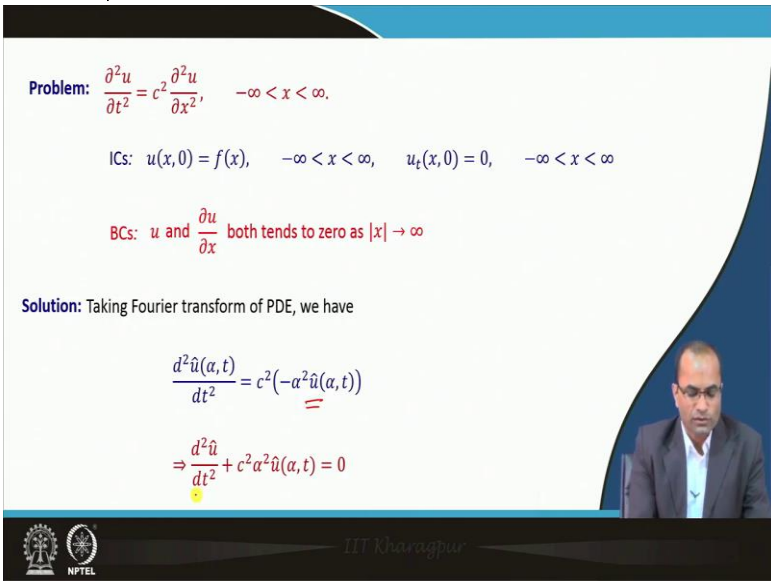

So coming to the problem on wave equation, so we have this double derivative now del 2 u over del t square and is equal to c square del 2 u over del x square and the range here for x is given from minus infinity to plus infinity.

So naturally, given this range for x we will apply the transform, the Fourier transform, not the cosine or sine transform. The initial conditions must be supplied to this problem. So here we have the initial condition u x, 0 is given as f x and for all values of x and also this u t x at 0 is given as 0. So here since we have the double derivative now for u with respect to t then we have two initial conditions, one is u x at t equal to 0 and the other one is its partial derivative with respect to t and again at t equal to 0. So the boundary conditions, we have these standard boundary conditions that u and del u over del x both tend to 0, so both tend to 0 as x tend to plus infinity or minus infinity.

So coming to the solution, so it is clear here because of the range and these boundary conditions that we will apply now the Fourier transform to the equation, to the given wave equation, so now taking the Fourier transform of this given wave equation, what we will get now? So this del 2 over del t square that is because t we are not taking with respect to t, we are taking with respect to x, so this t will remain as it is and we have here the Fourier transform of u. And the right hand side, we have c square already there, and that is the Derivative Theorem we have applied that when we apply the Fourier transform on this del 2 u on del x square, on second derivative so we will get minus alpha square and the Fourier transform of u.

So having this ordinary differential equation now from the partial differential equation, we got the ordinary differential equation which is much easier to solve for this u hat. So this can be solved now easily.

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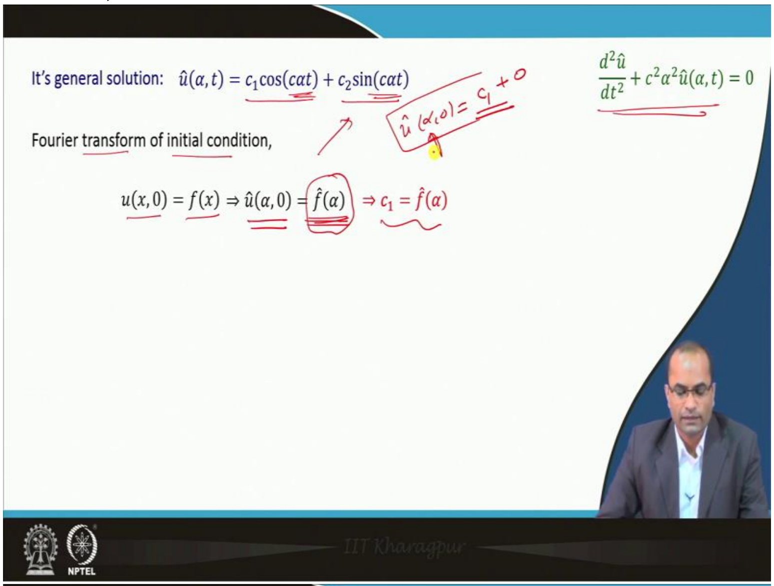

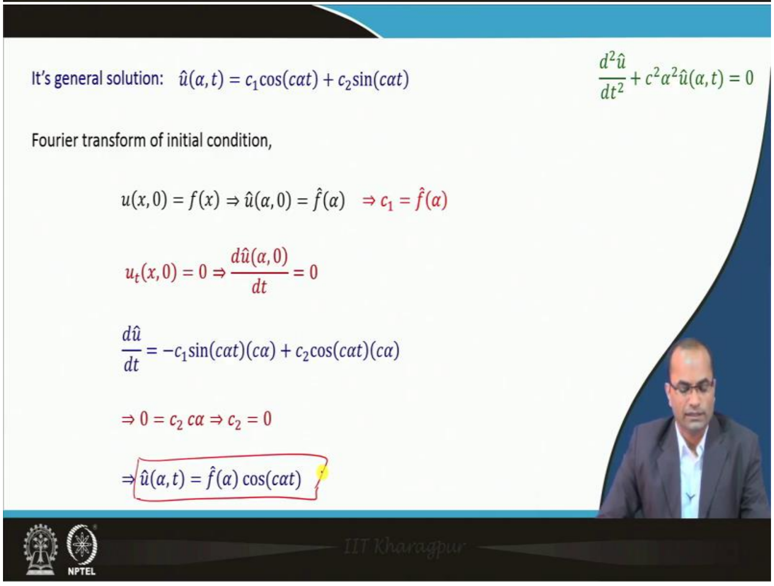

And its solution will be c1 cos c alpha t and c2 sin c alpha t, because this is a second order linear differential equation with constant coefficient. So we know the solution. This is the complementary function we got, c1 cos c alpha t and c2 sin c alpha t. So the Fourier transform of the initial condition we need to apply because the c1 and c2 are unknowns here.

So we have u x, 0 as f x, that is the first initial condition given. So if we take the Fourier transform of this given initial condition that mean u hat alpha 0 is equal to this f hat alpha and that means once we apply this here so that means that t equal to 0 we have to set so that means u hat, this alpha, 0 will be here c1 and then the cos 0, so it is 1, and in case of sin that will be 0.

So we have just c1 equal to u hat alpha naught. So u hat alpha naught is f, this f hat alpha and therefore this c1 is f hat alpha. So having the c1 we have to now get c2 using the second initial condition that was u t x, 0 equal to 0. So here also we will apply the Fourier transform and we will get the derivative with respect to t of this Fourier transform u hat alpha at 0. So here then we have to differentiate the given solution with respect to t first. So we will get minus c1 sin and then c alpha will come.

