Engineering Mathematics II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur
Lecture 49 - Applications of Fourier Transform to PDEs (Part I)

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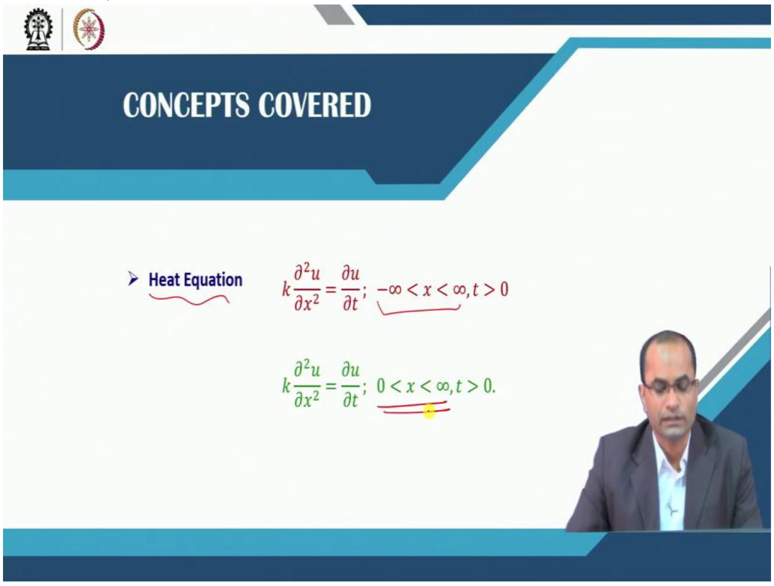

So welcome back to lectures on Engineering Mathematics II and this is lecture number 49 on Applications of Fourier Transform to Partial Differential Equations. So this is part I so there will be one more lecture on this. So in this lecture, we will, in particular focus on heat equation and that too in two forms. So we have already discussed this heat equation in the previous lecture. So in one case, we will go for this domain which is from minus infinity to infinity and in the another situation, we will go with this half of the range from 0 to infinity.

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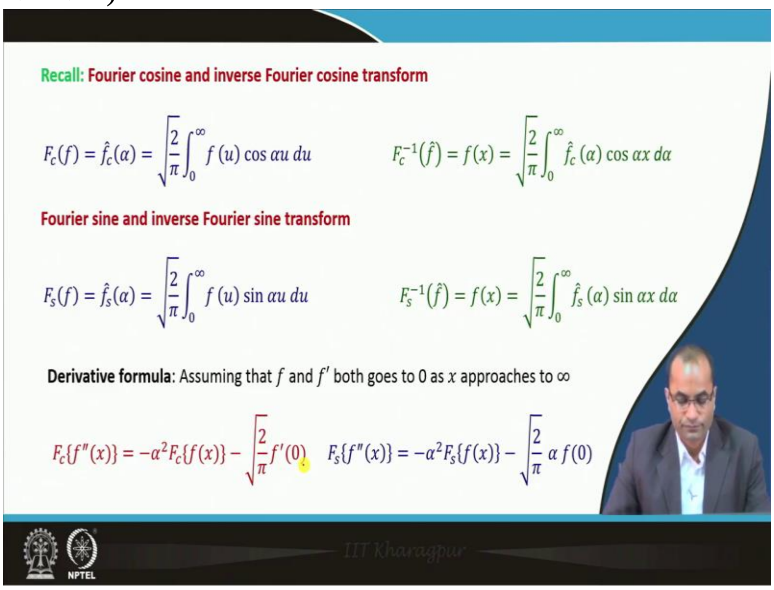

So just to recall, the all the formulas that will be used for solving these differential equations and in particular, we have already talked about Fourier cosine and inverse Fourier cosine transform because they will be also used in this lecture.

So the Fourier cosine transform was with this factor square root 2 over pi f u and then cos alpha u, and then we have the corresponding inverse which get back to f with, if we use this Fourier cosine transform here and this cos alpha x integrate over this alpha then we will get back to this function f. So this is called the inverse Fourier cosine transform.

Similarly, we have for the sine one, where instead of this cos, the sin appear and we, the important for the point of view or from the point of view of solving differential equation is the derivative formula. So in the derivative formula, we have learnt that if f and f prime, the derivative of f prime and this f they both go to 0 as x approaches to infinity, in that case the Fourier cosine transform of the second derivative because we need most of the time second derivative, I am focusing on the second derivative, we have minus alpha square, the Fourier cosine transform of f x, and minus square root 2 over pi and then we have the first derivative evaluated at 0.

Similarly, when we take the Fourier sine transform of this double derivative then what we get, minus alpha square the Fourier sine transform of this f and then we have this extra factor square root 2 over pi and alpha into f 0. So what is the difference basically in two which we have to keep in mind? Then in this cosine transform this f prime is appearing in the formula whereas in the sine transform f 0 appears in the formula.

So based on the given information if in the problem, f 0 is given then we will use the sine transform because that value will be directly used here f 0, we need f 0 if we want to apply the sine transform and if f prime is given in the problem then we have to apply the cosine transform because we can use this f prime 0 the given information directly in this formula. So that point we will keep in mind while discussing the partial differential equations.

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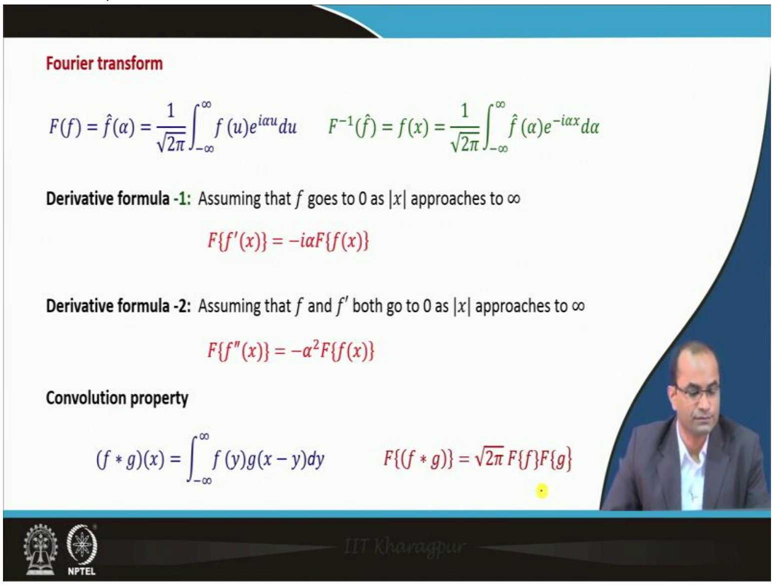

Then we have the Fourier transform and the major difference between this cosine and sine, so here the Fourier transform appears from minus infinity to plus infinity, the other two were 0 to infinity. So depending on the given domain, or the given range of x, we will choose whether we go for Fourier sine or cosine transform or we have to apply the Fourier transform that depends on the problem we will discuss in next slides.

