Engineering Mathematics II
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Lecture 48 - Introduction to Partial Differential Equations

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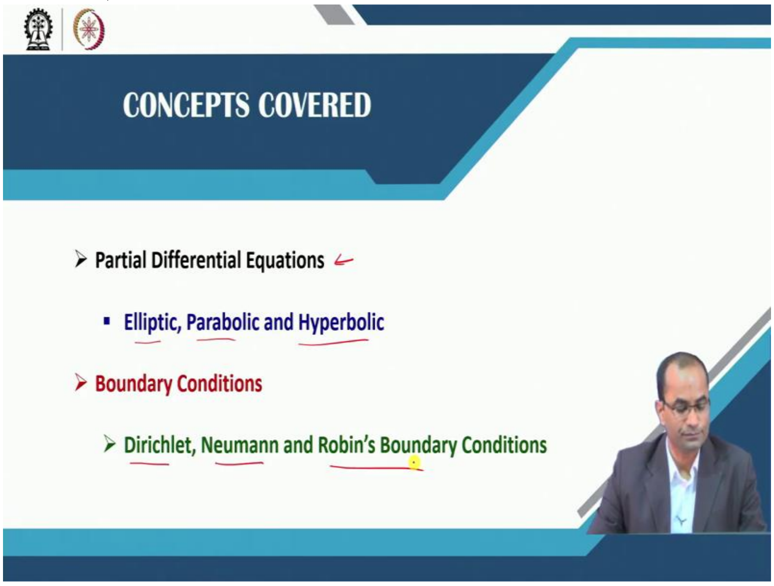

So welcome back to lectures on Engineering Mathematics II and this is lecture number 48, where I will give you a short introduction to partial differential equations because after this lecture there will be two lectures on the applications of Fourier transforms to partial differential equations.

So today I will introduce you the partial differential equations. Basically, there are three types of differential, such differential equations. It is so-called elliptic, parabolic and hyperbolic and then later on, I will explain what type of boundary conditions are used to define problems which are explained by these partial differential equations. So there are three types of boundary conditions again, so Dirichlet, Neumann and Robin's boundary conditions.

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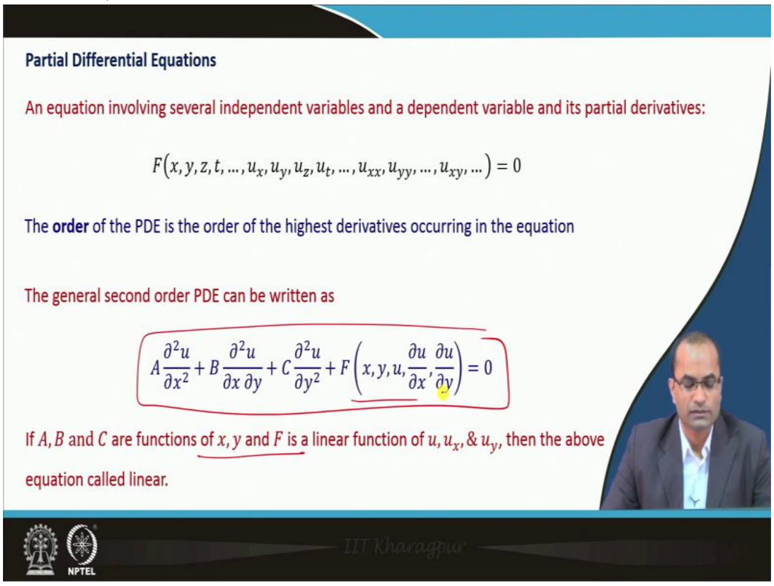

So coming to the partial differential equations, so an equation involving several independent variable and a dependent variable, so just recall when we are talking about the ordinary differential equation, we have only a single independent variable and then if there is one equation, only one dependent variable, or there are equations, system of equations then there will be more dependent variables.

So here in PDEs, we have several independent variables. If there is one PDE then we have one dependent variable; otherwise there could be also several dependent variables in case of a system of differential, system of partial differential equations. So here the important point which differs from the ordinary differential equation is to have several independent variables because now we will be talking about the partial derivatives. There is no ordinary derivative anymore because of more independent variables.

So here this is an equation which involves several independent variables and a dependent variable and of course, its partial derivatives. So such an equation where we have a relation between this x, y, z, t, for instance, these are the independent variables. So we have several independent variables and u here is dependent variable. So we have their partial derivatives. So first order partial derivative and then we have second order partial derivatives and so on and it may also depend on u of course.

So such a relation is called partial differential equation and the order here of partial differential equation is the order of the highest derivative occurring in the equation. So for instance, in the equation if the order of the highest derivative is 3 then the order of the partial differential equation we call as 3.

So the general second order partial differential equation can be written as, here we have some A and then it is the second order partial derivative, then B the mixed partial derivative and then here C times, so these are some coefficients here, A, B and C. So with C we have the partial derivatives with respect to y and there all other terms with lower order are combined into this function F and the right hand side is 0.

So if these coefficients here, A, B, C are functions of x and y and this F here is a linear function that means overall, the whole equation, because if these A, B and C are linear, are functions of x, y, z then we have a linear term in u and if this function F is also linear that means here also either u appears or del u or del x but there is no product appearing in this expression. So this is a linear expression then in u. So if A, B and C are the functions of x and y and this F is linear then we call this equation a linear partial differential equation.