Similarly, for the second term we will have c cos and then the c alpha term will appear and then again, when we put t equal to 0 the first term will become 0 and we will get from the second one so this is will be 1, cos 0 will be 1. So we have c 2 and into c alpha is equal to this derivative of u hat. And the derivative of u hat at t equal to 0 is given as 0. So we notice here that c 2 c alpha is 0. That means the c 2 is 0 and c 1 is f hat alpha. That means our, this solution for u hat is much more simplified now and we have only the f hat alpha and the cos term with c alpha t.

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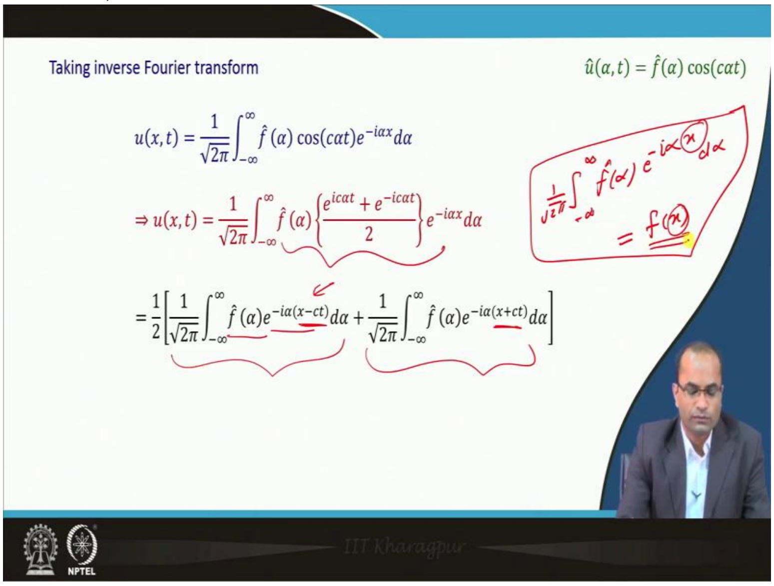

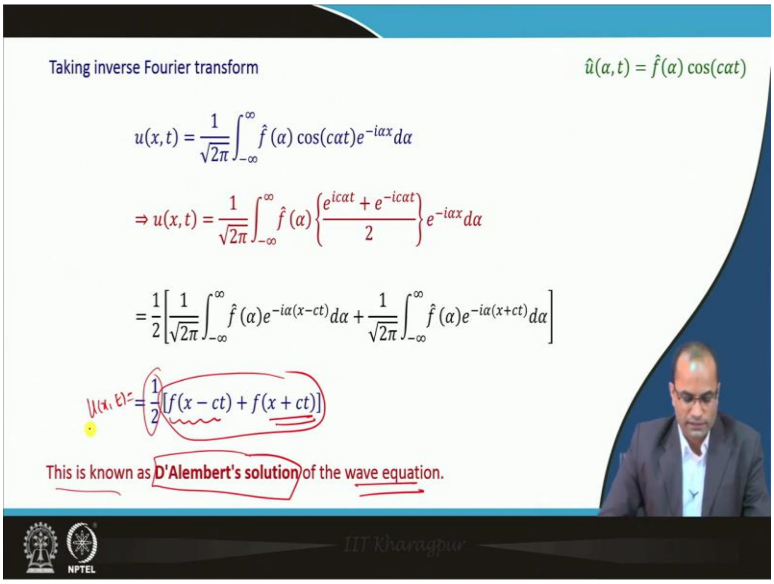

So now we need to take the inverse Fourier Transform of this u hat alpha t to get back to u x, t. So this u x, t will be 1 over square root 2 pi and we have this inverse Fourier transform that means f hat cos c alpha t and then because of this inverse now we have e power minus i alpha x and then it has to be integrated over d alpha.

So now this u x, t what we can do here, so given this cos c alpha t we can use this Euler formula to set here i c alpha t plus e power minus i c alpha t divided by 2 and then we have here e power, the exponential minus i alpha x and integrated over this d alpha. So we can rewrite now, so with this f hat alpha we have e power i c alpha t and then we have here e power i alpha x.

So we can club this to have this i alpha x minus c t over d alpha and the second case we will get this minus i alpha x plus c t integrated over this d alpha. So here this integral is splitted into these two integrals and now it is easy to recognize that f hat alpha and exponential i alpha, so instead of this x we have here x minus c t and instead of this x we have x plus c t.

So being this x here just 1 over square root 2 pi minus infinity to plus infinity f hat alpha and e power minus i x d alpha, this is the definition of f x, the inverse Fourier transform. So now the only change is that this x here at both the places is replaced in the first term by x minus c t whereas in the second term it is x plus c t. So we can now simplify these two.

So the first term is simply f x minus c t because of this x minus c t and in the second case it is x plus c t, so that is the solution now of the given differential equation, the given wave equation and this solution is known as the D' Alembert solution of the wave equation. So here just the average, so 1 by 2 of this f, the value evaluated at x minus c t and x plus c t, so all this information is available and this is exactly the unknown u x, t we have evaluated.

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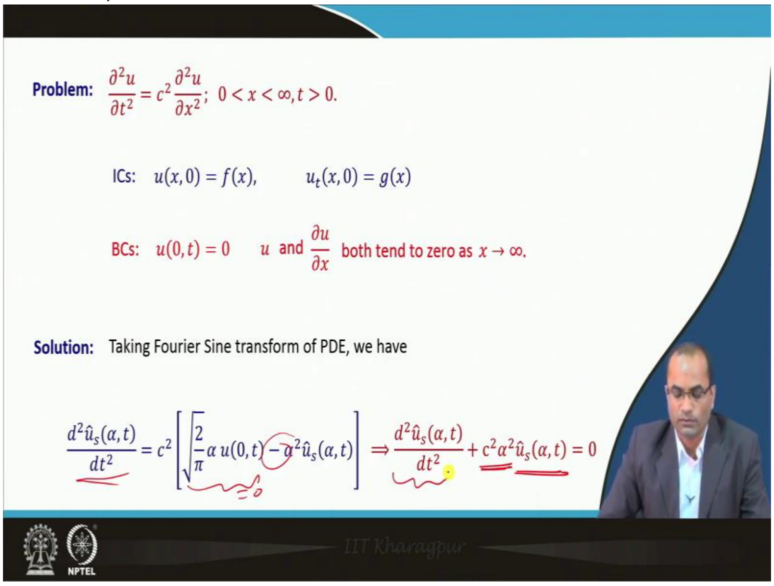

Well, getting to a new problem now. So we have here again, the wave equation but the domain is 0 to infinity. So once the domain we see, it is 0 to infinity, naturally, we cannot apply the Fourier transform. We will be applying either Fourier sine transform or Fourier cosine transform. So the initial conditions are given as u x, t is equal to 0 f x and then here it is g x.

