Engineering Mathematics II

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Lecture 47 - Evaluation of Fourier Transform (Part - II)

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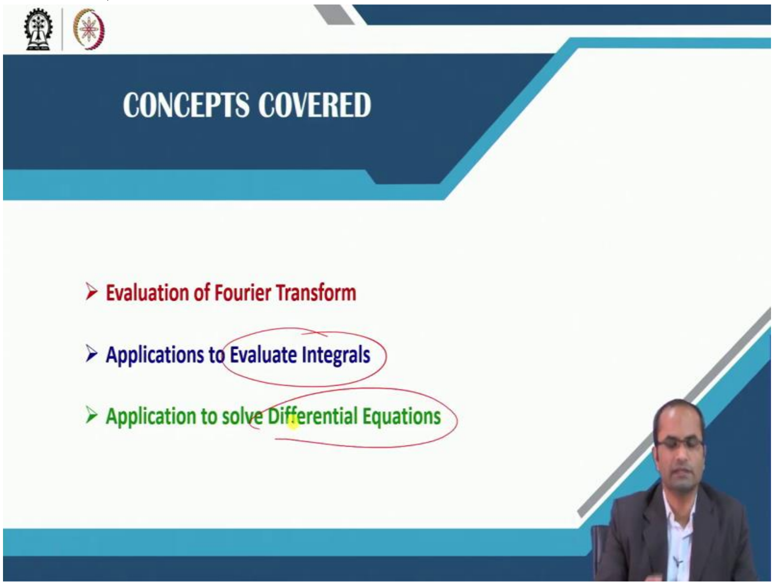

So welcome back to lectures on Engineering Mathematics II and this is lecture number 47 on evaluation of Fourier Transform and this is part II. We have already discussed in part I many evaluation of different functions like exponential or Dirac-Delta functions etc.

So today, basically, we will focus more on the evaluation which leads to the some kinds of applications like integral, evaluation of integrals, or then integral equations or differential equations etc. So this will be focused mainly on the evaluation of integrals and solving integral and the differential equations.

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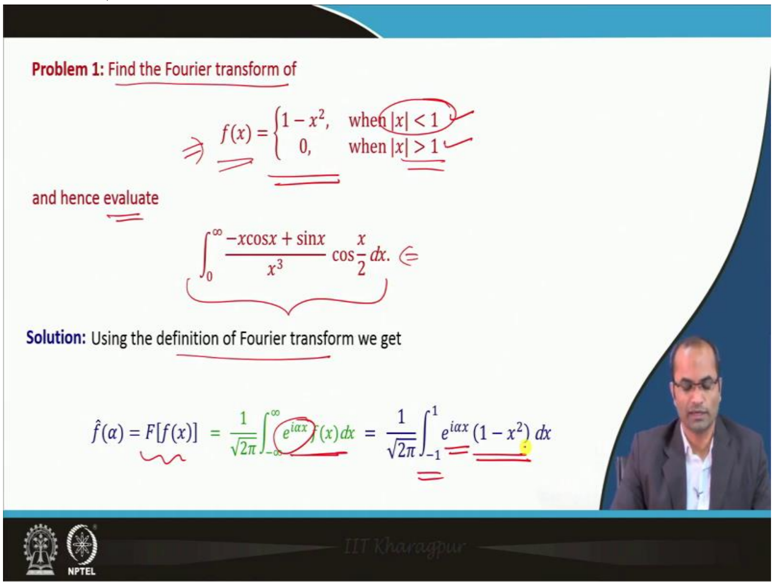

The idea will be explained here. So coming to the evaluation, so we have for instance, this problem that find the Fourier transform of this function which is 1 minus x square when this x is less than minus 1 and when absolute value of x is greater than 1, the function is 0.

Once we get the Fourier transform of this function, we will evaluate this following integral, the value of this integral so that is a kind of application. This Fourier transform has to evaluate such kind of complicated integrals. So coming to the solution, so we will first find the Fourier transform of the given function; that means so we will apply the definition of the Fourier transform of this f x which is this integral f x and multiplied by e power i alpha x and then dx. So our function is defined only for this minus 1 to 1, other than this it is 0. So the integral will take place from minus 1 to 1 e power i alpha x and then we have 1 minus x square dx.

