Engineering Mathematics - II

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Vector Calculus
Lecture 05
Conservative Vector Field

So, welcome to the lectures on engineering mathematics 2 and this is lecture number five on Conservative Vector Fields.

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So, today we will discuss that what are these conservative vector fields and the second again is very important topic in a vector calculus we will discuss Independence of Path. So, there are certain vector fields which if they are integrated over a path in the domain, then they are this, these integrals the line integrals are independent of path. So, we will also discuss this independence of path and the connection between these conservative fields and independence of path will be explored in this lecture.

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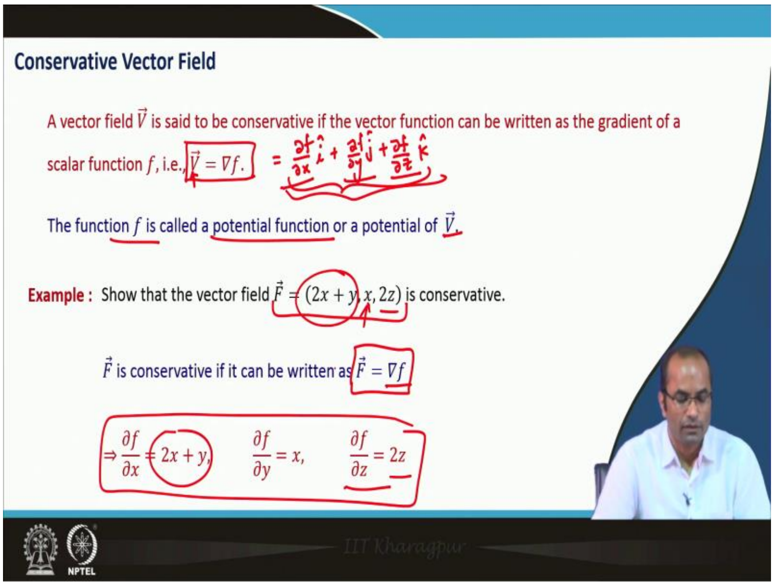

So, what is a conservative vector field? We will see now, so a vector field we said to be conservative, if the vector function can be written as a gradient of a scalar field f that means, if we can write this V that given vector field as this gradient f, the del f. So, meaning this del f if we can recall from the previous lecture. So, del over del x. The i component del over del y, the jth component and with for f here again f and del f over del z the third component, so, if this V can be written in terms of a function f such that this partial derivative with respect to x the first component, partial derivative with respect to y the second component and the partial derivative of f with respect to z is the second component

So, if we can find such a scalar function f for a given vector field V then we call that V is a conservative vector field. So, the function this f is called the potential function or the potential of the vector field V. And for example, if we take we want to show here that the given vector field f, where there are 3 components to x plus y x, and this 2 z is conservative.

So to prove that this given vector field is conservative, we have to now find f, such that we can write this f as the gradient of small f. So if this is conservative, if we can write down F as, the gradient of f. So the what is the gradient of f we have just seen the partial derivative with respect to x must be equal to the 2 x plus y which is the first equation here and del F over del y, the second component must be equal to the x the second component here and del F over del z must be equal to 2 z. So, now, with the help of these 3 equations, if we can find a small f such that this gradient f is equal to the vector F, then we will call that this f is a conservative vector field.