So we have much more general situation that both the places, at one here we have f x and here at this place we have g x, it is none of them is 0. So the boundary conditions u at x is equal to 0 is prescribed at 0 and both u and u x they tend to 0 as x approaches to infinity, so again this natural condition which we need for applying the Fourier transform.

So here this boundary condition which is given as u 0, t is 0 suggest us to apply the sine transform because directly u is given, not its derivative with respect to x. So given this we will apply this sine transform because of the reason that u is prescribed at this x is equal to 0, not the partial derivative of u.

So taking this Fourier transform of this PDE what we have, we have this second order term, d2 over d t square and this u s alpha t and then the right hand side we have c square and this is exactly the Derivative Theorem we have just written here and as we can see that u 0, t is sitting here at this place, which we can substitute directly from the information provided with the problem. So now this equation is simplified because this is 0 now here, so we have plus this c square. So this we can bring to the left hand side. We have c square alpha square and this Fourier sine transform of u and this is the second derivative of this Fourier sine transform of u.

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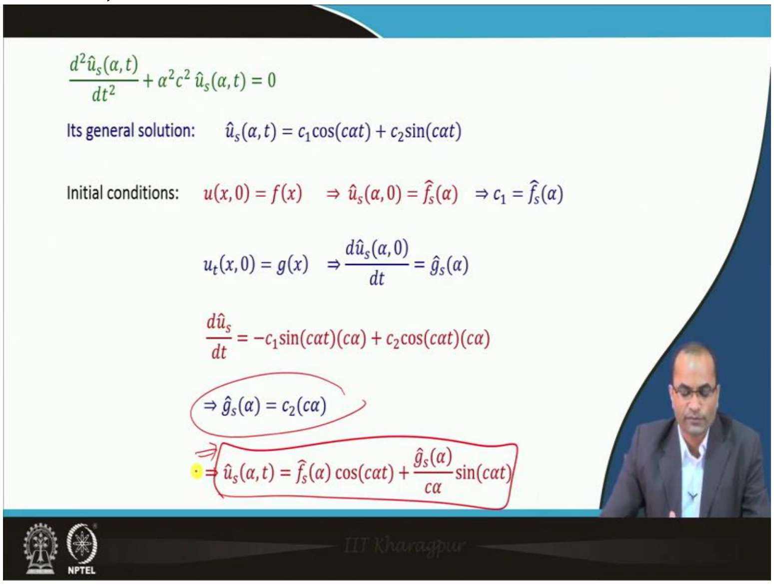

So having this equation, we can again easily solve this because this is a homogenous equation with constant coefficient. So its solution again will be written in terms of sine and cosine, so we have c1 cos c alpha because the roots going to be c alpha i and minus c alpha i, so here we have c1 cos c alpha t and c 2 this sin c alpha t. Initial conditions if we take a look here at t equal to 0, this is given as f x and which we can take now the Fourier transform, naturally the Fourier sine transform we are talking about, so u hat s alpha 0 is equal to the f s, so this f s alpha is the Fourier transform of this, Fourier sine transform of this function f x.

So with this information if we set here t equal to 0, the c 2 term will disappear and then we have c 1 with this 1, so therefore we are getting directly c 1 as f s at, f s hat alpha. Now the u t x, 0 is given as g x, so there also we can apply the Fourier sine transform and the Fourier sine transform now is appearing here for g s, and this is the derivative of this Fourier sine transform of u.

So again, now we have to differentiate the equation, the general solution we got for the Fourier sine transform of u and then you will substitute t equal to 0 and apply or use this result. So when t equal to 0, so cos 0 this is 1 and this will get cancelled this time, so we have c 2 and the c alpha and that is equal to this g s, so basically we got this c 2 as g s hat over c alpha, which is substituted here for c 2 and for c 1 we have f s hat alpha.

So we got the solution here which incorporates all boundary conditions, all initial conditions. The only question now that how to get back to the u. So we will take naturally the inverse Fourier sine transform and then there will be some difficulty to deal with the second term.

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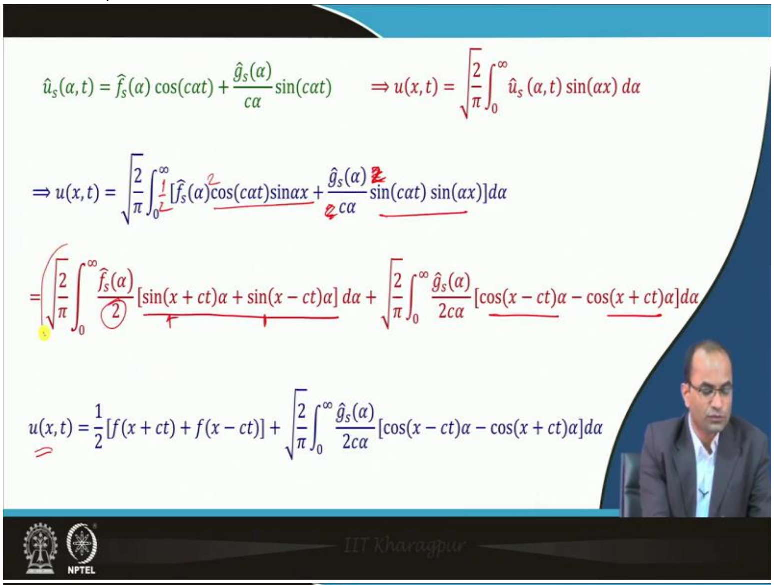