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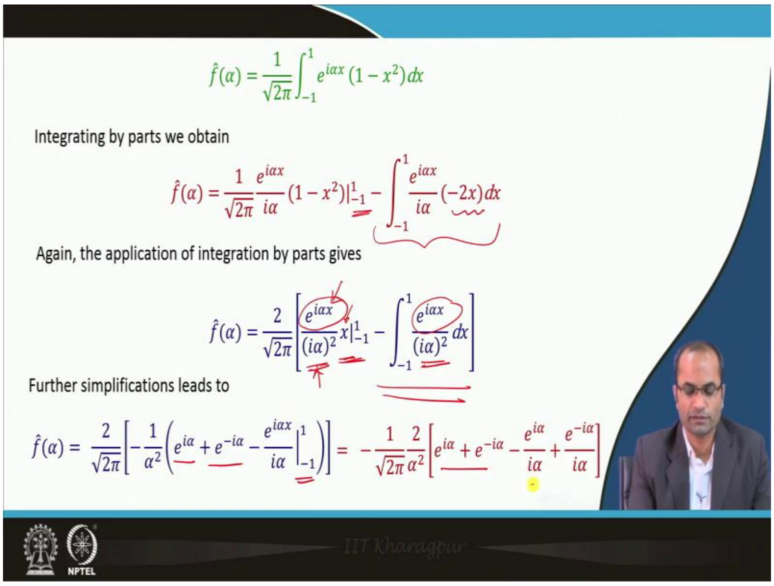

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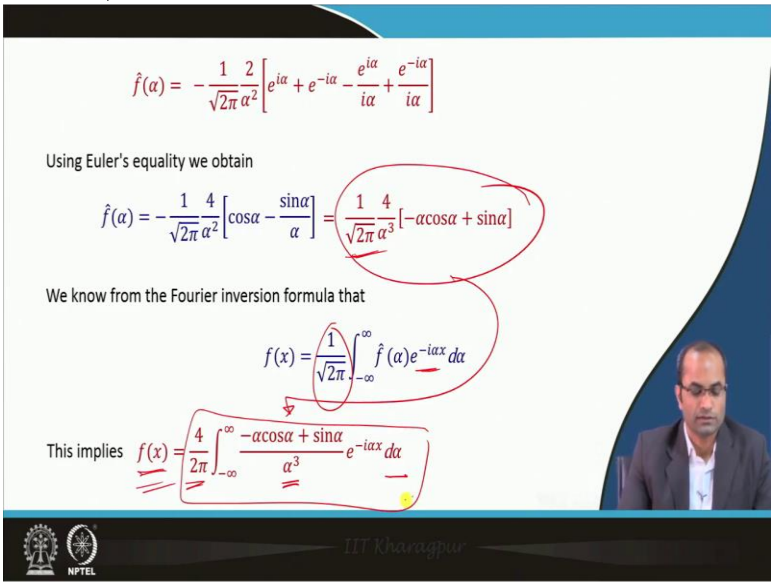

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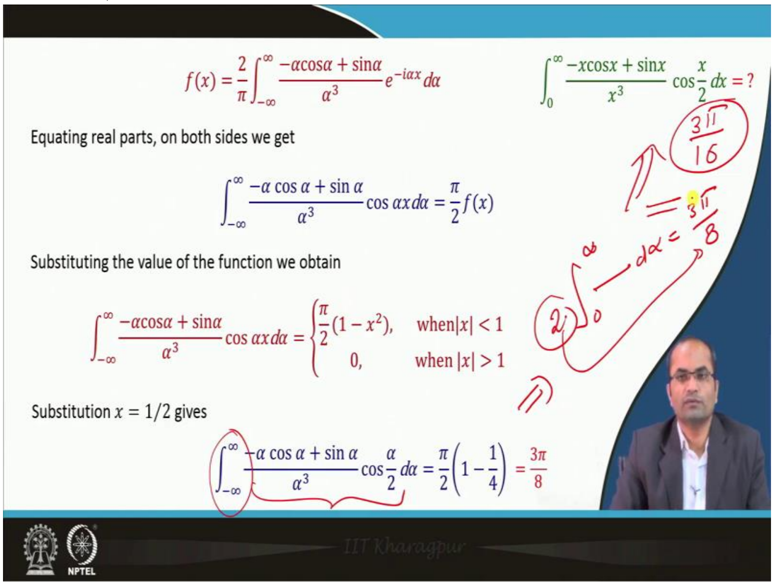

So this integral we can now evaluate. So we have to integrate this by parts, so first e power i alpha x will be integrated so we have i alpha and then 1 minus x square will remain as it is, and then we have to put these limits here from minus 1 to 1 and then second part again we have the integral of this e power i alpha x and then differentiation of 1 minus x square that is minus 2x.

So we need to integrate by parts again but this first term here when we put x is equal to 1, this will become 0 because of 1 minus x square and same thing will happen when we put x is equal to minus 1, so again this will become 0. So only the second term will survive and we have to again apply this integration by parts because we have x into this another function.