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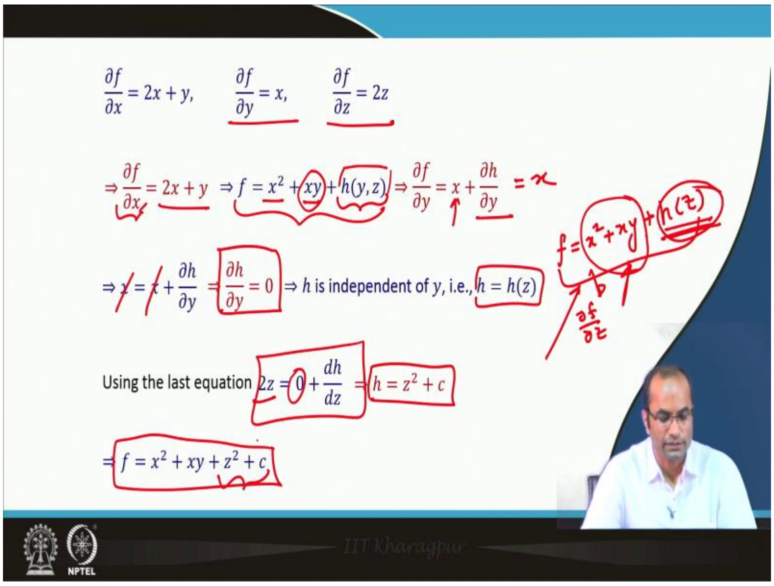

So, having these 3 equations we will start with the first one. So, the first equation is a partial derivative with respect to x is equal to 2 x plus y. So, if we integrate this partially with respect to x, then what we will get here we have 2 x. So, that will be 2 into x square by 2. So, we have x square and then y that will be x y and plus some this constant of integration term with what we use.

But in this case, since this f depends on the 3 components x, y, and z 3 independent variables. So, here as a constant of integration we can write as a function of y and z because we were different integrating with respect to x. So, having this expression for f. Now, we can use the second equation, that means from here we will get the partial derivative of f with respect to y. So, here del f over del y would be… so x square that is become means zero here and with respect to y, so from x y, we will get simply x and then we have partial derivative of h with respect to y. And this should be equal to from the second equation should be equal to x.

So having this relation this x, x get cancelled and then we have the partial derivative of x with respect to y is 0. So what this relation says that h is independent of y h does not depend on y though here we have assume that h can be a function of y and z, but this relation makes sure that h is a function of that alone, it is not a function of y that means h can be a function of z only not the function of y. So, now our… this expression for f reduces to x square plus this x y and then we have a function of z alone, not the function of y.

So, from this now, we will use the third equation that means, from here we will get del F over del z and that will be z equal to 2 z. So, using this last equation we have 2 z equal to when we do the partial derivative here with respect to z. So, there is no term here in first 2 terms there is no term of z. So, that will be 0 and from the third we will get dh over dz. So, we have this relation that h equal to when we integrate this z square plus a constant.

So, h is now is z square plus a constant which we can substitute there and we have f x square plus this y square and then this h z that is z square plus c. So, this is the potential function whose gradient is the given vector field that means, the given vector field is a conservative vector field.

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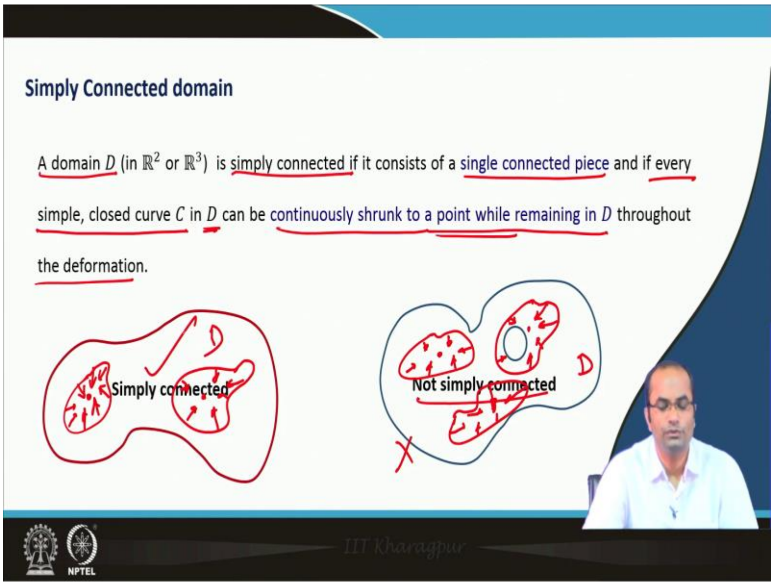

So, now, we will go for the test with the help of that test we can find out whether the given vector field is conservative or not. So, for that we need some preparation. So, this is the simply connected domain. So, what do we mean by the simply connected domain? A domain D is simply connected if it consists of single connected piece. So, there will be only one piece of the curve not the 2 different pieces.