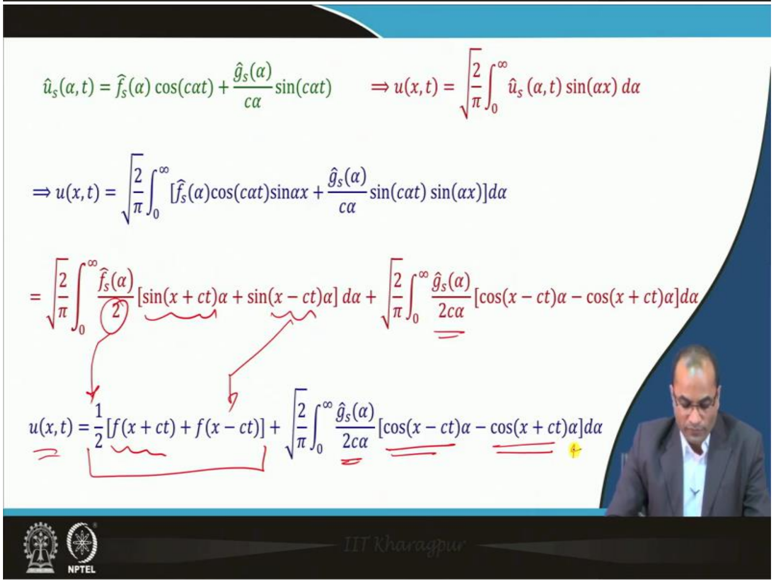

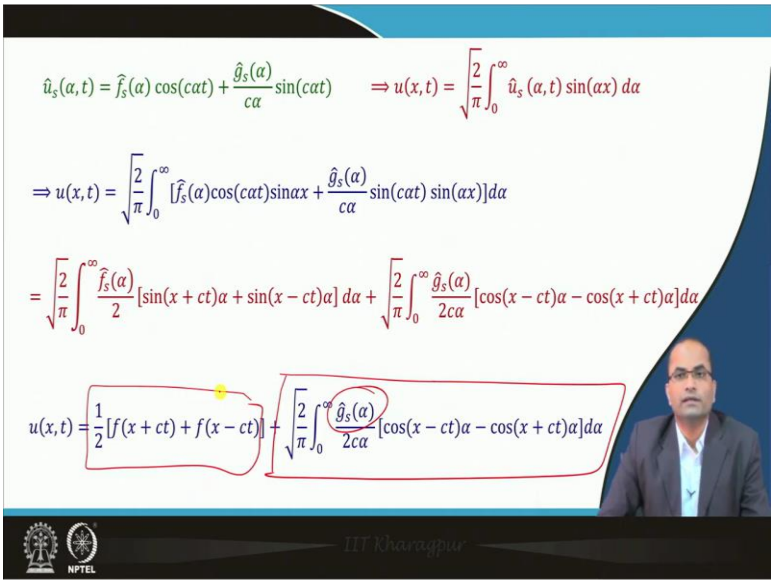

So we have this Fourier sine transform, the expression for the Fourier sine transform given here and then we can take the inverse Fourier transform, that means this u s hat alpha t will come here and then we have sin alpha x d alpha term because of this inverse sine transform. And then we have this u x, t; then we have substituted this value of this u s hat alpha and then sin alpha x, sin alpha x was already there in this formula.

So we got here this cos a sin b type term here, we got sin a and sin b type terms. So we can make this two times cos a sin b and here 2 times sin and then 1 by 2 we can keep it outside, so and then here again 2 times will be there. So we have 2 cos a cos b and here also we have 2 sin a sin b which we can now use this trigonometric identity.

So 2 cos a sin b we will get only sin sin terms, so sin x plus c t and sin x minus c t with alpha and the second place here we will get cos x minus c t and then minus cos x plus c t. So u x, t this is half there and with the square root 2 over pi and this term here, so square root 2 over pi and sin x plus c t alpha d alpha and then we have here f hat s alpha.

So again this is the Fourier, the inverse Fourier sine transform of the, this f s hat alpha that means of f, just the x has to be replaced by x plus c t. So with this half sitting here we have the first term f x plus c t and the second term will give this x minus c t. So we are fine with this first term now, u x, t is half of f x plus c t plus x minus c t, this is exactly the D' Alembert solution earlier we got using Fourier transform and then because here this was not 0 so we have the g s hat term over this 2 c alpha and then cos x minus c t and minus cos x plus c t with this alpha.

So this, here we need some more simplifications because this g s hat is sitting there and we do not want to see the Fourier sine transform of g, we want to have a solution directly given in terms of g. The first term is fine because this is given absolutely in term of this f now. So for the second term we have to again do some calculations because this g hat alpha which is the Fourier transform of this g x.

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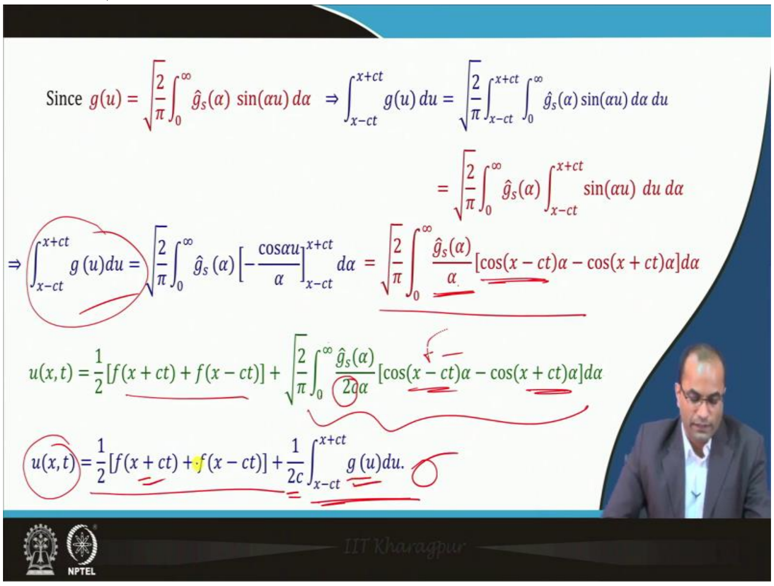

So here we will use this inverse formula that g u we can get as g s hat alpha and sine alpha u d alpha. So with this inverse formula for this g s hat if we just integrate both the sides from x minus c t to x plus c t, x minus c t to x plus c t then we will get from this right hand side the desired term which needs to be replaced in terms of g.

So here if we just change this order of integration that means du d alpha, so we have here 0 to infinity and then here x minus c t to x plus c t because with respect to alpha and u these limits are constant, so we can simply just interchange them. So we have g s hat alpha and then sin alpha u over du but now after interchanging, this term can be simplified because sin alpha u du that can be integrated now. So on integration, we got this minus cos alpha u over alpha and x minus c t to x plus c t, integration is done over d alpha.

So we can put the upper limit then we can put the lower limit to get this g s hat alpha over this alpha there and cos x minus c t and here cos x plus c t alpha. And now just get back to this u x, t the first term as we discussed is fine, only the second term has to be simplified because of this g hat s. And now if we look at this one, this is exactly, the precisely the same term we need here.

So it is cos x minus c t, cos x plus c t and d alpha. We have the integral over this g s alpha over alpha integration over 0 to infinity. So this everything here can be replaced by g u and the integral from x minus c t to x plus c t. So this is what we have done now. So x, u x, t is equal to this half x plus c t, x minus c t and then we have here 1 over 2 c, 1 over 2 c will be coming from here and the rest everything is integral, this x minus c t to x plus c t g u du. So this is the solution, the desired solution we got here in terms of g and in terms of f there.