So the Fourier transform of a function f is defined by this integral, this improper integral where you have this function and exponential e power with the plus sign alpha u and i. And its counterpart, so the inverse that means if the Fourier transform is given then we can use this improper integral to get back to the f x.

So these two will be used there and the most important, the derivative formula where we assume that this f goes to 0 as this x approaches to infinity and in that case the derivative formula was f of this f prime minus i alpha, so this factor i alpha comes out of this Fourier transform and then we have the Fourier transform there.

The derivative formula 2 which assume that f and f prime, the both go to 0 as x approaches to plus infinity or minus infinity and this is what we will use now in this lecture for the second order derivative. So for the second order derivative, it is even much simpler so we have just minus alpha square out of this Fourier transform of f and we can get rid of the derivative term which is the strength of this Fourier transform in connection to the solution of partial differential equations.

So here, and we will also use the Convolution property which just to recall, it was the convolution integral where we have this f y and g x minus y as the integrand and the Fourier transform of this f star g is given as square root 2 pi, the Fourier transform of f multiplied by the Fourier transform of g. So that was the Convolution property and we have already discussed in previous lectures.

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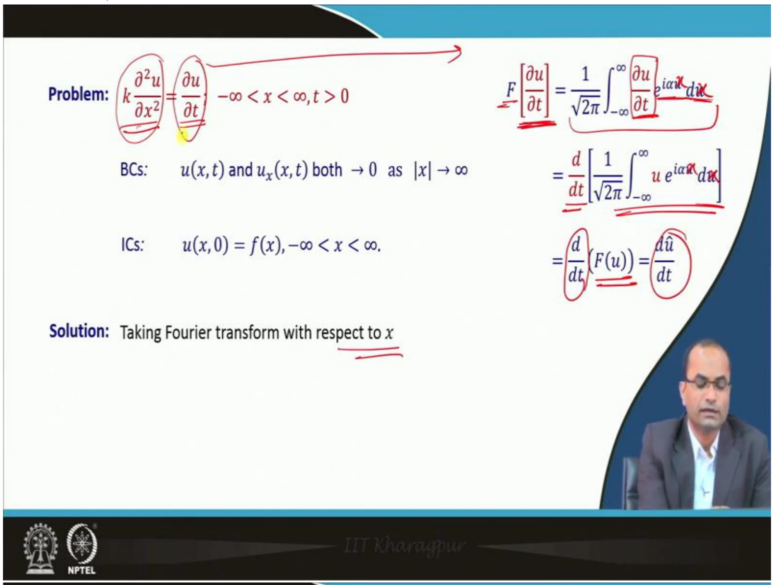

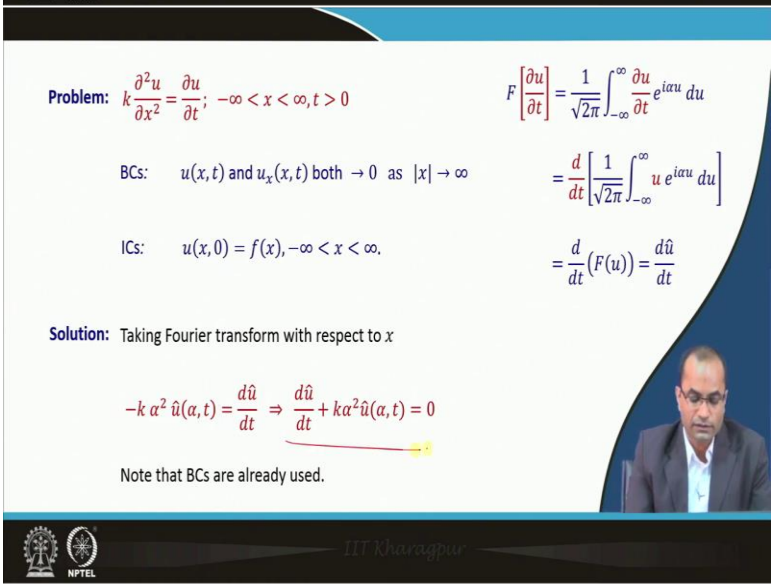

So now we will get back to the problem. So we have here the heat equation. So the first we are considering the heat equation and given in this domain from minus infinity to infinity. So the x is given from minus infinity to plus infinity and then we have here time which is positive, and there are more information here.

So the boundary conditions for this problem is prescribed as, or are prescribed as u x, t and u x so the x derivative with respect to x both goes to 0 as x goes to plus infinity and minus infinity. And this is exactly the condition we need for the application of the derivative formula we have just seen before. Once we have these two conditions then the derivative formula is very simple and we can get rid of the derivative terms from the differential equation.

So naturally, since these informations are given here which are directly related to the Fourier transform and in particular to this derivative formula, we will naturally apply the Fourier transform with respect to x because now we have two variables here, x and t. So we have the possibility of basically applying with respect to t or with respect to x but it is natural now given such conditions because if we look at the initial conditions, we have the information that u x, 0 is equal to f x that is the initial condition given.

So we do not have these conditions with respect to this time there, naturally we have with respect to x. So we can use the Fourier transform with respect to x obviously. So in this case, if we apply the Fourier transform now with respect to x then what will happen here with respect to t? So when we apply with respect to x, the t will be treated as constant, the second independent variable will be treated as constant.

So we will apply just with respect to x. So for instance, if we look at the right hand side which is with respect to t here and we are not applying this Fourier transform with respect to t, we are applying with respect to, so applying this definition there del u over del t this function and then we have e power i alpha u du. So we are not applying here with respect to, or this should be now better to have this x because u is already there.

So here in this definition now, with respect to time we can take, we can bring this out there and then here we can say that this is the Fourier transform of this u. So this d over dt term will just come out from the Fourier transform of u or we can say that the right hand side will still have this del over del t and this u will become u hat because we are taking with respect to x, we are not touching t there.

But the left hand side when we have the derivative there with respect to x, there the derivative formula which we have just discussed will be applicable and we have a simple term corresponding to this second derivative or the second derivative will be removed once we apply the Fourier transform.

So applying this Fourier transform and just to recall the formula here, we had this minus alpha square so that factor will come and the Fourier transform of u and the right hand side we have d over dt and the Fourier transform of u hat. So we applied the Fourier transform to this equation and now we got the equation in this transformed variable u hat.

The steps are the same which we had discussed in previous lectures that if we have a differential equation we have to apply just the Fourier transform, we will get the new equation which is in transformed variable which is normally much easier to solve. So this is exactly happening here. We have this partial differential equation which is difficult to solve by other means here. But if we use this Fourier transform for instance, we got a simple ordinary differential equation in u hat and once we solve this u hat, we can get back to the u by taking the inverse Fourier transform. So these are the steps now.