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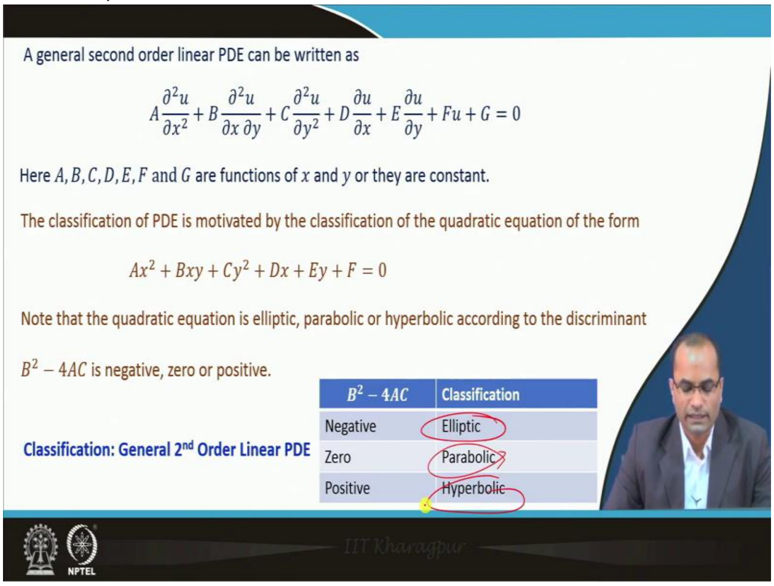

So a second order general linear partial differential equation then can be written as in this form where these A, B, C, D, E, F and G they are the constants, either constants or functions of x and y. They do not depend on u. So then we have only the linear expression in this u. There is no product of u and its derivative appearing in the whole equation.

So the classification of this PDE is motivated by the classification of the quadratic equation of this form which is again analogous to this equation. So here we have A then in this equation we had second order derivative but now we have here x square term. Similarly, the mixed order derivative then we have here this product xy with the B and then here C the second order derivative with respect to y, so we have here also C and this y square. Then we have this D and x term because here only the first order derivative, E also with the first order derivative so with y and then we have some constant here which is F equal to 0.

So this classification of this PDE is based on the classification of this quadratic equation because we know that this quadratic equation is elliptic, parabolic and hyperbolic according to this discriminant which is B square minus 4 AC. So if this B square minus 4 AC is negative then such a quadratic equation is called elliptic and if it is 0 then such a quadratic equation is called parabolic or it represents a parabola and then it represents a hyperbola so the equation is called hyperbolic if this B square minus 4 AC is positive.

So exactly the same classification so this is again easier to remember if we relate with this quadratic equation and the classification of this second order partial differential equation is exactly based on this terminology.

That means the second order, this linear differential equation is negative if this B square minus 4 AC is negative then this partial differential equation is called elliptic partial differential equation, and if this B square minus 4 AC is 0 then it is called a parabolic partial differential equation. Similarly, if this B square minus 4 AC is positive then this equation is called hyperbolic partial differential equation.

So this is the classification and actually, we go into the detail, so based on these classifications all these elliptic PDEs, they have common properties, similarly the hyperbolic PDEs and the parabolic PDEs they share many common properties and therefore, they fall into this one of these categories.

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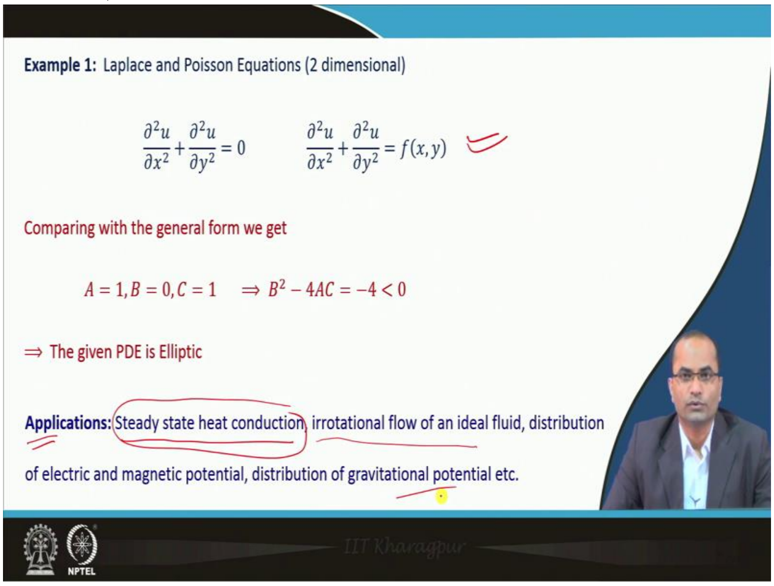

So coming to the example, the first example we are stating here that is the Laplace or the Poisson equation. So this equation is called the Laplace equation where we have the first order derivative with respect to x, sorry second order derivative with respect to x and here the second order derivative with respect to y that is adding to 0. So this is the Laplace equation.