And every simple closed curve in C, every simple closed curve C in this D can be continuously shrunk to a point while remaining in D. So, I will explain you what do we mean by and throughout the deformation. This is what the definition of the simply connected domain just look at this domain here.

So, if we take any simple closed curve in the whole domain, then this closed curve can be continuously shrunk to a point without or while remaining in D, we will not leave D and without leaving D we can shrink this curve to a point here and any curve we can take any closed curve we can take, we can shrink this curve to a point without leaving the domain D. If this is the property hold in a domain then we call this a simply connected domain.

Whereas, for instance, if we consider this domain which is not simply connected domain in that case, if you take a curve here at at this place you can it can be shrunk to thz point to a point there is no problem we can take a curve here that has also the the same property that it can be shrunk to a point, but if we take for instance this closed curve which is also in our domain D, but without leaving the domain we cannot shrunk this to a point that is the problem here.

So, every closed curve every simple closed curve cannot be continuously shrunk to a point while remaining in the domain D does not have that property. So, therefore, this is not simply connected domain, but this one is simply connected domain because it has that property. So, this terminology will be used in next slides.

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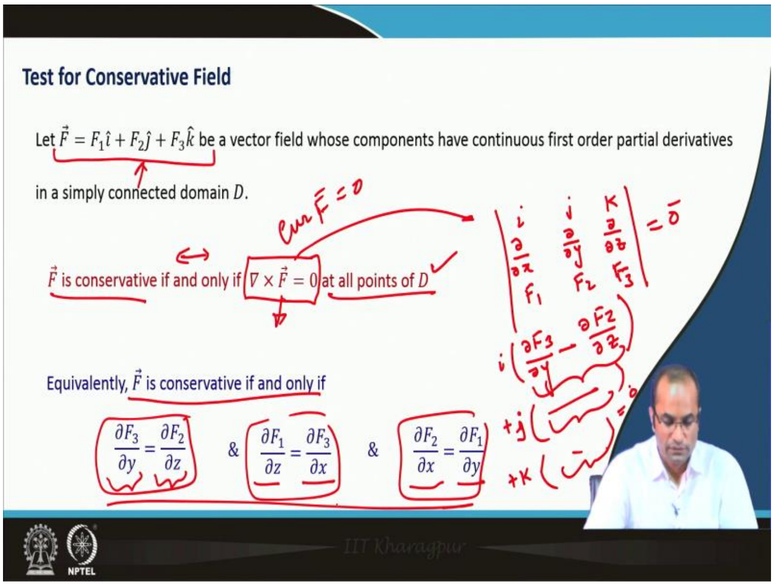

So, now, we will come to the test of the conservative field we have seen that a conservative field can be written in terms of a potential function, so, that the conservative field will equal to the gradient of that scalar which is called the potential function.

Here we will see that how to test if a given function for example, this f is equal to f one f 2 f 3 is given, then how to test that the given field is conservative or not without finding the potential function as we have done in the previous example that we, we found the potential function and if we are able to find the potential function we will conclude that okay this is the conservative field, but without finding the potential function how to conclude that the given vector field is a conservative vector field.

So, here F is conservative if and only if this is the result that del cross F, del cross F is 0 at all points of D, then f is conservative and this is the either way round, if f is conservative, then this del cross F has to be 0 or this del cross F is 0 then F has to be conservative, this is the idea so, this is the curl of F. So, this is the curl of f is zero, then f is conservative and if F is conservative then the curl F has to be at all points of the domain D.