So we will solve this differential equation. So we have del u by del t plus k alpha square and u hat alpha and the boundary conditions are already used. That we should also notice here that these boundary conditions are already used when we have used this derivative formula because that derivative formula we got this minus alpha square this u hat using these boundary conditions. So that is already used here. Only the initial conditions are not used, that we will take care or we will be utilizing them little later.

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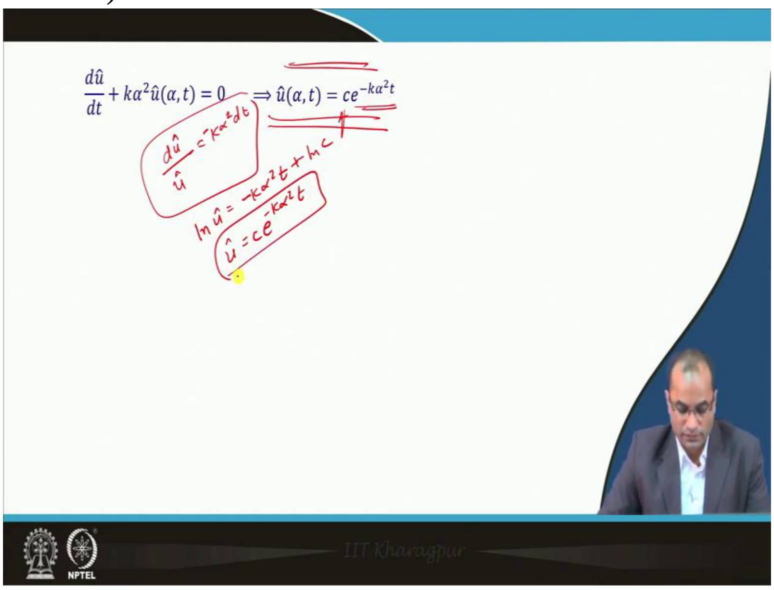

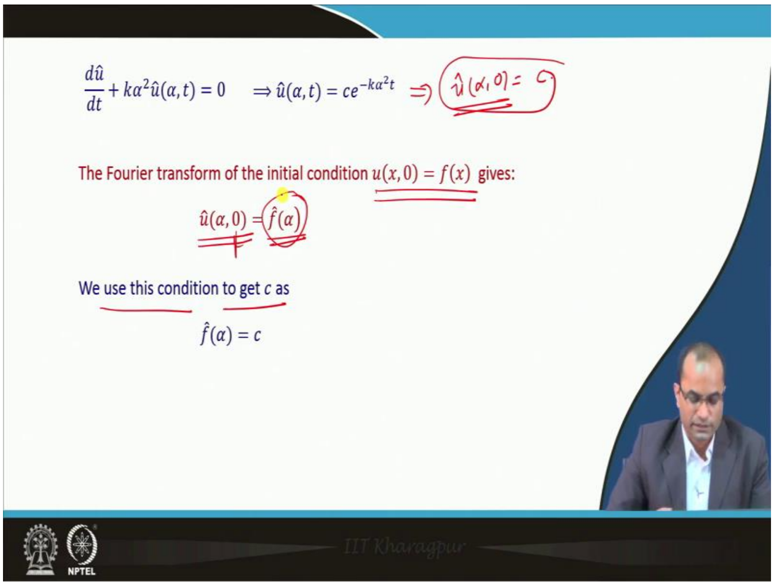

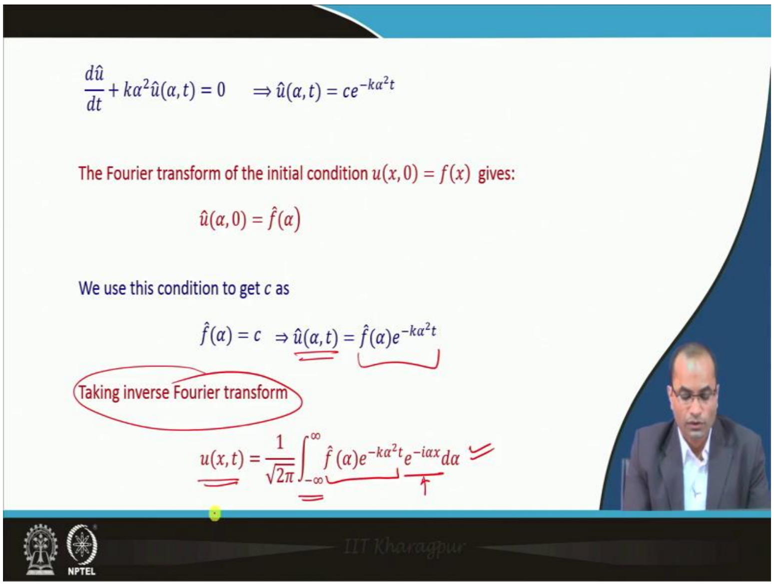

So here we have this differential equation which we can solve easily. Having this differential equation, the solution of this differential equation will be simply some constant times e power k alpha square and t, because we can separate this. So the left hand side we have du hat over the u hat and then we have k alpha square and then dt and once we integrate this, so we have ln this u hat and there k alpha square and t then the u hat will be, so there is minus sign there, e power minus k alpha square t with this constant term there. So we have to also add a constant there, so we can have this ln some constant.

So we will get this solution out of this simple differential equation. So we got this u hat as a function of alpha and t, c e power minus k alpha square t. But the c is unknown here which we can get using the initial condition which is given in the problem that u x, 0 is equal to f 0.

So having this information u x, 0 is equal to f x, 0 we can take the transform here, the Fourier transform and then we will get the condition in terms of the u hat alpha 0.

So having this u x, 0 is equal to f x, if we take the Fourier transform we will get u hat alpha 0 is equal to f hat alpha. So we use this condition to get the constant c there. So if we put t equal to 0 so what we will get, u hat alpha and t is 0, so c. And u hat alpha 0 is f hat alpha that means the c is f hat alpha. So we have now this u hat alpha as f hat alpha which is the Fourier transform of f and then here we have this e power minus k alpha square t.

So now we can apply this inverse Fourier transform to get u x, t. That means the u x, t will be 1 over square root 2 pi minus infinity to plus infinity, this transform here the Fourier transform and then we have e power minus i alpha x and integrated over this d alpha.