Sometimes this is defined in terms of this Laplace operator, Laplace u equal to 0. So this is the so-called Laplace equation, and a slight variant of this equation when the right hand side is not 0, it is some function of this x and y then this equation is called Poisson equation. So this is more general equation and if we set this f x, y to 0 then this becomes Laplace equation.

So if we compare with the general form which we have written in the previous slide with A was the coefficient of the second order derivative term, then B with the mixed and C was with the second order derivative with the second variable.

So here also if we compare this so this is the A sitting here, there is no B because the mixed order pairs derivative is not there and then we have again C is also 1. So if we compute this B square minus 4 AC, so this number will be negative as minus 4. So this is an elliptic equation. So based on this classification we have to just look at these coefficients A, B, C and then we can compute this discriminant. Based on this we can classify the equation. So this is an elliptic equation or a typical example of elliptic equation. So these both equations the Laplace or Poisson are elliptic. It has several applications.

For instance, steady state heat conduction problem or irrotational flow of an ideal fluid then the distribution of electric and magnetic potential, distribution of gravitational potential etcetera. So it has several applications starting from this steady state heat conduction equation to the distribution of the gravitational potentials etcetera.

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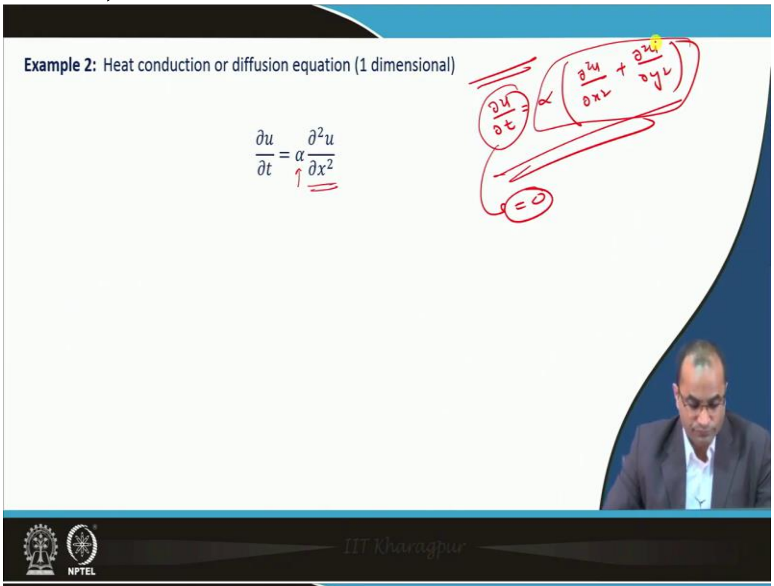

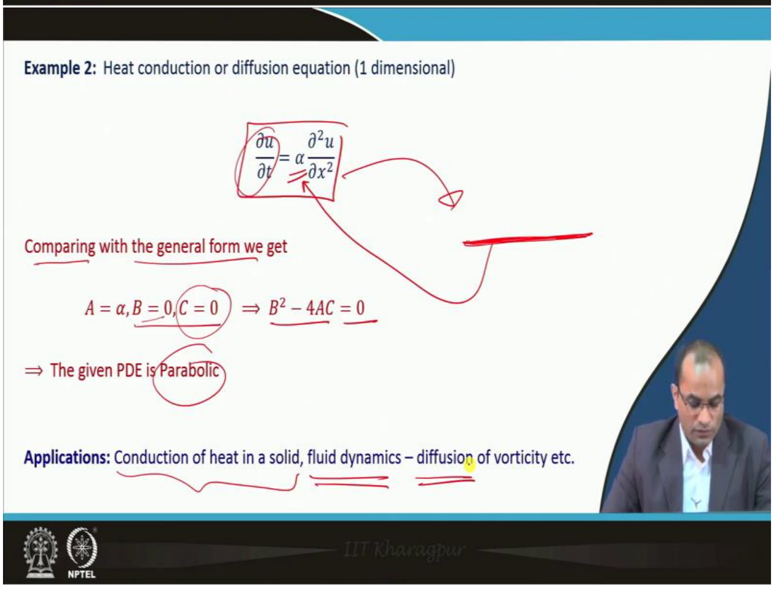

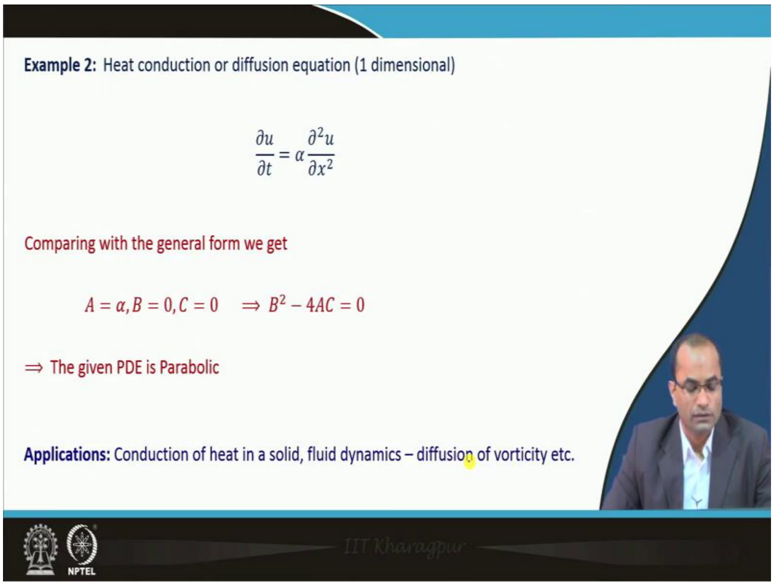

The another example which is heat conduction or diffusion equation, this is we are stating here in one dimension. So this is del u over del t is equal to alpha, some constant which depends on the material properties etcetera or the mechanism. So here we have the partial derivative with respect to x square. So if we write for instance this in two dimension, this takes the form of del u over del t then some constant there and then partial derivate with respect to x square and then we have the partial derivative with respect to y square.