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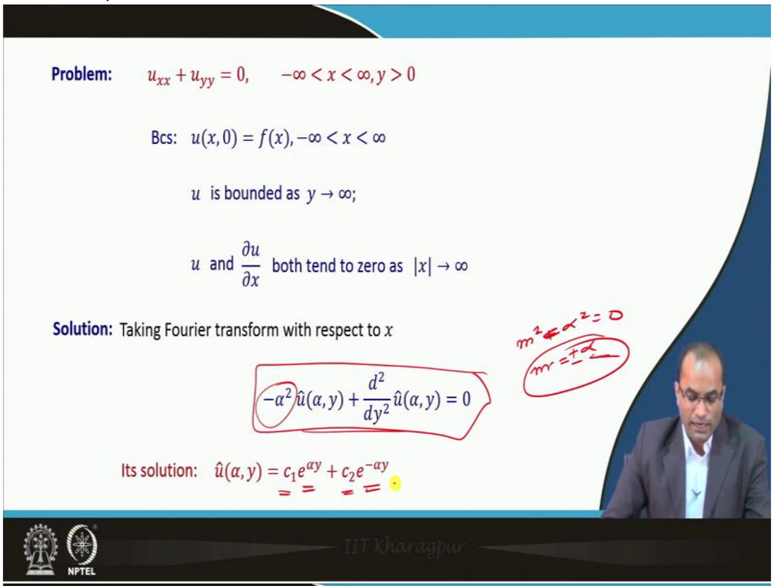

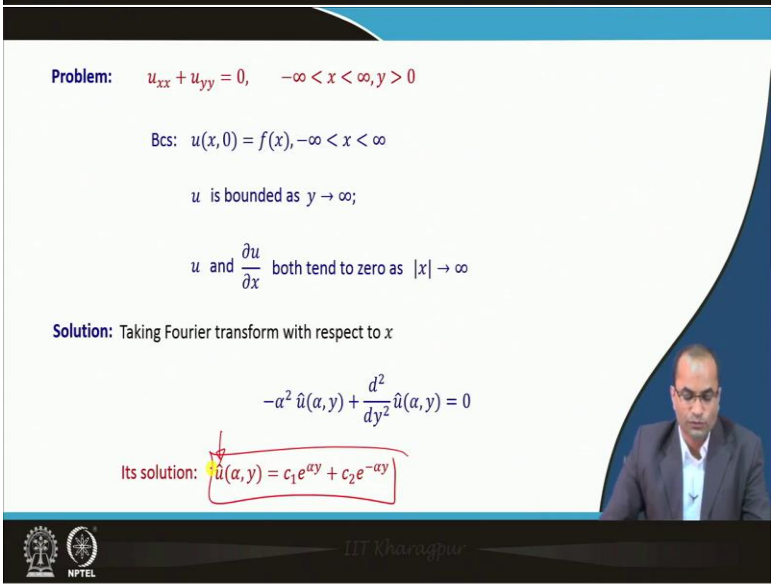

Well, so here we consider the next problem where we have u xx plus u yy equal to 0, so this is the Laplace equation, Laplace equation which we had discussed in earlier lecture. So this is the Laplace equation defined in this domain. So for x we have minus infinity to plus infinity and the y is taken as positive.

So coming to the boundary conditions, we have u x, 0 is given as f x in the range minus infinity to plus infinity and u is bounded as y approaches to infinity. So that is the other information given here that u is bounded when y approaches to infinity. And there is standard conditions that u and u x both tend to 0 as x approaches to plus infinity and minus infinity. So we have these boundary conditions because there is no initial condition as such there.

There is no time. So we have only the boundary conditions in this problem. So one, here we have for all x at y equal to 0 because y is from 0 to infinity so y at equal to 0, y equal to 0 we have this f x and as y, so the other boundary here when y approaches to infinity u is bounded and then we have as x approaches to infinity, we know that both u and u x tends to 0.

So we can solve this again using this Fourier transform. We will apply Fourier transform with respect to this variable x because this can be directly incorporated in the Derivative Theorem. So the solution, we take the Fourier transform and with respect to x, that means we have here minus alpha square u hat alpha y, so y will remain as it is. We are not doing with respect to y. Here we have d 2 over d y square and then the Fourier transform of u with respect to this variable x.

So we have this ordinary differential equation, which again it can easily be solved because with constant coefficients. So its solution will be in terms of the exponential function because the auxiliary equation will have, so if we write the auxiliary equation we have m square minus alpha square equal to 0, that means m, the roots are plus minus alpha. And then we have the solution, some constant e power this root alpha y c 2 e power minus alpha y. So we have the solution of this given differential equation but this is in terms of u hat naturally.

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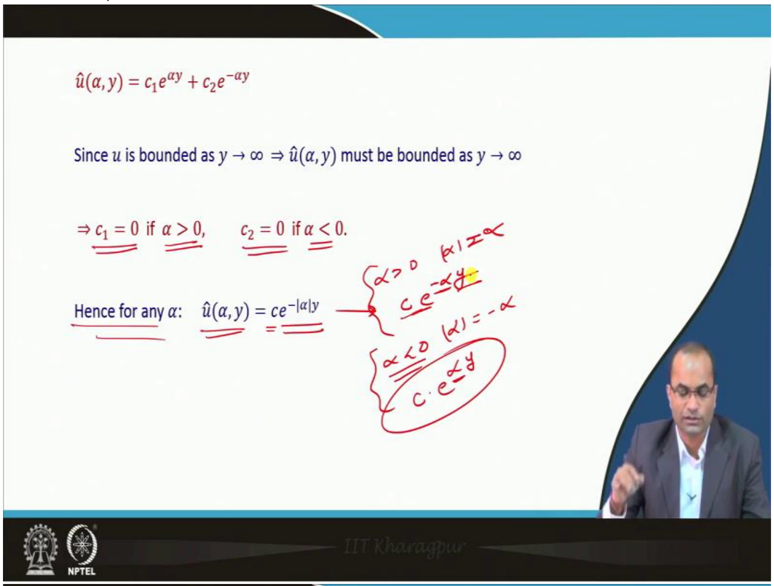