By doing so, so we will again end up here with x and then this e power i alpha x and i alpha will come because of this integration at this place and also at this place and now x will disappear in the second integral. So in the first part, we have this x and then we have to put this minus 1 for this e power i alpha x, so x is at these two place so first we will put 1. So we have e power i alpha and then with the minus sign and then minus 1 so we have e power minus i alpha and then because of the second integral, we will get again e power i alpha x over i alpha and then the limit of x from minus 1 to 1.

So this can be simplified further and we have such terms there which can be again combined to have this cos alpha for the first part when we divide by 2, so we can multiply here also by 2 and in the second case also, we have the same situation. So this will become the sin function with the minus sign and divided by alpha. So here we have now this alpha times this cos alpha and then we have minus this sin alpha but this minus we have incorporated here, so minus alpha cos alpha and then plus sin alpha.

Now having this Fourier transform of this given function and that is the trick we usually use. That is again the Convergence Theorem of this Fourier transform or in other words, what we say now we will take the inverse Fourier transform of this function and then we can get the desired integrals.

So having this formula for the inverse Fourier transform that means we will get back to f x and we know already f x and here in the formula the Fourier transform of f will appear and then e power minus i alpha x dx.

So this implies that we have f x equal to, we know already the Fourier transform which is given here so we can substitute that in this place. So we have 4 over this 2 pi because it is square root 2 pi and then 1 over square root 2 pi will be coming because of this Fourier transform. So we have 2 pi and then minus cos alpha plus sin alpha divided by this alpha cube and then this factor e power minus i alpha x d alpha.

So this is the integral now which has the value this f x and this is what the motivation for getting these such integrals with the help of this Fourier transform and then taking this inverse Fourier transform we can get the value of this integral as this f x.

So we can now further simplify this because this 2 over pi can go to the other side. So we have basically this pi by 2 f x the right hand side now and then this integral minus infinity to plus infinity, with this integrand integrated over this d alpha the value of this will be pi by 2 f x. Now substituting the value of this f x because the function is known to us, so basically we have this integral value as pi by 2, 1 minus x square and 0; otherwise if x lies between minus 1 and 1 the value of the integral we can compute using this pi by 2 1 minus x square and when it falls outside the range then this is 0.

So in the question this was asked that what is the value of this x cos alpha plus sin x over x cube. So this integral exactly, quite similar to the integral here we have on the left hand side. Other than this factor here, so here we have x cube and here the integrating variable is alpha, so that is not a problem. But here we have this x by 2 so this x if we put half there then we will exactly match with this desired integral.

So putting x is equal to half and x is equal to half lies in this range, that means we have to evaluate our integral from this pi by 2, 1 minus x square. So substituting this x is equal to half now in this integral, what we will get? We have minus 1, minus infinity to plus infinity and minus alpha cos alpha plus sin alpha and then we have alpha cube there and cos and the x is now half, so we have cos alpha by 2 d alpha, which is the same integral as asked in the question and then it is equal to now this value here when we substitute x is equal to half, so it is pi by 2 and then 1 minus 1 by 4.

So it is going to be 3 by 8 because this is 3 by 4 and then 4 into 2 we have this 8. So the value of this integral here is 3 by 8 and in the question it was asked that the same integral, the only difference is this 0 to infinity for this alpha.

So if we notice here with respect to alpha, so this is odd function in terms of alpha and here we have also, sorry the cos is even function, so here also we have even function but this alpha is sitting for, that will make it odd, sin alpha is also odd and here alpha cube is also odd. So the minus sign from this numerator will cancel out from the denominator and as a whole this integrand is an even function.

So this minus infinity to plus infinity, this limit we can now get as the 2 times and then this integral 0 to infinity this whatever integrand here d alpha and here we have 3 pi by 8. That means the value of this desired integral will be because this 2 will go to the denominator other side that means this value will be 3 pi by 16.

So in this way, we can get the value of the integral. First, we have to get the Fourier transform and then taking the inverse Fourier transform we can get the value of some interesting integrals.