So this is the second dimensional heat conduction equation and exactly the steady state heat conduction equation that means this, there is no change with respect to time. That means del u over del t is 0. In that case, this equation is called the Laplace equation or steady state heat equation which we have just explained in previous slide.

So here we are considering the one dimensional this heat conduction equation and if we want to go for the clarification, the classification so we compare again with the general form. So this A will be alpha in this case and there is no more term for B and C so B and C will be 0 and now we compute this B square minus 4 AC so since C is 0 and the B is 0 then B square minus 4 AC will be 0.

So that means this equation is a parabolic equation because of this discriminant. Here it is 0 so this equation falls into the class of parabolic equation. It has of course several applications and one, the most important popular one is the conduction of heat in a solid.

So if we have for instance a solid bar and the heat conduction in this bar can be modeled with the help of this heat equation. So this material property and etcetera will be modeled in terms of this alpha and also it also has application in fluid dynamics, for example the diffusion of this vorticity etcetera. Well, so this is the example of an parabolic equation and this is well-known equation of this heat conduction.

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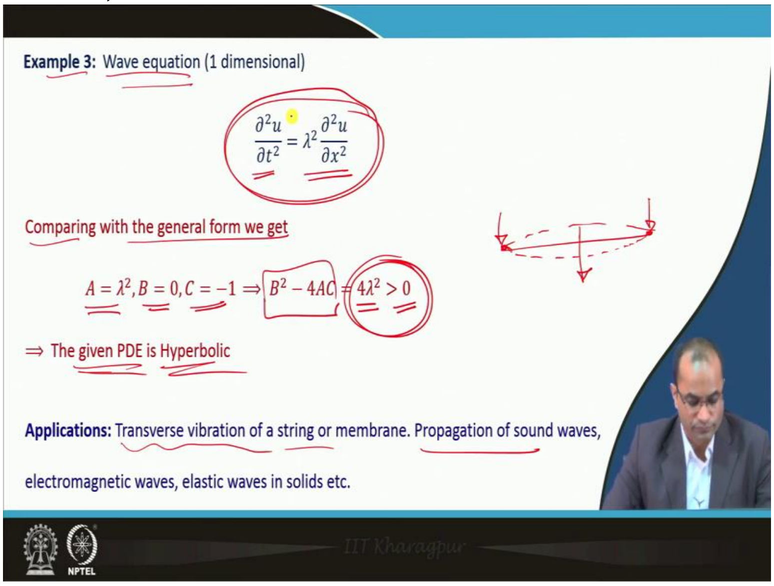

Now the third example we have this wave equation, which again in one dimension we are writing here del 2 u over del t square is equal to this lambda square del 2 u over del x square and if we compare this with the general form what we will get, this A is lambda square, the B is 0 and C is minus 1.

So if we compute this B square minus 4 AC which is coming for lambda square so here, this is positive, so this quantity is positive now, this discriminant B square minus 4 AC and hence this equation fall into the category of this hyperbolic equation so this given PDE is hyperbolic.

The applications for this equation, the most important is this transverse vibration in a string; so if we have a string for instance fixed at these two places and then we initiate some movement of this in this direction then this displacement from the original state can be modeled with the help of the so-called wave equation. So here it has many applications related to wave, for example propagation of sound wave, electromagnetic wave, elastic waves etcetera. That can be modeled in using this wave equation.

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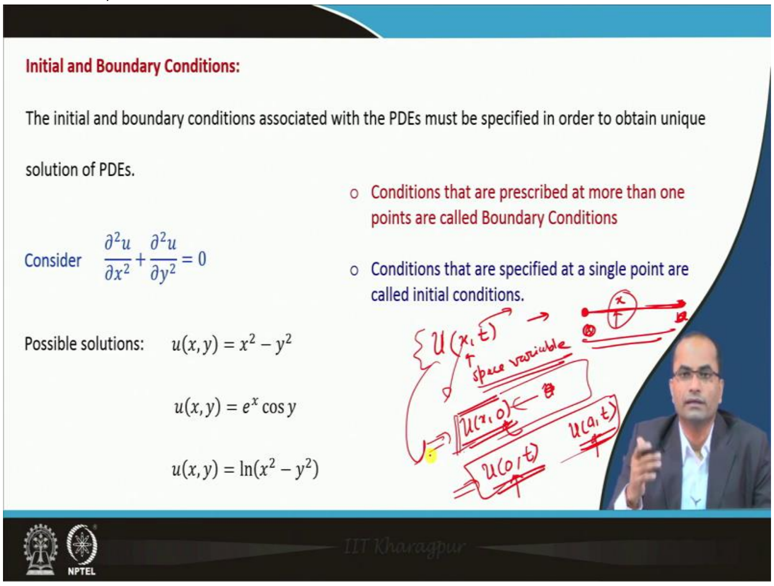

So now we will discuss initial and boundary conditions because many of them will be used now in next two lectures, where we will be discussing how to solve such differential equations with the help of Fourier transform.