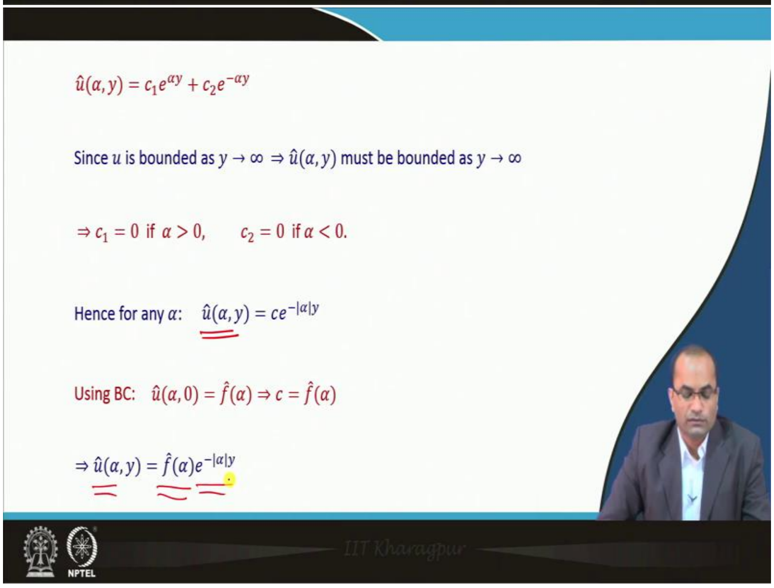

So having the solution now, if we look at the condition which is prescribed for u as y approaches to infinity, so this is given that u is bounded as y approaches to infinity and therefore in this when we take the, when we take the Laplace, sorry when we take the Fourier transform of this u, that means u hat alpha y, this must be also bounded with respect to y because y will remain as it is there. We have taken the Fourier transform with respect to x.

So if this u is bounded, the u hat y must be also bounded and having this information here in hand what we observe now, because if y approaches to infinity this is bounded. In that case, suppose this alpha is positive then this cannot be bounded, so the c 1 has to be 0. In the other case, if alpha is negative then this cannot be there in the solution and c 2 has to be 0. So one of them has to be 0 and only the exponential with negative power will supply. So if this alpha is positive then naturally, this part will survive and if alpha is negative then this part of the solution will survive. The other one has to disappear; otherwise this cannot be bounded there.

So we have the situation that if this alpha is positive then c 1 has to be 0 because this term should go to 0 and when the c 2 will be 0 if alpha is negative, because if alpha is negative this term will be positive and when y goes to infinity, this will be unbounded. So to have this condition the c 2 has to be 0.

So we have these two conditions that either c 1 will be 0 in this case when alpha is positive or c 2 will be 0 when alpha is less than 0. So to satisfy these two conditions that means for any alpha, whatever alpha is whether positive or negative, we can write a combined term for both, so u hat alpha, y will be some constant times this c e power minus absolute value of alpha.

So if this alpha is positive, the absolute value of alpha will be just alpha, so in that case this will become c minus alpha y and on the other hand, if this alpha is negative then this alpha will become some minus alpha and then again, we have the c times e power alpha y. So constant times alpha y, where alpha is negative or c constant e power minus alpha if alpha is positive, so we have this exponential decreasing behavior there with some constant. So this is for any value of alpha, we can write down that u hat alpha, y will be of this form, c e power minus absolute value of alpha y.

Now we can use the boundary conditions, the other one so which was u hat alpha, 0 is f hat alpha. So we have taken this Fourier transform which was u x, 0 is f x. So here we have u hat alpha is equal to f hat alpha. So if we put this y equal to 0 there, we can determine that c has to be now f hat alpha. So what we have now, the solution, the Fourier transform of the solution with respect to y is given as f hat e power minus alpha y.

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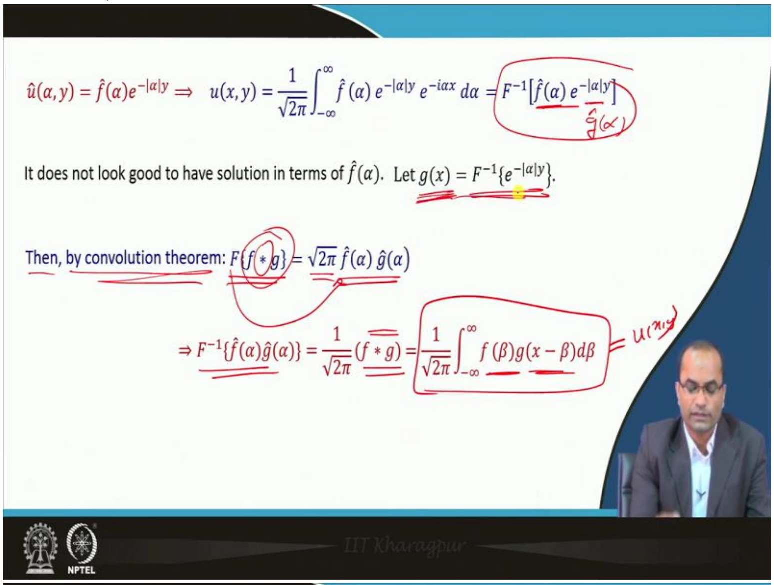