So, if F is conservative if and only if this is equivalent condition because this curl f we can see that this curl F equal to 0 is nothing but having this partial derivative of f 3 with respect to y equal to the partial derivative of f 2 with respect to z and the partial derivative f 1 with respect to z and equal to f 3 with respect to x.

And there is another condition, the partial derivative of f 2 with respect to x should be equal to the partial derivative of f one with respect to y. So, these conditions are equivalent, because this curl F, we know that we can find with the help of this determinant. So del over del x and del y del over del z and then we have this f 1, f 2 and f 3. So this is the condition here that this should be equal to 0. So if we expand this, so we have with the i component, we have del f3 over del y and minus del f 2 over del z. This is the first component with i similarly, we will have with the j and then with the k.

So, if this everything is equal to zero 0, then what will happen that this is possible when the individual terms here are 0. So, that means this del f3 over del y equal to del F to over del Z. So, that is the precisely this the first condition the similarly from the second one we will get the second condition and from the third component we will get this third condition. So, these 2 are equivalent whether we say that the curl F equal to 0 or we check these 3 equations the 3 partial derivatives must be equal to 0 both are equivalent.

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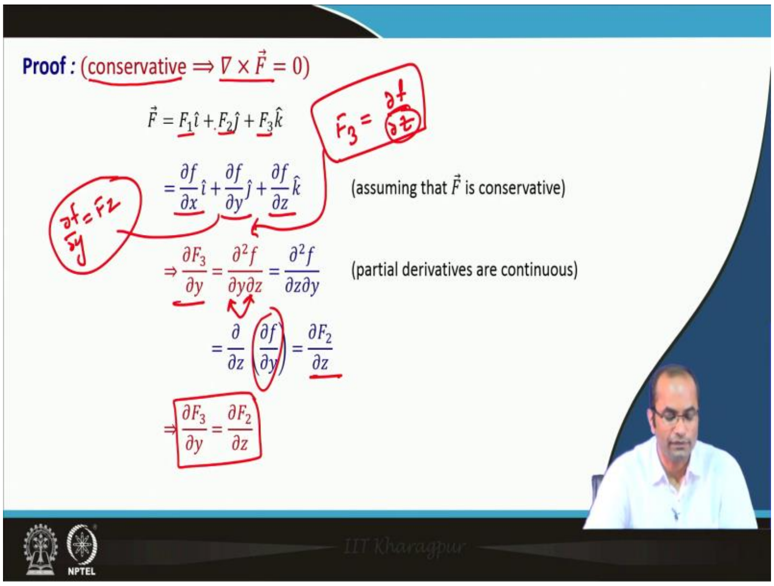

We will see just a short proof but only in this way that we will assume that the given vector field is conservative and then we will prove that it's curl is zero. So, if the given vector field suppose we have F 1 F 2 and F 3 is conservative, then we can write down this as a gradient of f. So, what is the gradient of f ? we have here, these partial derivatives constitutes the 3 components fx, fy and fz.

So, if this is the case we can compare now, because this is equal that means del f del x must be equal to f1 del f over del y must be equal to f 2 and del F over del z must be equal to f 3. So, from the third equation we have del F 3 over del y.

So, we have the third component equal means f 3 equal to del f over, del z the all others are also equal. So, let us just start with this one. So, if we differentiate this with respect to y del F 3 over del y, then we have here del 2 F, the second order mixed derivative with respect to y and with respect to z it was already there.

So out of this, we can just change the order here, so del 2 F and then del z first and then del y and this is possible if he assumed that these partial derivatives are continuous. And having this we can again rewrite this del over del z of del F over del y and what was del F over del y the del f over del y, the second component equality of the second component says that this is equal to f 2. So this is what we have written here del over del z F 2. So what we get the partial derivative of f 3 with respect to y must be equal to the partial derivative of f 2 with respect to z.

So this one of these conditions we got just by looking at this third component and using the second equality, similarly, you can check all other components that they are also these partial derivatives are equal four, if we have the conservative vector field.