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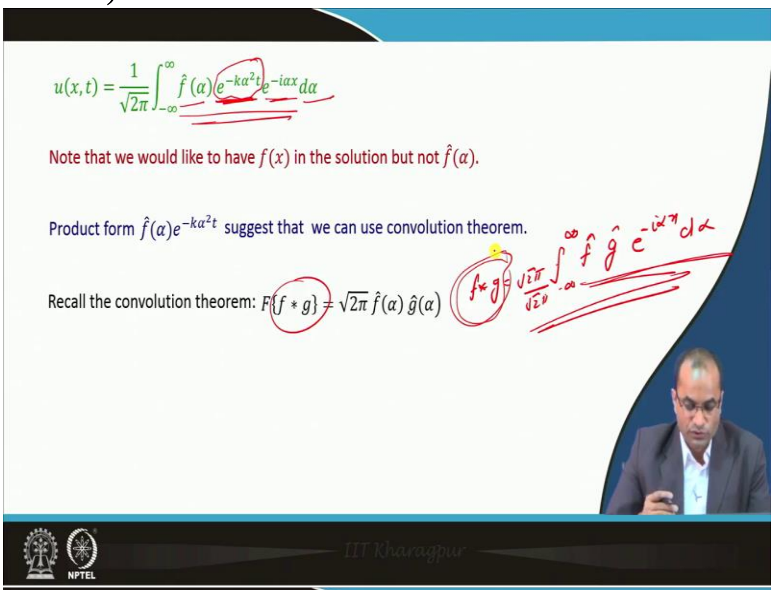

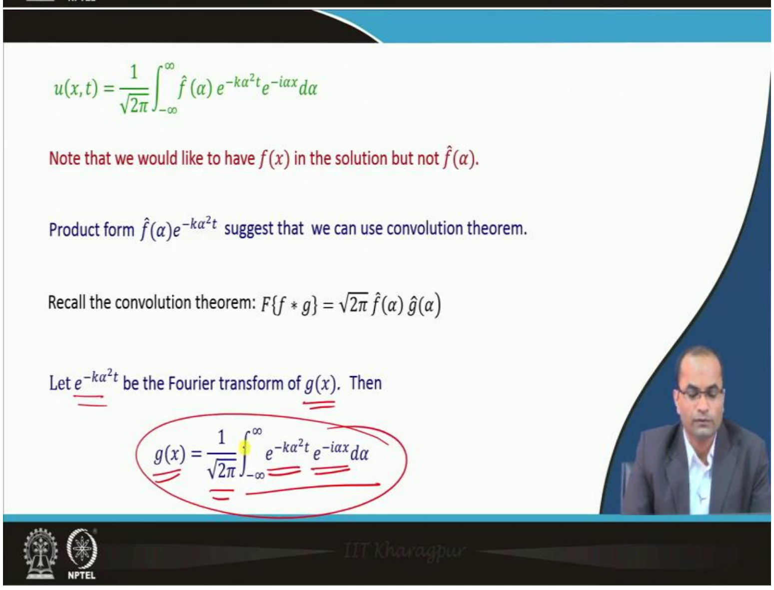

Well, so having this u x, t now in this form e power minus i alpha x term is there, though we have this u x, t but this is written in terms of f hat alpha. f hat alpha is not given in the directly in the problem, the f was given in the problem. So we would like to have f in the solution, not its Fourier transform f hat.

So we have to further work out a bit to get a better solution, a better form of the solution which is directly related to the given conditions; so here we do not want to have this f hat but we rather would like to have this f x in the solution so we will work for that now. This product here, these are the two Fourier transform, one is f hat another one is e power minus k alpha square t.

This suggests that we can use the Convolution Theorem and exactly we will use it because here we have the Convolution Theorem which says that Fourier transform of this convolution is equal to square root 2 pi the product of the Fourier transform, and we are getting that product there and because if we take the inverse here, so we have f star g is equal to square root 2 pi and then the f inverse, f inverse means 1 over square root 2 pi then minus infinity to plus infinity f hat then g hat and e power minus i alpha and then we have this d alpha.

So this is exactly the structure we have in this case here. So f hat then we have something here which is g hat and then we have e power i alpha x and then we have d alpha. So if we know that whose Fourier transform is this e power minus k alpha square t then we can write down this integral here as the convolution of f and this g.

So the question is now that whose Fourier transform is this e power minus k alpha square t because this is the Fourier transform of f we know, the second part here is e power minus k alpha square t. So if we know that this is, for example we assume here that this is the Fourier transform of g x and once we get this g x, we can apply this Convolution Theorem and we can write down this integral in terms of the convolution and then we can get rid of this f hat.

Well, so assuming that this is the Fourier transform of this g x, we can apply the inverse Fourier transform for g x, 1 over square root 2 pi, the integral e power minus k alpha square t and e power minus i alpha x d alpha. So we need to solve now this integral; so how to solve this integral?

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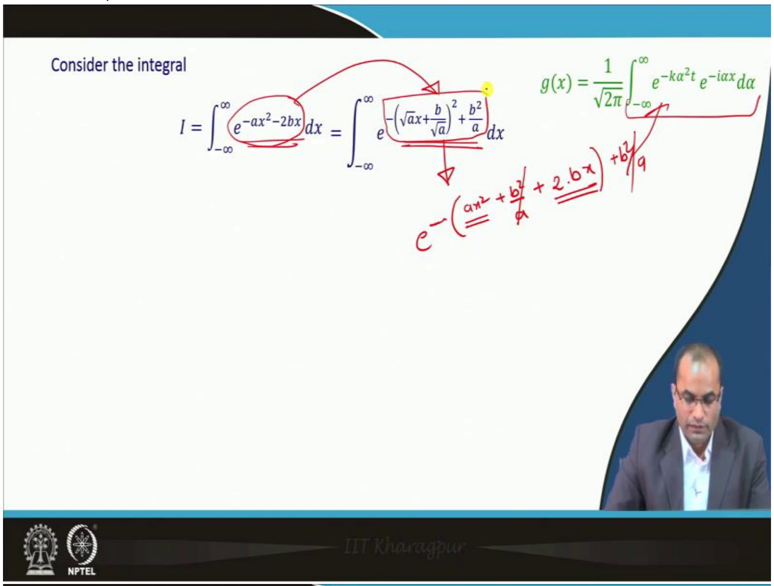

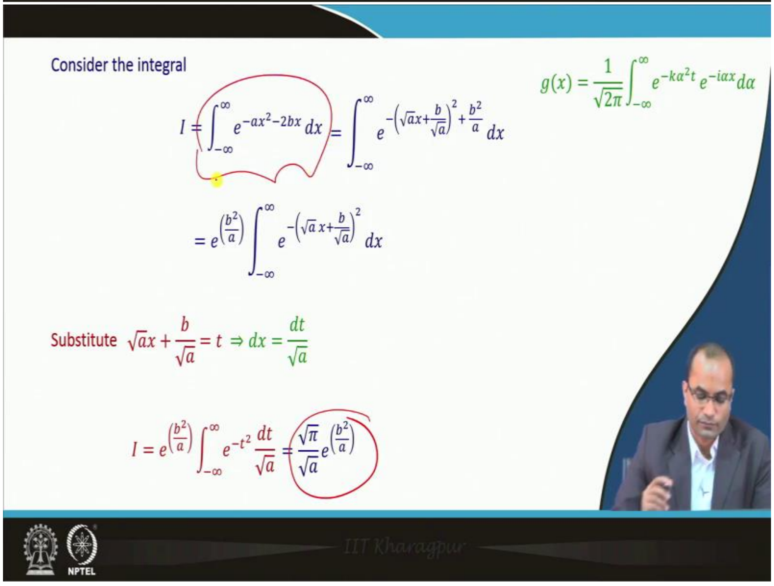

Let us assume this I, we consider another integral and later on we will see that this integral is actually related to the integral in hand here which e power minus k alpha square t minus e power i alpha x. So we consider this integral first and then we will see how this is related to the integral in hand. So this is minus infinity to plus infinity, this minus a x square minus 2 b x we have written as a whole square term.