So the initial and the boundary conditions are always associated with the partial differential equations which we have just stated now, and then so these conditions they are the so-called initial and boundary conditions must be specified in order to obtain a unique solution of the PDE because if we just give the PDE we can find the general solution for instance but that will not be the unique solution because, and in reality in applications when we model equation then we, our interest is to have the unique solution of that problem.

So for instance, if we have this equation, the Laplace equation, the possible solutions this u x, y is equal to x square minus y square when we plug in this equation this function will satisfy the given differential equation. Also, u x, y is e power x cos y will also satisfy the given differential equation. And u x, y is equal to ln x square minus y square this function will also satisfy the given differential equation. So there we have so many functions which can satisfy this given differential equation. Then which one is the solution of this given differential equation?

So there is no uniqueness in the solution. So we have to specify some extra conditions with this partial differential equation to have a unique solution of this problem and then only we can discuss about that unique solution later on.

So there are two types of conditions which are prescribed along with such partial differential equations. The conditions that are prescribed at more than one points are called usually the boundary conditions and on the other hand, the conditions that are specified at a single point are called initial conditions. So let me just explain here a bit more.

So for instance, we have a dependent variable u which depends on x and t. x is the space variable, space variable. For example if this u define the temperature in a solid bar then the x is the position where we are interested getting this temperature, so for example this is the position x. So here this x is a space variable and the t is the time variable because with respect to time the temperature there varies.

So we have 2 variables there, one is the position which is from here to here, can be any point on this solid, and then t is the time which is varying from 0 to whatever infinity. So there are two types of boundary, initial and these boundary conditions are prescribed. Here the initial condition means with respect to basically time.

That means u must be prescribed at 0, or in many cases when such this variable appears in a second order there with respect to time then we have to have two conditions initially. That means one will be like u x, 0, another one will be the derivative condition again at this x equal to, sorry t equal to 0. So at the same point at t equal to 0, if there could be many conditions depending on the order of the differential equation may be prescribed but this is usually called the initial condition. So at initial time we have prescribed the profile of the temperature in the body.

So the second conditions what we call the boundary conditions, they will be prescribed with respect to the space variable. So here we have this s, for instance this we are naming at x equal to a to x is equal to b or x is equal to 0 to x equal to a for instance. Then the condition, the u must be prescribed at 0 all the time and u must be prescribed at the other end all the time.

So here there are different types of boundary conditions possible, either directly the value of this dependent variable is given or in some other forms that we will discuss in next slides. But as far as the initial conditions are discussed, so they will be prescribed at this 0 point. But here we have a different point.

At 0 it is prescribed and at a again it is prescribed so these two boundaries of this given bar then this is called boundary conditions and when the conditions are prescribed at one point then we call this as initial condition.

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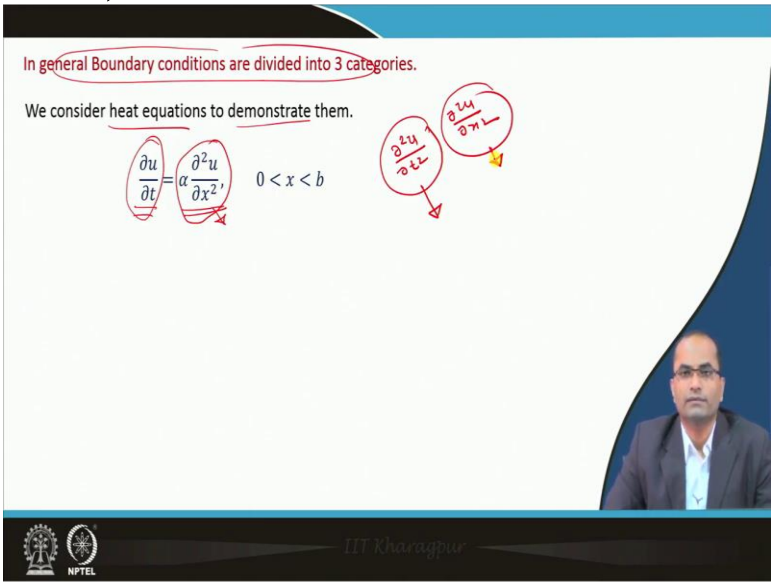