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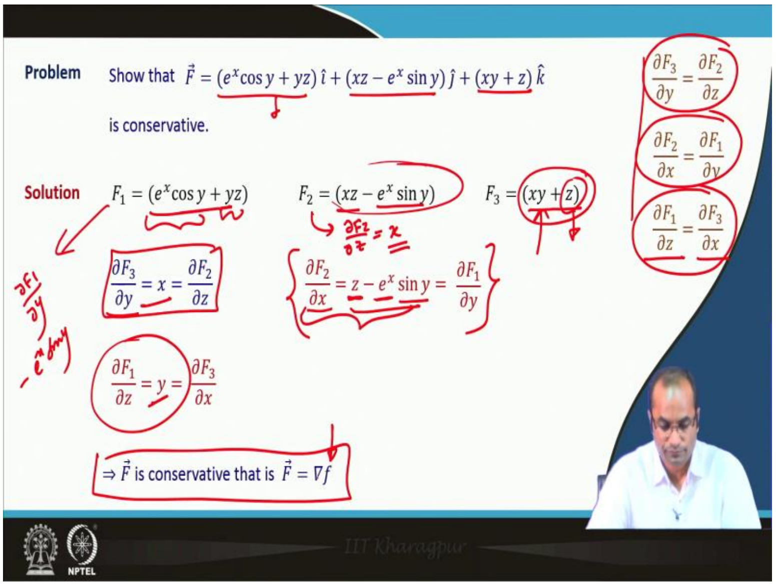

Well, so, here we will show that this given vector e power x cos y plus y z it and the second component and then we have the third component as x y plus z is conservative. So, to show that this is conservative, we will apply the idea which was developed now. So, we have the F one the first component which is e power x cos y plus yz, the second component we have xz e power y and sine y the third component here x y plus z So, f 1 f 2 f 3 and we know these conditions that if these conditions hold then we have the conservative vector field.

So, one way could be that we directly straight away as in the previous example we have done, we can try to find the potential function, but here we want to just show that this function is conservative so we do not have to just directly find the conservative potential functions here we can with the help of these conditions we can prove that this is a conservative vector field. So having this let us try for the first condition del F 3 over del y.

So here we have the F 3 and we with respect to y if we differentiate this, so we will get just x so del f 3 over del y will be equal to x from this condition and now it should be equal to del F 2 over del z. So if we get out of this here, del F 2 over del z. Then what will happen? So here we have just the X and there is no z there, so we will get X. So which is equal.

So, this the first condition is satisfied the first equality of the derivatives is satisfied. The second one if we get del f 2 over del x, so del F 2 over del x from here, so, we have xz that means, from here the z will survive and then e power x sine y. So, with respect to x, if we differentiate, we will get e power x the derivative of e power x is e power x and then we have this sine y which will be treated as constant and then this partial derivative should be equal to del F 1 over del y.

So, here we have F1. So, if we conclude the partial derivative of F 1 with respect to y then so, the first term will here with respect to y we will get minus e power x sine y and plus z. So, we have exactly the same relation that del f 2 over del x equal to dz minus e power x sine y, which is equal to the partial derivative of F 1 with respect to y the third condition we have the partial derivative of f 1 with respect to z must be equal to the partial derivative of f 3 with respect to x.

So, the partial derivative of f 1 with respect to z if we compute we will get only y because first term does not have z. So, only the second term has z which will give us y and this del f 3 over del x will also get this y the same because here this is just z which will become 0 because we are making this partial derivative with respect to x.

So, this F is conservative because all these conditions are satisfied. And we have in that case once we show that these conditions are satisfied, one can get such a potential function with the process which we have seen in the previous example.