So if we just look at here what is this e power minus, so here the square that means a x square then we have b square over a and then the two times the product that is bx and then we have b square over a. So this b square over a will get cancelled because here the minus sign is sitting there and we have minus a x square and minus 2 b x so exactly this term.

So this term is rewritten here to have this format because we want e power minus some y square whose value is known to us. So here the b square over a can go outside, so e power b square over a and then we have e power minus square root a x plus b over square root a whole square. And now we can substitute this to exactly have e power minus y square form.

So if we substitute this ax plus b over a as t what we will get after this substitution? Our integral will become e power b square over a, this minus infinity to plus infinity e power minus t square and then this factor dt over square root a will appear. So this value now we know that e power minus t square dt, the value of minus infinity to plus infinity e power minus t square dt that is square root pi. So having that we have this value square root pi over square a e power b square over a, the value of this integral.

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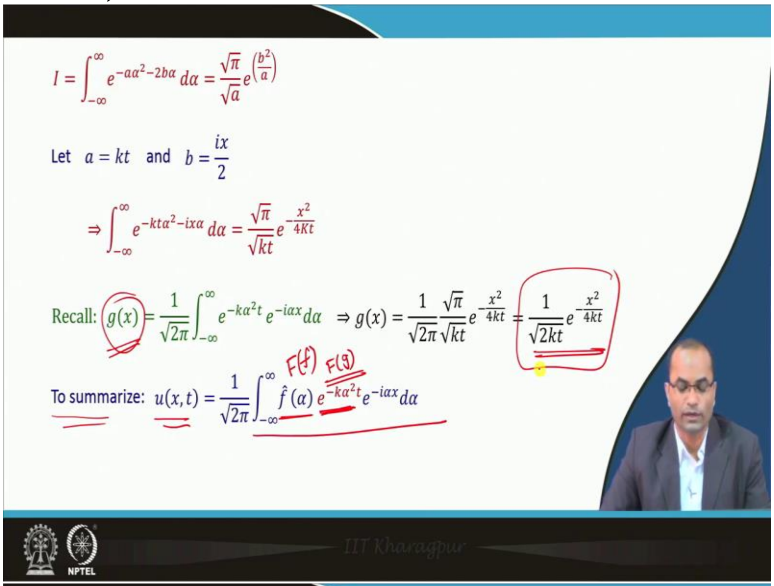

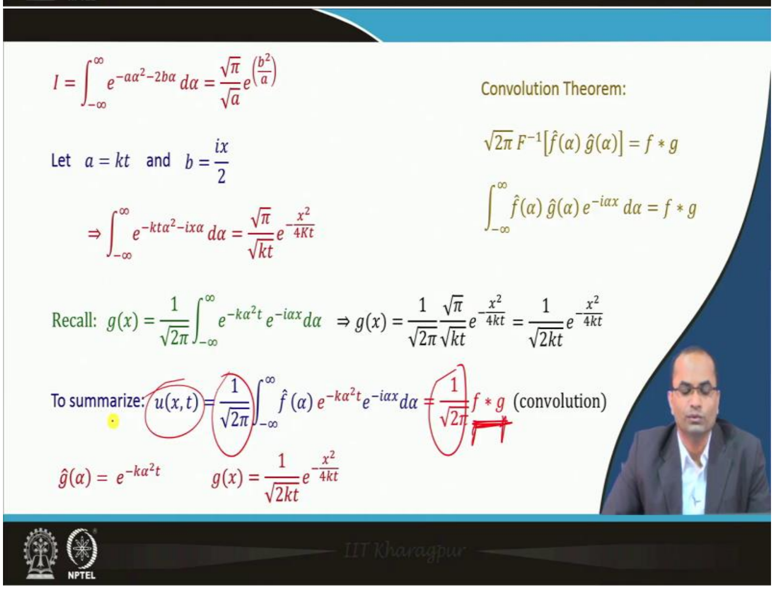

Well, so having this value of this integral if we, if we set here a as kt and then b we take i x by 2, why we are taking this I will clarify in a minute, so we have here minus infinity to plus infinity, a is kt so we have substituted kt and b we have, or 2b is i x so that we have used there and then similarly, the right hand side also b and a are used.

So we have the value of this integral now. And just to recall our g because first we are getting the g and then we will apply the Convolution Theorem to get u. So this g was this integral here and if we compare with this integral, we are exactly matching with, so we have minus kt alpha square then minus i alpha x d alpha and, so the value of this integral here, it is square root 2 pi g x, or rather we will substitute this value here now and get g x.

So our g x will be 1 over square root 2 pi and from here we have pi over square root kt and e power minus x square over 4 kt. So in this way, we got our g x as 1 over square root 2 kt multiplied by e power minus x square over 4 kt.

Just to summarize again, what we have done our u x, t was 1 over square root 2 pi and this integral. So for this integral to apply the Convolution Theorem we have this f hat already which is the Laplace Transform of f, so this is the, sorry Fourier transform of this f is already there and this is the Fourier transform of g and we have found now that the g whose Fourier transform e power minus k alpha square t is this function here 1 over square root 2 kt e power minus alpha square by 4 kt.

So we have now all the desired information and then the Convolution Theorem we can apply here to write down now a better form for this integral. The Convolution Theorem says that f inverse of this; the product which is exactly here, the f inverse of this product is equal to f star g. So we have to just now get this convolution of f and the convolution of this g. So that means this integral here which is written just this Fourier inverse and then square root 2 pi will get cancelled naturally, we have f star g. So we have now, for this integral we will write down f star g, this factor will remain as it is.

That means this g alpha this implies that we have the g x which we have already evaluated. That, so we have 1 over square root 2 pi and then f star g, that is the Convolution Theorem. So g we know, f is already given in the problem at the beginning so this convolution only will give us the function u x, t.