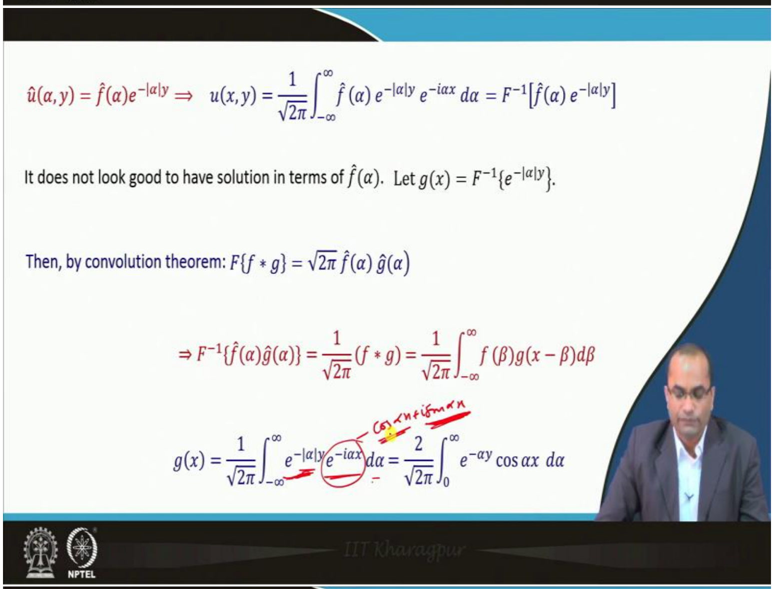

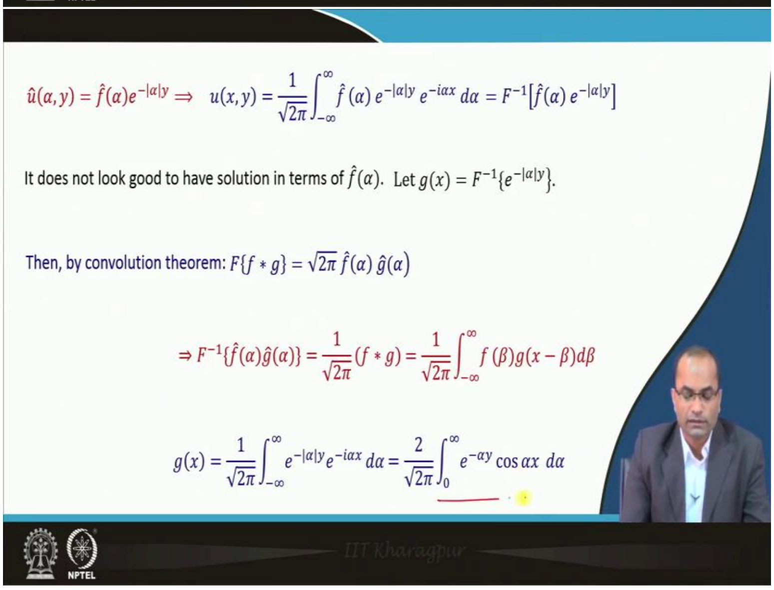

So having this form of the solution, we can take now the inverse. So taking the inverse, we got this u x y which says f hat and then e power minus absolute value of alpha y and e power minus i alpha x d alpha. Now the question is that how to simplify this one because we have the product here of two transforms, so one is f hat alpha and there is also a transform of some function we will find out which function.

So this means that this is the f inverse of this f hat alpha and e power minus this mod alpha y. Obviously, we do not have to leave here the solution in terms of f hat alpha. As usual, we are doing we will also get back to f finally and that will be the solution. So here we take that g x because we need to know that whose Fourier transform is this e power minus absolute value alpha y. So that is, means we need to know what is this g x f inverse of e power minus mod alpha y.

So then by this Convolution Theorem we know that the Fourier transform of f star g, the convolution of two functions is square root 2 pi and f hat alpha g hat alpha. That means we have this structure here, f hat alpha and then we have here this g hat alpha and this inverse will be just 1 over square root 2 pi and the, this convolution of f and g.

That means we are going to have the solution in this form because we are here with this u x, t so this is, this is going to be our u x, y and written in this integral form where we have f and g. But we do not know exactly what is g now, we can evaluate that. So g x is e power minus this absolute value alpha y, e power minus i alpha x d alpha.

So this we can write here as cos alpha x plus i sin alpha x and this with sin alpha x having this exponential function, power mod alpha y it is an odd integrand for this integral because this is an even one and then we have odd there so product will be odd. So that will be 0, the value of the second integral having the sin there, so only the first one will survive and that too with a even integrand. So that means we can have this two times 0 to infinity e power minus alpha y. So now alpha is just 0 to infinity, the positive, so we have removed the absolute value. So we have e power minus alpha y cos alpha x and d alpha. So this integral now we can evaluate.

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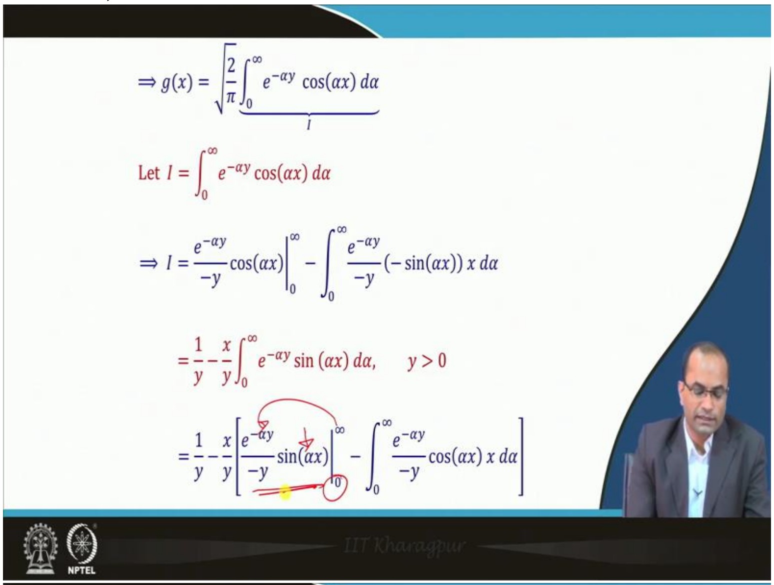

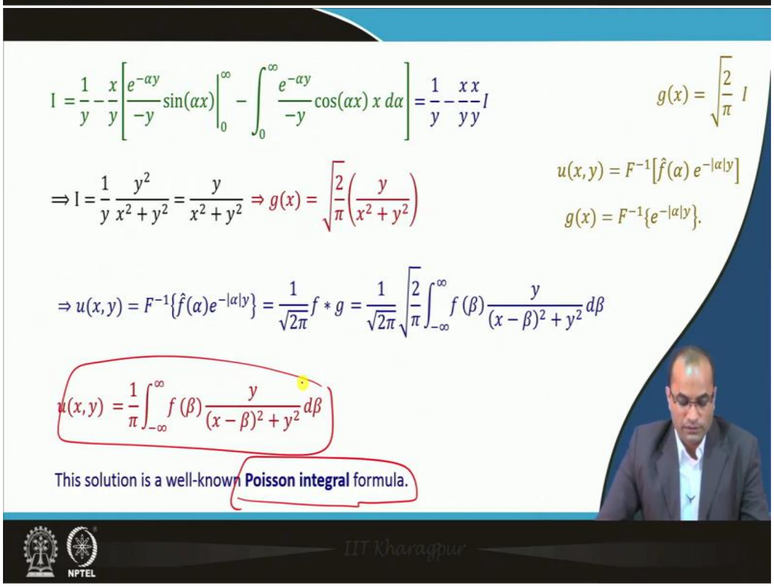

So let us take this I and having this I now, we have the integral we can integrate this by parts, so this e power minus alpha y and divided by this minus y we have this cos alpha x and here also. So with respect to alpha we are integrating, therefore this minus y has come after the integration of this e power minus alpha y.

Here also the same situation and the differentiation of this cos will be minus sin alpha x and then there will be x term. So after the simplification, because when this y goes to, sorry alpha goes to infinity and y is positive so this will go to 0, so only this when alpha goes to 0 that will survive. So this is 1 and then here we have 1 over y. So because of this first term we are getting 1 over y. Here we will get this x over y outside and then again we have e power minus alpha y and then sin alpha x d alpha.

Well, so this can be again differentiated. So this can be again integrated by parts so we have e power minus alpha y over y and then this sin as it is, sin will become cosine, so we can do this and then here in this place when this alpha is 0, this sin will become 0 and when alpha goes to infinity, this e power minus alpha y will make it 0.