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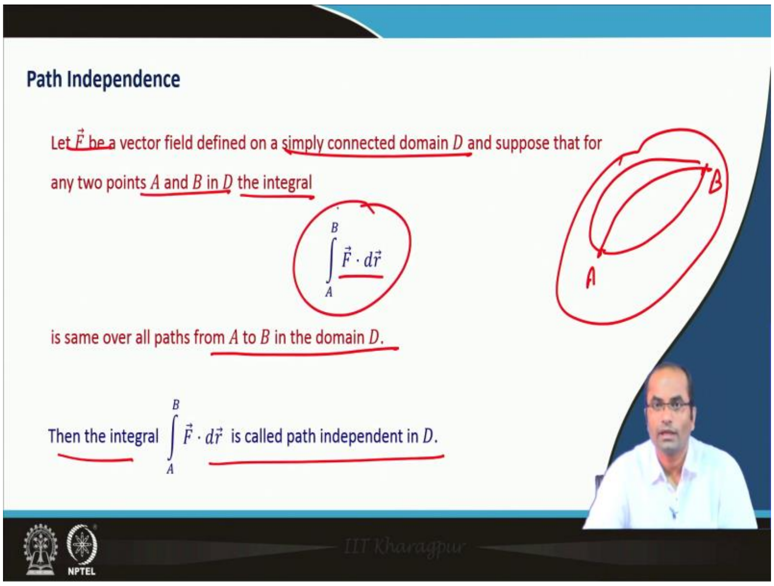

Now, the concept of this path independence So, let F be a vector field defined over a simply connected domain and we suppose that any 2 points A and B in D so, we have some domain here take 2 points A and B and this integral consider this integral from A this point to this B the curve integral.

So, there is some path here, where we are evaluating this line integral. So, what this says or we will prove also indeed, that the given vector field this integral is same over all the field from A to B in the domain D and then this integral is called the path independence of, the path independence of this given integral in D.

So, if any part we take along any path if this integral is same, then we call that this integral is Path Independent in D. So, this concept will be used to relate this path independence with D conservative vector field in the next slide.

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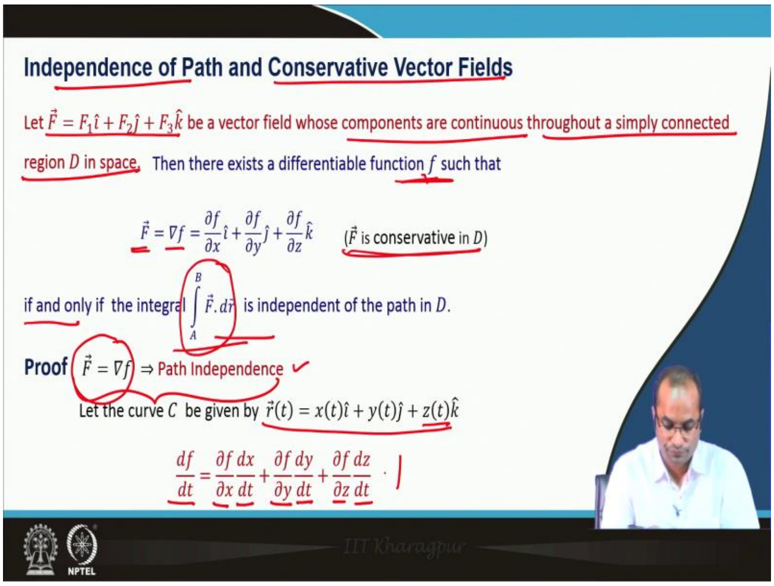

So, we have the independence of path this relation with the conservative vector field. So, suppose we have F 1, F 2, F 3 this a vector field whose components are continuous throughout a simply connected region D in space. Then there exists a differentiable function f such that f is conservative in D that means f can be written as a gradient of f if and only if this integral is independent of path in D.

So, if we prove that this integral is independent of path in D that means, that f is conservative or the other way around if F is conservative one can prove that this line integral is path independent. So, the only this simple proof we will see that we will assume that F is a conservative vector field and then we will prove that it is path independent a given integral here is path independent.