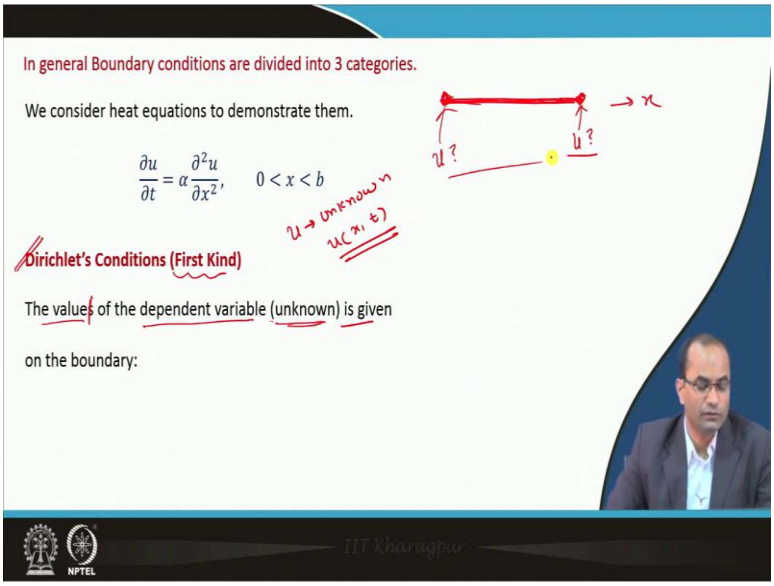

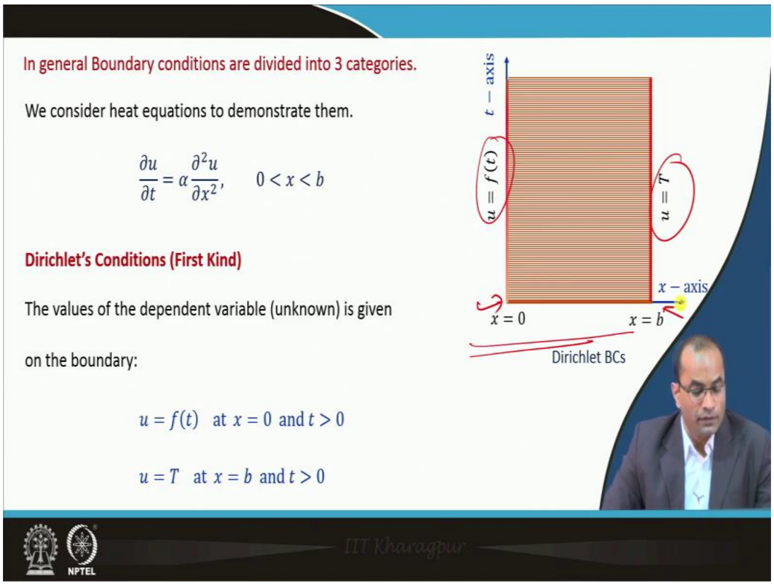

So in general, these boundary conditions are divided into 3 categories and the first one we will consider here with the help of the heat equation. So we will demonstrate all these 3 types of boundary conditions using the heat equations but the idea can be applied to the wave equation, for instance or the steady state heat equation or the Laplace equation.

So we consider this heat equation del u over del t is equal to some constant alpha and del 2 u over del x square. So here we have second order derivative in x, so we need two conditions with respect to x that means two boundary conditions are needed, and we have here del u over del t then we need one initial condition.

So for instance, in the wave equation we will have del 2 u over del t square together with this del 2 u over del x square. So in that case, we need two initial conditions here and we also need, sorry two initial conditions and we need two boundary conditions again. So depending on this derivative, the order of the derivative we have to prescribe the boundary conditions to, in order to get the unique solution.

So the first type of conditions which we will discuss here is the so-called Dirichlet's conditions and these conditions are also called first kind boundary conditions. So the values of the dependent variable which is unknown, so in our case the u is the unknown variable and this u depends on x and t, on two variables. So u is the unknown here.

So if the value of the dependent variable is given, so the value of the dependent variable is given on the boundary, so depending on what kind of boundary we have, so if we are talking about one dimensions then we have only for instance a situation where the given material, the given bar or given whatever string this is in the direction of x for instance, so just in one direction we can describe this.

So this is, then we will have definitely these two boundaries of this bar here, for instance. So we have to prescribe now the value of the dependent variable. That means what is u here and what is u there. That means the direct value of the, direct value of the dependent variable is known at the boundaries, given at the boundaries then we call such conditions as Dirichlet condition.

So u is given as some, it could be a function of time, it could be constant but the u is prescribed. The important here is that u is prescribed at x equal to 0 so at one boundary if we are taking domain from this x to b for instance, so at 0 and for all time because now this may depend on time also, it may be constant, so for all time then this u must be prescribed at this boundary x equal to 0, and this u equal to t is prescribed for example, at x equal to b. So at one end, we have some fixed value and at the other end the value is depending on the time t for instance.

So that is the possibilities but in either case the u is prescribed at both the end. However, at one end it is as, it is given as a function of time and at the other end this is prescribed as a constant value. But at both the end the direct value of the dependent variable is known and such type of boundary conditions are called Dirichlet boundary conditions.

So just to describe again, so for instance we have this here, a rod from x equal to 0 to x is equal to b and where we are finding for instance the temperature if you are talking about this heat conduction equation.

So we have a bar from this x equal to 0 to x is equal to b and our interest is to find the temperature in this given bar and given these conditions here, in this direction we have the time. So with respect to time the distribution of this temperature will change and then according to these boundary conditions, what is given that at x equal to 0 we have, the u is given as the function of time and x equal to b, u is prescribed as constant value.