So in either case this term will go out and then we have 1 over y there and then from here we have x over y and then again this x over y will come and then we have again, get back to the integral which we have started, that means I there. So now we can compute this I. It is y over x square plus y square. So the g x was square root 2 over pi and with this I. So we got the g x and having this g x in hand we can easily write down the solution because u x, y is f inverse of this one, and using the, using this Convolution Theorem, what we have, we have that this is 1 over square root 2 pi f star g. And we know g and we know the f, so the solution is simply f of some beta y over, the shift will be for the argument of g, that means x will be now x minus beta and then y square d beta.

So that is the solution now u x, y written in terms of f exactly. So this is the final solution for this equation and this is known as the Poisson Integral formula. This solution is known as the Poisson Integral formula for this Laplace equation.

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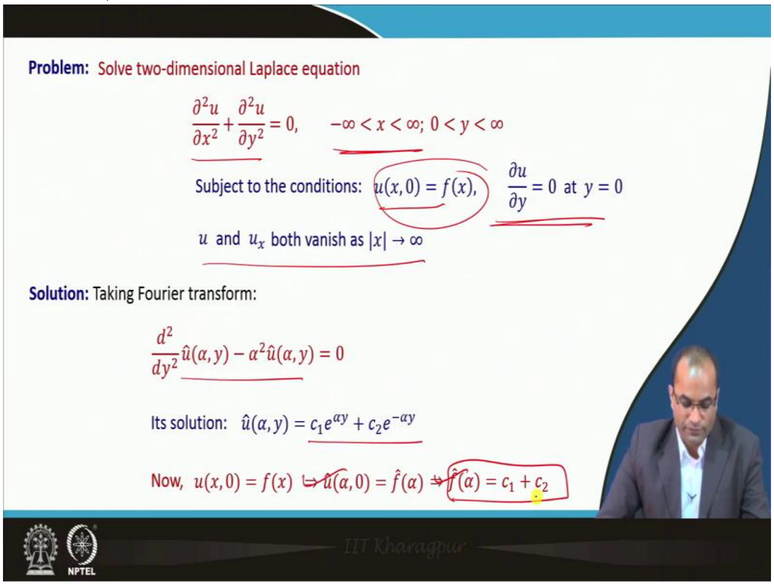

Well, so we will go through quickly with this problem, where we have again this domain minus infinity to plus infinity, so there is no much change here. u x, 0 is f x and this partial derivative with respect to y is 0 and again this is standard condition. So we will take the Fourier transform and then we get this here, its solution will be c 1 y and c 2 e power y. Having this initial condition, we can now get back to this one equation which says c 1 plus c2 is f hat alpha.

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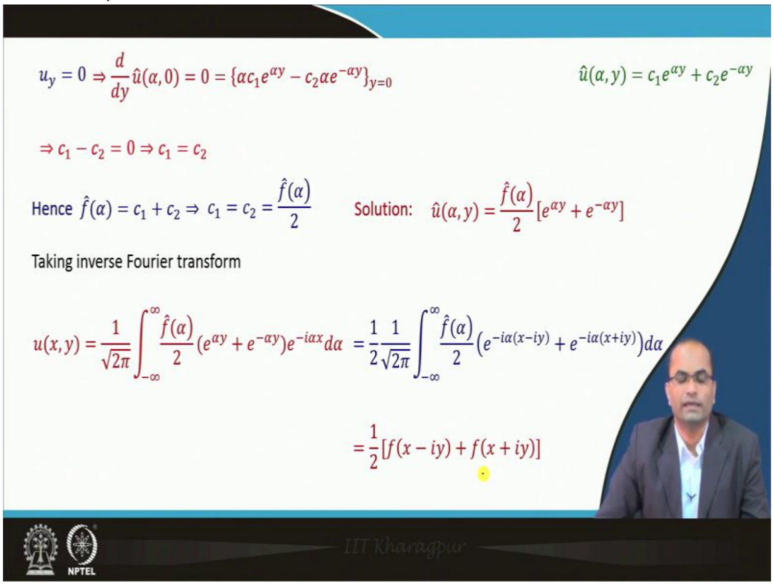

Then the second condition we can use now for dy over dt so which can give us again one more relation which says c 1 equal to c 2. So we can get this c 1 plus c 2 equal to f hat alpha that means c 1 is equal to c 2 is equal to f hat alpha. So its solution we can write down now in this form u hat is equal to f hat by 2 e power alpha y and e power minus i alpha.

So taking now its inverse Fourier transform, we can just club these exponential functions because this time now we have the exponential function so x minus y and here we have this x plus y and this can be written now directly with the inverse formula so f x minus y and here it will be f x plus y. So u x, y is equal to this half of x minus i y plus f of x plus i y. So it was a similar step what we have already done in many problems.

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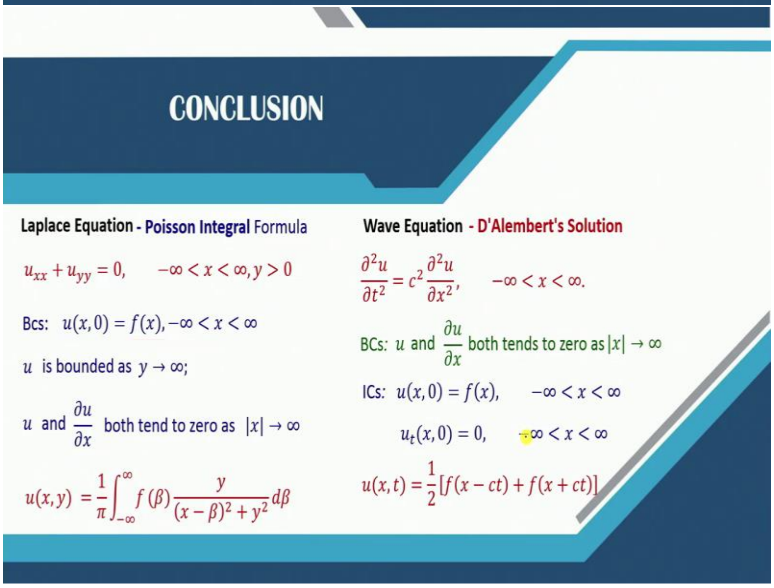

These are the references we have used for preparing the lecture. And just to conclude again, so we have discussed the Laplace equation and mainly, this Poisson Integral formula which is very well-known for this Laplace equation. So for this, given these conditions we got this formula which is known as this Poisson Integral formula. It was derived using the Fourier transform, and for the wave equation which was considered in the whole domain here and other variants also were discussed in this lecture. So having these boundary and initial conditions we got this well-known formula which is called the D' Alembert's solution of the wave equation.

So with this we are done with this Applications to the Fourier Transform, two applications and we have considered mainly three types of equations, heat equation, wave equation and the Laplace equation. So in many variants we got the solutions using this Fourier transform. So that is all for this lecture and thank you for your attention.