So let C be the curve given by this x t yt and zt and in this case, we know already from the chain rule that since f is a function of x, y z and x is again a function of t, and y is a function of t and z is a function of t. So with the chain rule we can get this derivative df over dt as the partial derivative of f with respect to x dx over dt partial derivative of f with respect to y, and dy dt, the partial derivative of f with respect to z and dz over dt. So with this chain rule, let us continue here with the next slide.

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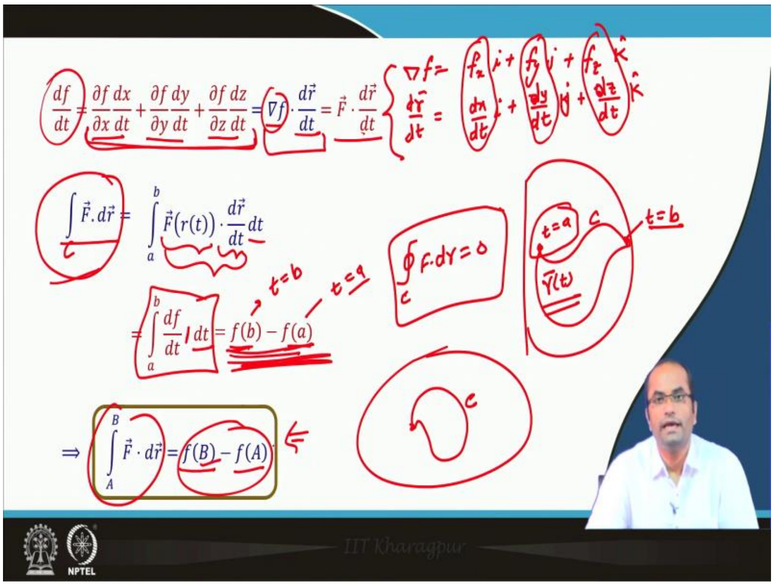

So we have this chain rule to get this df over dt, which can be written as the gradient f the dot product of dx dt just recall so the gradient f is given as the partial derivative of f with respect to x, the partial derivative of y with respect to j and the partial derivative of f with respect to z and k and this dr over dt is this dx over dt, the first component than dy over dt, the second component and then we have dz over dt, the second component or third component, so this is the second one j and then we have the k.

So, the dot product of these 2 vectors will be the fx into dx over dt the first component here fy, the partial derivative y with respect to y and then dy over dt and fz and dz over dt the third component. So, this chain rule here can be written as the gradient f dot product of dr over dt or we can write down because we have assumed that this F is a F is equal to this grade f, f is a conservative vector field.

So, that means we have f dot dr over dt. So, now, if we takes this integral as f dot dr and f dot dr we have seen that this is equal to df over dt, so, which we can replace in a minute. So, we have here f dot dr and then dt, but this F dot dr dt can be written as df over dt. So, here we can replace this by df over dt and then we will integrate over so, there is no dot product here we can integrate now over dt and then we immediately observed from this relation that this is nothing but Fb minus Fa. That means F evaluated as t is equal to b. And this is evaluated at t is equal to a, that 2 ends of the curve.

So, here in this domain, we have some curve, which varies from t equal to a to t equal to b, and the equation was given by our rt, which we have written before. So, the value of this integral of this line integral over this given curve C is fb minus fa, that means, the F is evaluated for t is equal to b and minus F is evaluated at t is equal to a. So there is no path dependency here if we take any other path, the same value we will obtain fb minus fa.

So this value does not depend on path, but in general, we will see also later that these line integral depends on the path. But if f is a conservative vector field, then this line integral will not depend on the path, this is what we have seen here because the value of this line integral if your F is conservative is simply fb minus fa and this is small f is the scalar function corresponding to this f so, that is f is equal to gradient of this scalar function f that means, what we get out of this, that this integral from the point a to b in that domain is equal to the f evaluated at the point b minus f has to evaluated at the point a.

The path does not play any role. So, here just another observation that if we have a closed path, we have the simply connected domain and there we have taken a closed path C, so, what will happen here the b and a are the same the initial and the endpoints are same. So, in that case this value will become 0.