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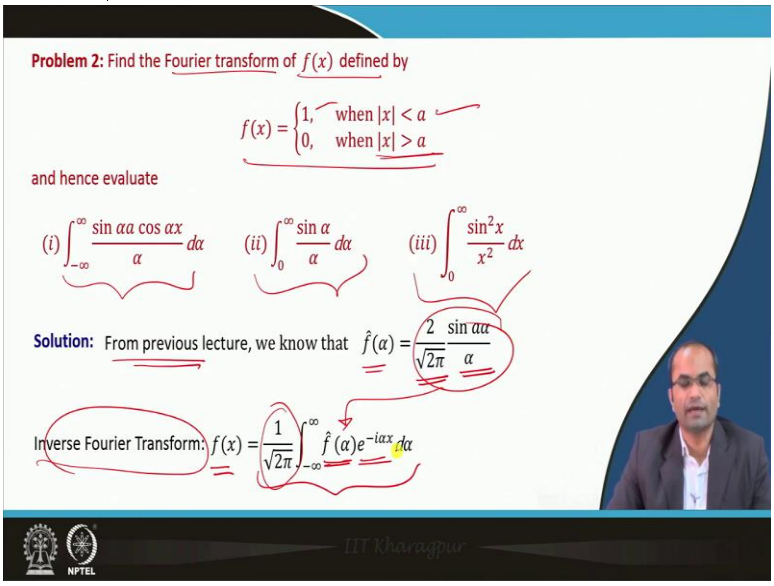

So coming to another problem where we have this Fourier transform of this f x which is defined as this pulse function, so x less than equal to a we have 1; x, mod x greater than a, we have 0 and then we will evaluate once we get the Fourier transform these various interesting integrals. We have already done in the previous lecture that the Fourier transform of such rectangular function is given by 2 over square root 2 pi and sin a alpha over this alpha.

So we will use this result now. We do not have to evaluate again and then we can apply this inverse Fourier transform so the same trick what we have done in previous example. So we have f x equal to this integral, where this integral will be 1 over square root 2 pi f hat alpha where this, the Fourier transform will come and then e power minus i alpha x d alpha.

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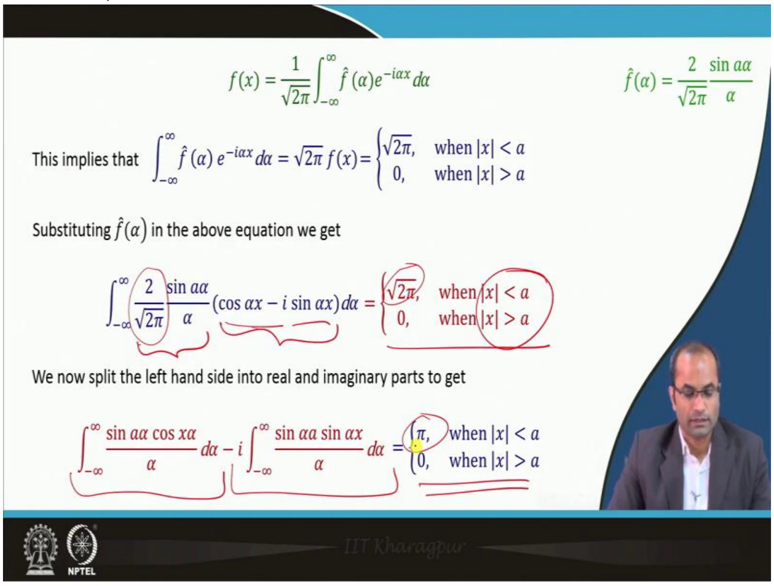

So having this now we can just take this square root 2 pi to the other side so that is the value and we know already the function value, so we have substituted here 1 and 0 depending on this x, and this f hat alpha we can now substitute there. So f hat alpha was 2 over square root 2 pi and sin a alpha over alpha, so by substituting this in this integral now and having this exponential there in terms of this cos alpha x minus this sin alpha x d alpha, we have this integral.

So this is the f hat alpha and then e power i alpha x we have written as cos alpha x minus i sin alpha x. The value is square root 2 pi 0 when this x lies in these intervals. Now we can split this integral into this real and imaginary parts. So we have these two integrals basically.

First one gives the real one and the second one is the imaginary, pure imaginary value with this i and equal to again we have just simplified here because square root 2 over pi can now give us here exactly pi there. So here we have square root 2, so square root 2 gets canceled and then we have this pi the other side, okay.