So at this end throughout the time, so here the time varies but at this end of this bar here u is always fixed and at this end u is given as a function of time. So this is the situation we have for this Dirichlet boundary conditions that the direct value of the dependent variable of the unknown is prescribed at both the ends at x equal to 0 also at x is equal to b throughout the time.

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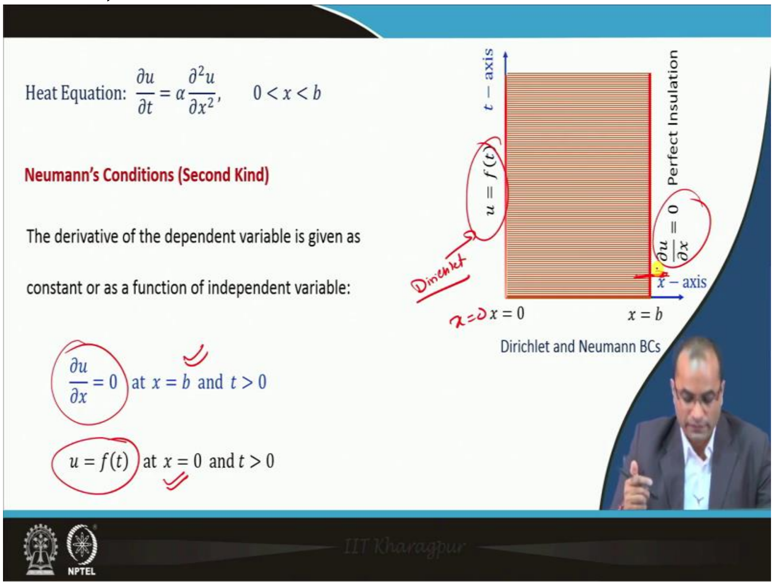

The next situation we will explain again with the help of this heat equation and these boundary conditions are called Neumann boundary conditions or the Second Kind boundary conditions. The derivative of the dependent variable is known. Now not the dependent variable directly known at the boundaries but its derivative is known. So this is the so-called Neumann boundary conditions, where the derivative information is not the directly, not directly the information of the dependent variable.

So if the derivative is known as the constant or a function of again independent variable for instance time there, so situation could be like partial derivative with respect to x is given as 0, is set at 0 at x is equal to b, and for all time again. So this is because the derivative information is known we call this as Neumann boundary condition. So one end we have this Neumann boundary condition, for instance at x equal to 0 we can still have the Dirichlet boundary condition.

So in this example, at this end x equal to 0 here we have the Dirichlet boundary condition and at the other end we have del u over del x equal to 0. So what do we mean by del u over del x? That in the direction of this x the u does not change. That means there is perfect insulation here. There is no loss at this boundary.

There is no heat loss at the boundary x is equal to b because del u over del x is flux in this direction of x. It is set to be 0. So this situation is like this end is kept at the perfect insulation and at this end we have given the temperature, which is varying with respect to time and then our interest is that what is the temperature with respect to time in this bar, which is from x is equal to 0 to b. So here we have these Dirichlet at one end and the Neumann boundary conditions at other end.

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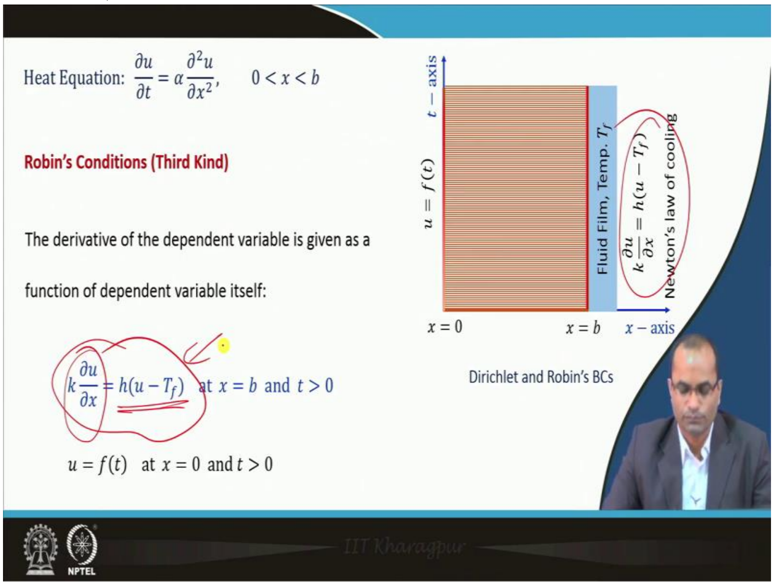

The third kind of boundary conditions we will discuss here, these are called Robin's conditions or the third kind of boundary conditions. The derivative of the dependent variable is known in this case as a function of dependent variable itself.

So if we consider for example this information del u over del x is given as h u minus some constant this T f, so the derivative here of the dependent variable is a function of dependent variable again. So such boundary conditions are called Robin's boundary conditions and they again often appear in applications.

The second end for instance at x equal to 0, we are still keeping as Dirichlet boundary condition. So the situation in this case could be from the modeling point of view now, x equal to 0 this is the end where we have the Dirichlet boundary condition.