So, for the closed path, the same integral the line integral from over a closed path C f dot dr will become 0 that is another consequence of this path independence, which we have proved here that if a vector field is a conservative vector field, then the integral, this line integral does not depend on the path, it depends on the only the end points of the curve.

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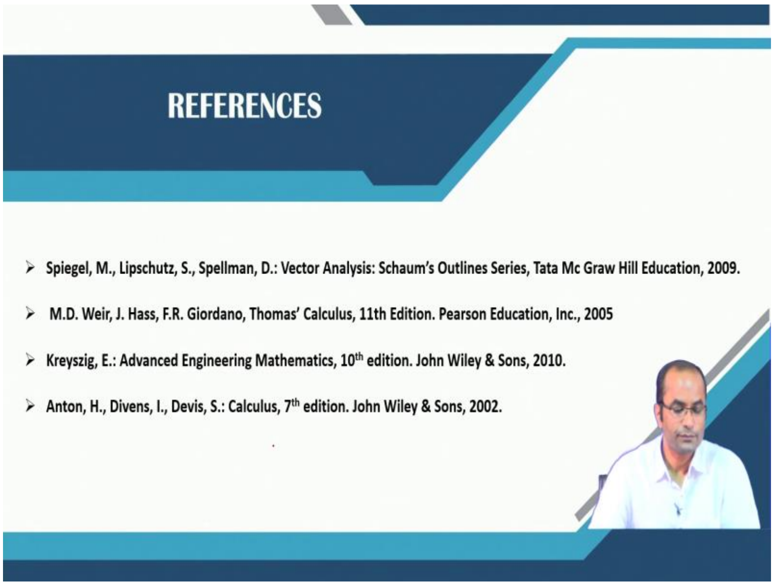

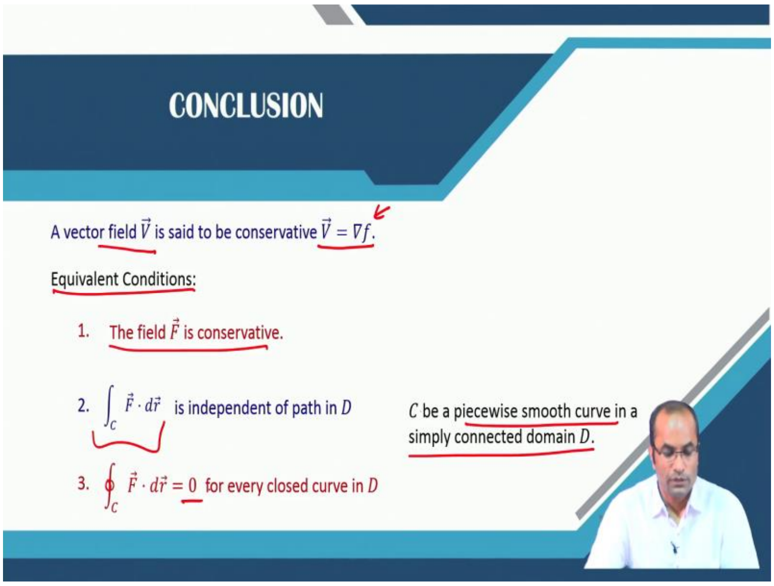

So, these are the references we have used for preparing this lecture and to conclude, so, what we have learned that a vector field V set to be conservative if it can be written as the gradient of f. This f is a scalar function and the equivalent conditions which we have already discussed, that the vector field F is conservative. This is equivalent to that, this line integral is independent of Path in D, where the C is a piecewise smooth curve in a simply connected domain D.

So, we have simply connected domain and you take any, any part between the 2 points A and B, that value of the integral will be the same. The third consequence we have seen that if we are talking about a close curve, then the value will be 0 for every clause curve in D that is just a consequence which we have seen, because this integral depends only on the end points and if they are same, then naturally this integral will become. So, that is all for this lecture. And thank you very much for your attention.sssss