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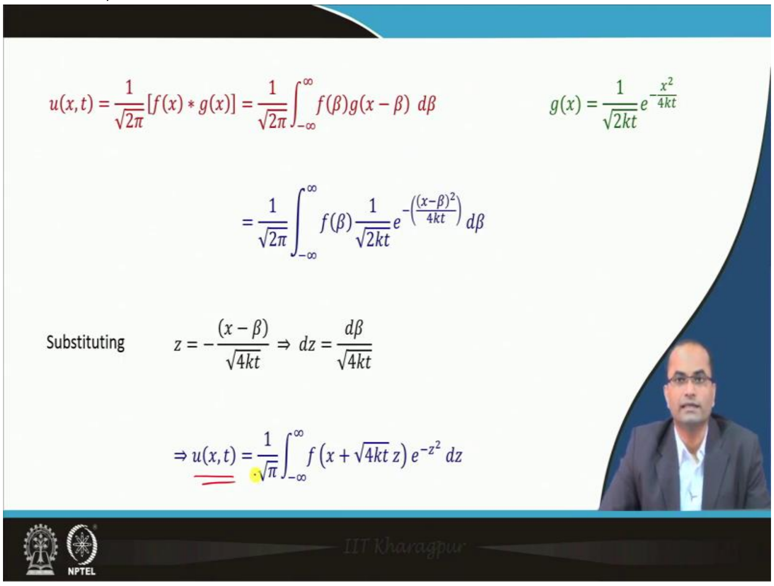

So u x, t is this convolution that means f beta and g x minus beta we have used this convolution product and d beta so where g is given as this one, we can put it there. So 1 over square root 2 kt and then this e exponential function with x minus beta. So now if we substitute this another simplification though we got here the value. So if we substitute this z as this x minus beta over square root 4 kt to have this nicer looking form here e power minus z square.

So we have this dz given by this one and then we can get back to this integral so we have 1 over square root pi and integral minus infinity to plus infinity f evaluated at x plus this square root 4 kt z e power minus z square dz. So this is the solution of the given heat equation, where f was given already in the problem and then we have e power minus z square we can integrate this and we can get this u x, t the solution of that heat equation.

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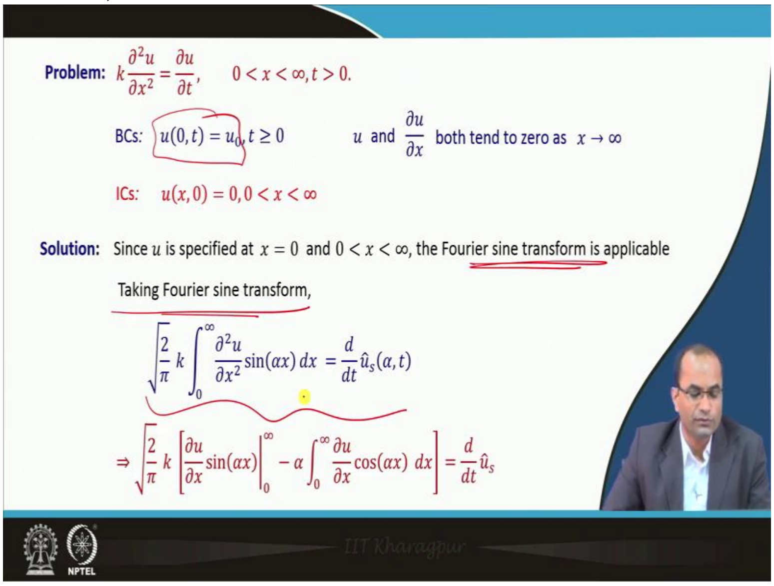

Well, so we can now discuss second problem, where we have the domain from 0 to infinity. So in the earlier problem, the domain was given from minus infinity to plus infinity but now we have 0 to infinity. So since the domain is given as 0 to infinity, we will apply either Fourier cosine transform or Fourier sine transform because in those two cases we consider the integral from 0 to infinity or rather the variable is defined from 0 to infinity.

So the boundary conditions here are u at x equal to 0, t is u 0 because now we have the boundary precisely at 0. In the earlier problem, there was just the infinity there. So those conditions were used that u and u x goes to 0 as x goes to infinity but in this case u and u x both tend to 0 as x tend to plus infinity, so we need exactly this condition to apply that Fourier cosine or Fourier sine transform to have that nice derivative formula.

The initial conditions are given as u x at t equal to 0 this is 0 for all x. So we have the initial conditions, we have the boundary conditions in hand and now we can apply the Fourier cosine or Fourier sine transform depending on this range here. So if the range is, the half range is given then we will apply Fourier cosine or sine, if the range of x is given from minus infinity to plus infinity then naturally we will apply the Fourier transform.

So in this case, we have this 0 to infinity and the second consideration we should have that u is prescribed at this t equal to 0 in the initial condition but if we look at the boundary condition the u is given at x equal to 0, and that is important because u is prescribed in the problem. And then we have just one choice out of two, whether we should use sine or cosine.

So we have to look in the derivative formula whether the sine is appropriate here or the cosine is appropriate because u is given, so since u is prescribed then the Fourier sine transform is applicable. If we apply Fourier cosine transform in this problem then we cannot solve it because we need this u x not the u at 0, t. So here the Fourier sine transform is applicable because directly we can use this u 0, t as u 0 in this first step itself.

So taking the Fourier sine transform now of this, so we are applying this direct definition though we can use the derivative formula directly. So having this I can quickly go through because this is the derivative formula which we have already discussed before.

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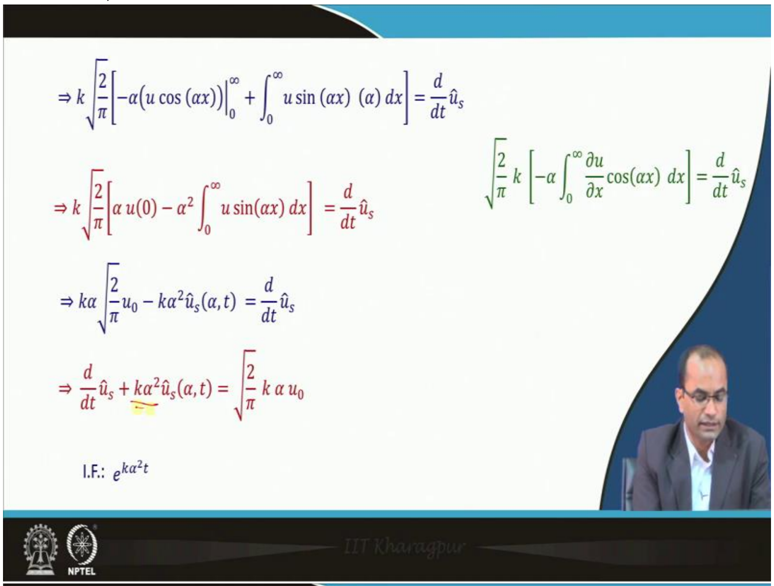