On the other hand, at x equal to b we have the so-called this Robin's boundary conditions. And here this in particular, these boundary conditions now which we have taken that some constant times the difference here of the two, the temperature u which is we have named the temperature of this bar here and just imagine that this bar is kept at this fluid film here at this end whose temperature is this T f. T f stands for the fluid temperature.

So there is a difference in the temperature at this end. In the earlier problem, this end was kept as insulation but now it is kept in this fluid whose temperature is T f. Then there will be energy flow, there will be a flux here, either the temperature, the energy, the heat will go from the fluid to the bar or from bar to the fluid depending on whose temperature is high.

So this is the very popular Newton's Law of Cooling that this del u over del x, this rate of change will be varying depending on the difference of the temperature between u and this T f. So such boundary conditions often appear in applications and these called the Robin's boundary conditions.

From the handling point of view, the Dirichlet boundary conditions are the easiest one because directly the value of the dependent variables are prescribed at the boundaries. The next comes to this Neumann boundary condition where the derivative information is prescribed as a function of independent variable and this is the third one, the most difficult one where the, again this derivative is function of dependent variable. So the handling of this boundary condition is most difficult among these three.

So these are the 3 types of boundary conditions we have discussed, associated with heat equation but we can extend this idea for any other equation, for example wave equation, or the Poisson or Laplace equation.

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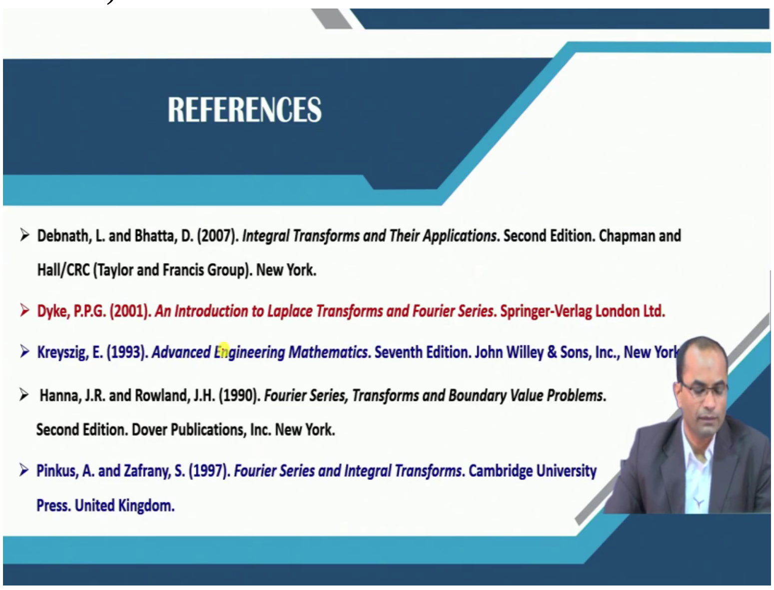

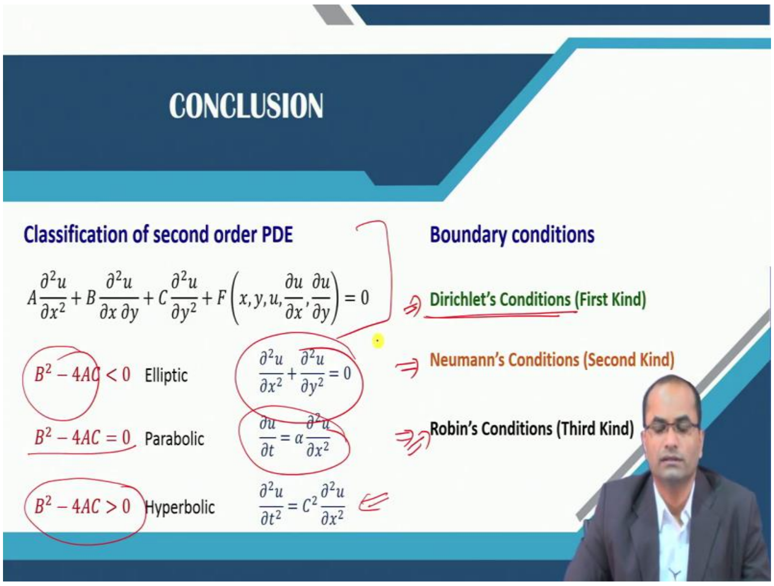

So these are the references we have used for preparing this lecture. And just to conclude that the second order partial differential equations, we have discussed the classification which was based on this value of this discriminant which is negative then it is called elliptic, and this was the example of an elliptic equation, the Laplace equation and if this discriminant is 0 then the equation is called parabolic and this was the example, heat equation of this parabolic differential equation and if this B square minus 4 AC is positive then the equation is called hyperbolic equation and this was the example, the so-called wave equation of this category.

We have discussed various boundary conditions, the one was the first one was the Dirichlet boundary conditions, where the value of the dependent variable is prescribed directly as a function of the independent variable or a constant. In the second type of Neumann boundary conditions, the derivative information is provided at the boundaries or it can be a function of independent variable.

The second, the third one is the Robin's boundary conditions, where the derivative of the dependent variable is prescribed in terms of again, dependent variable. So this is the difficult one to handle from the solution point of view and the Dirichlet conditions are the easiest one. Well, so that is all for this lecture, a very short introduction to partial differential equations and I thank you for your attention.