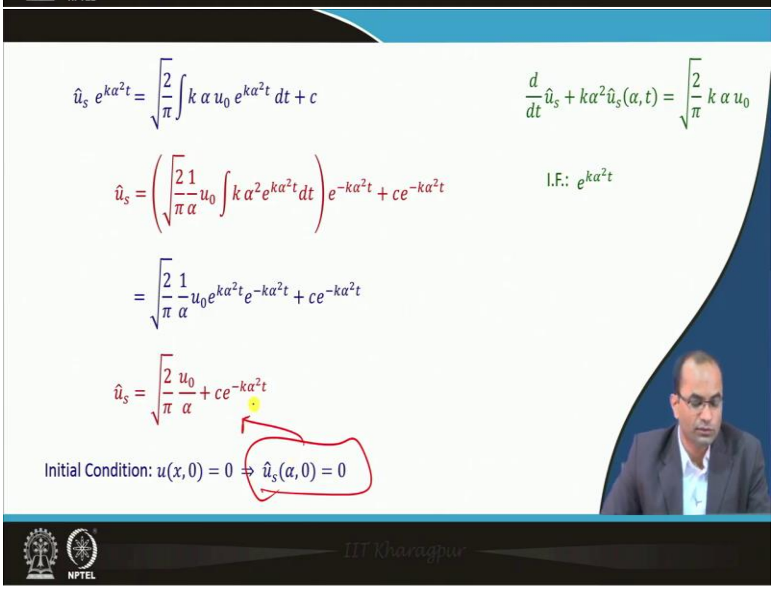

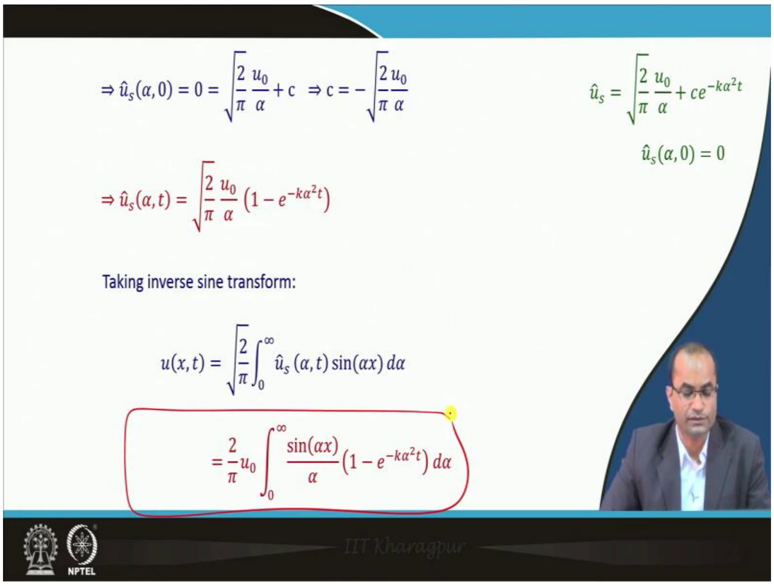

So having this integration by parts what we will end up, we will end up with this which is exactly the derivative formula and here this u at x is equal to 0 is used and that is the reason the Fourier sine transform is applicable here, not Fourier cosine transform. So having this now we need to solve this differential equation which is a linear differential equation in this u hat s. So we get the integrating factor e power this k alpha square and then this integration t.

So u power, e power k alpha square t we have the integrating factor for this differential equation and then we can write down its solution which is u s and this integrating factor e power k alpha square t and we have the right hand side here, square root 2 over pi k alpha u naught, k alpha u naught and then e power, this is the integrating factor here, integrating factor this is also integrating factor and then some constant c.

So having this, this u s hat we have now in this form, c e power k alpha square t, again some simplification out of this integration because e power k alpha square t we can integrate so we will get this term now with a power minus k alpha square t as it is and now we have this c, this constant which needs to be evaluated.

So having this u s hat and now the boundary condition u x 0, the initial, sorry the initial condition u x, 0 as 0 so if we take the Fourier sine transform here x 0 equal to 0 then we can get rid of this c; so having this Fourier sine transform of this u s, u x, 0 so since it is 0 we will get only the 0 right hand side. So applying this in this u s hat we will get the, our c here and c is going to be minus 2 over square root, 2 over pi square root and then u naught over alpha.

So the steps are again similar what we have done before. We got the u hat here in terms of this u hat sin now and then the c is evaluated based on the given initial condition. So finally, we got u s hat in this form and then the final step taking the inverse transform, so u s hat is given here already. So we substitute that there in the formula and we got the answer, we got the solution of the given initial boundary value problem or given heat equation.

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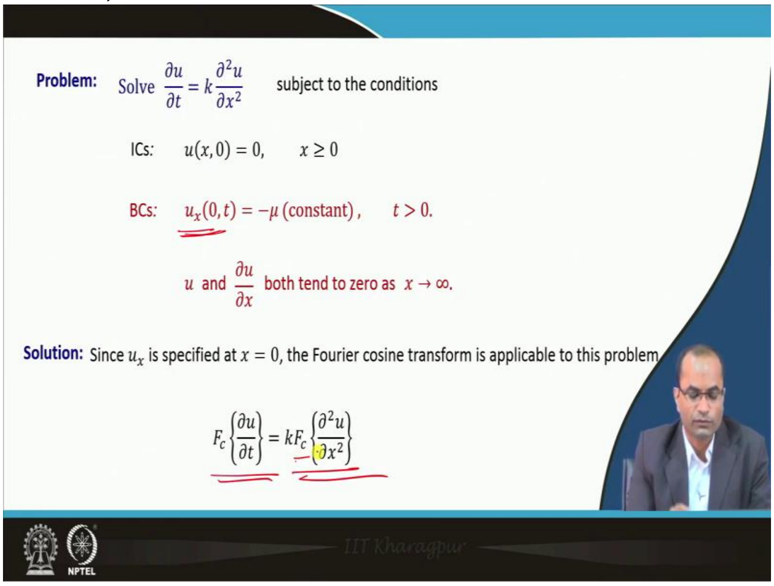

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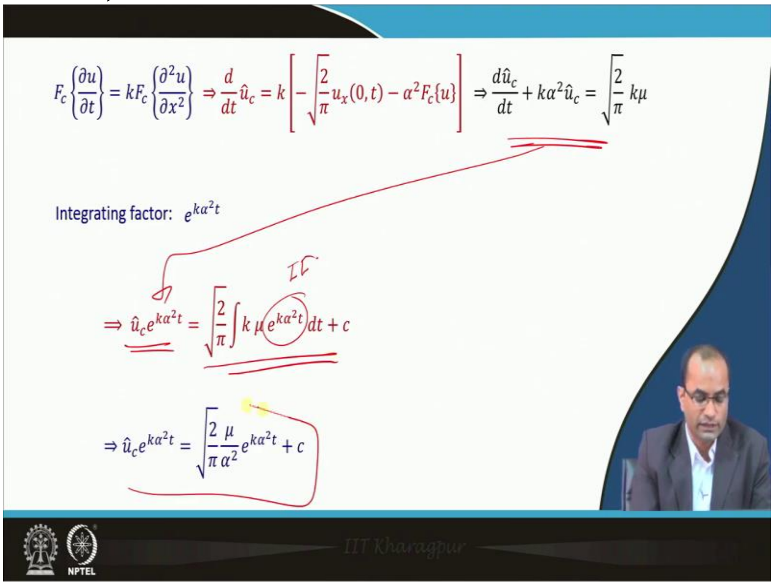

Well, so this is the last problem of today's lecture. We are going to discuss now one more problem where we have u t is equal to k, again the heat equation and subject to conditions, the initial condition is prescribed again at 0, the boundary condition now this time the u x is prescribed instead of u. In the earlier problem, u was given but now u x is given as minus mu, a constant and this is the regular condition we have that x goes to infinity then u and u x both tend to 0.

Now getting to the solution because u x is given and we have to apply now the Fourier cosine transform, in this case the Fourier sine transform will not work because that needs information of u which is not given here in this problem. So applying this Fourier cosine transform on both the side, so right side we have k Fourier cosine transform.

So having this we can apply the direct formula now this time, we are not deriving this here, so this u x, 0 t is exactly coming in the formula which we can use now. So we got this differential equation del u c over del t and k alpha square u hat c and then we have this square root 2 over pi and k mu because that mu was given there.

Okay, so the integrating factor for this differential equation is e power k alpha square t again similar to the previous example, where we can now write down its solution in terms of this integrating factor, here also we have this integrating factor and then finally, we need to evaluate that c.

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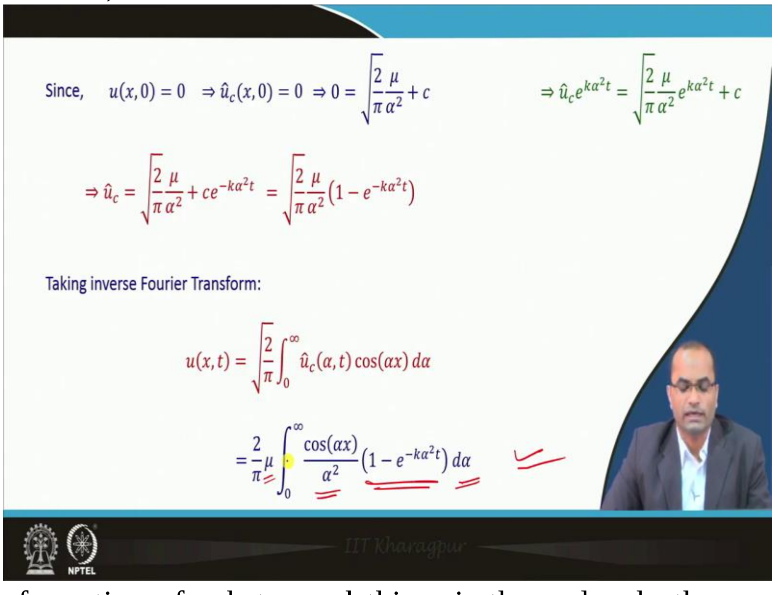

So having this information of u hat c and this c is there already the u x, 0 we have not utilized. So that is the initial condition. We can take this Fourier cosine transform so since it was 0, so we will get the 0 there and with this help we can get our this constant c. Substituting back now in this formula, we got the result for u hat c which is free from c now. The square root over pi mu over alpha square and 1 minus e power minus k alpha square t.

So now we can go for the final step taking this inverse Fourier transform. So we can write down this u x, t in terms of this cosine transform and we can now plug this value there into this integral. So we got cos alpha x over alpha square and 1 minus k alpha square t d alpha and this mu over pi and 2 times.

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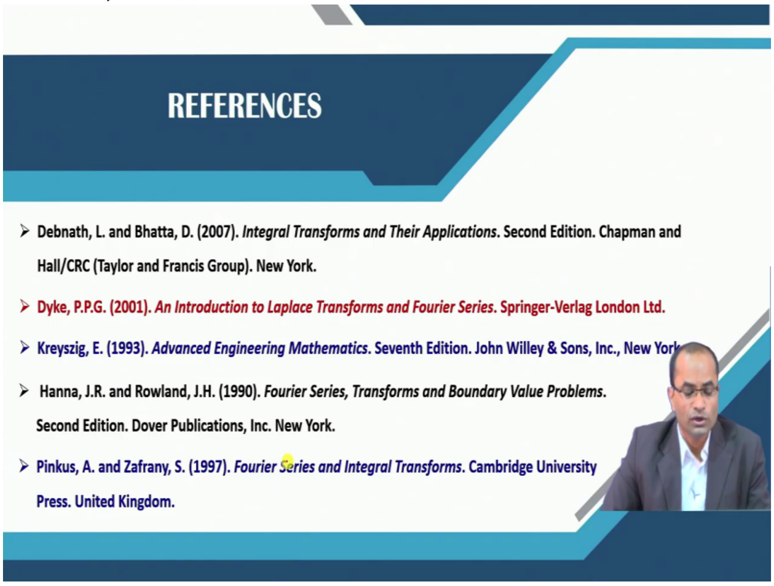

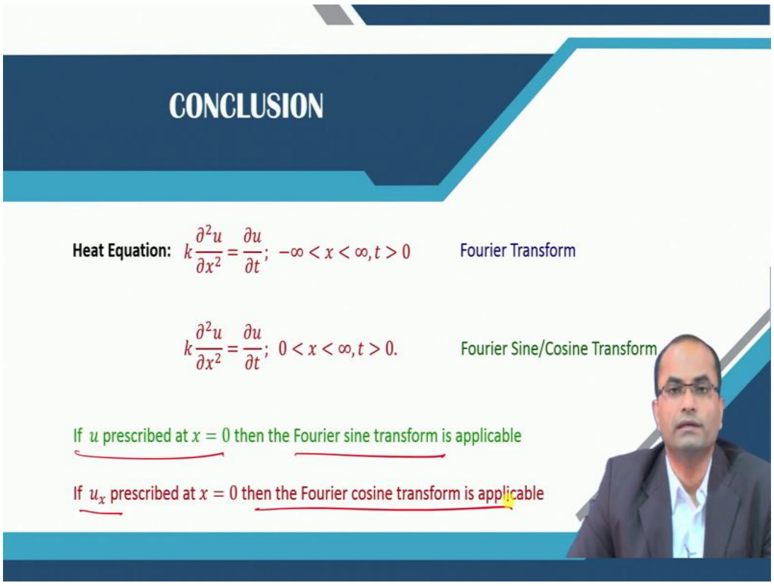

So these are the references we have used for preparing this lecture. And just to conclude that now we have discussed these two forms of the heat equation, in one the variable was from minus infinity to plus infinity and in the other one it was 0 to infinity. So once this range is given from minus infinity to plus infinity, we have to apply the Fourier transform. In this case, there are two possibilities depending on the conditions, the boundary conditions given.

If u is prescribed then sine transform is applicable, if u x is given then Fourier cosine transform is applicable. Well, so if u is prescribed then Fourier sine transform and if u x is prescribed then Fourier cosine transform is applicable to the problem. So that is all for this lecture and I thank you for the attention.