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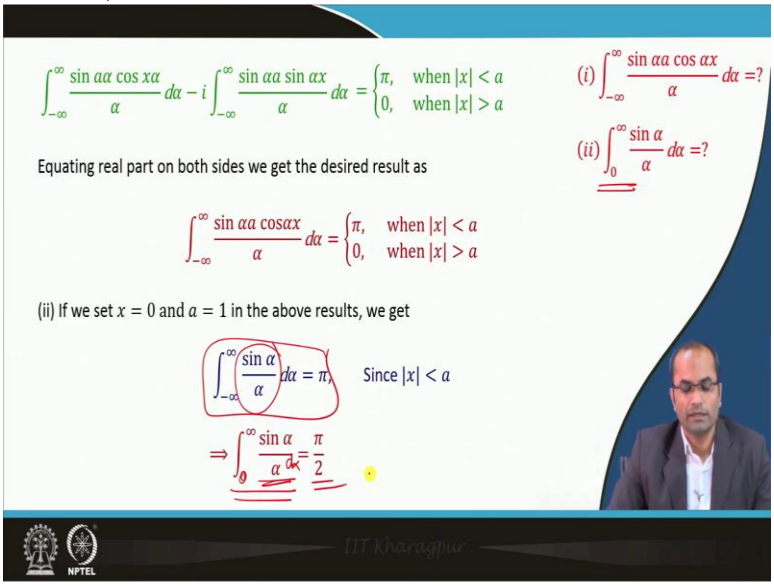

So having this now this combination of two integrals as pi and 0, it was asked that what is the value of this sin alpha a cos alpha x over alpha d alpha and if we look at this real part here, we have exactly sin a alpha cos alpha x divided by alpha and then d alpha.

So we equate this real part because the right hand side, the value of these integrals we know that it is pi or 0. So the right hand side is purely a real number. So the left hand side, the first integral should be equating this pi and 0, the other one the complex part, the imaginary part will become 0. So we are equating this real part now, so minus infinity to plus infinity and the sin alpha a cos alpha x over alpha d alpha, the value is again the same here, pi, 0 and then we have x, mod x less than a and mod x greater than a.

So the another question was that what is the value of this integral sin alpha over alpha so that we can deduce exactly from this itself. We have to set the a and what is the x so that we can get this desired integral. So if we set here this x is equal to 0 and a is equal to 1, so a equal to 1 will give us here sin alpha and x equal to 0 will make this cos alpha x as 1. So we will get this desired integral here sin alpha over alpha d alpha is equal to pi because here this x, it is falling in this region, x is 0 and a is 1. So this absolute value of x is less than a.

So we have this integral and it was 0 to infinity, so this integrand is an even function anyway, so we can just have this, we can use this property of the integral. So now 0 to infinity sin alpha over alpha and this d alpha is equal to pi by 2.

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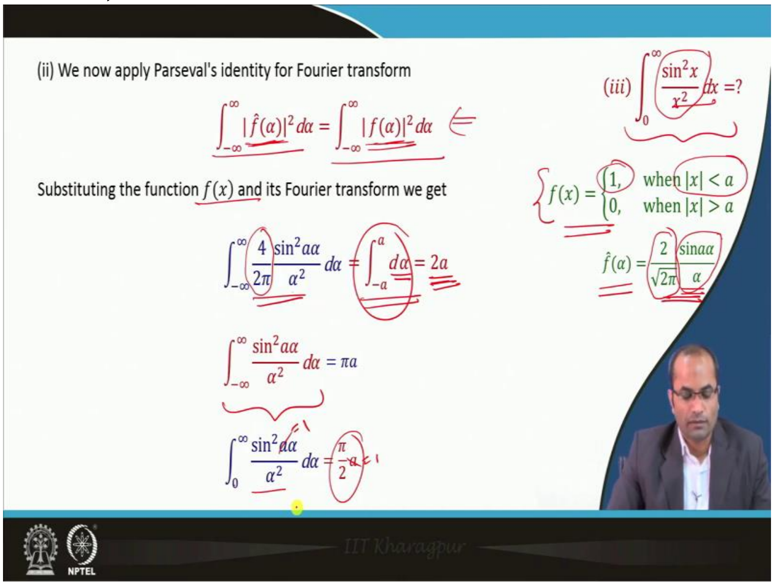

So the third question was that what is the value of sin square x over x square dx and we know already that we have this f x in the question and its Fourier transform was 2 over square root pi sin alpha a over alpha. And looking at this integrand of this integral, it seems that this is being squared here, sin alpha a or this over alpha, so that comes from exactly from the Parseval's identity which we have discussed already in the previous lecture.

So that Parseval's identity was this one, minus infinity to plus infinity absolute value of this f hat alpha square is equal to minus infinity to infinity and f alpha square d alpha. So we can now use this inequality to get such integrand which is the square of this sin x over x. So by substituting this f x and its Fourier transform in this, so first this Fourier transform we have sin square a alpha over this alpha square and then this will be 4 over 2 pi because it was 2 over square root 2 pi. The right hand side we know already it was just 1, the function was 1 in the range minus a to a; otherwise it was 0. So we will be integrating here minus a to a and then d alpha. This will become exactly at 2a.

So by this simplification what we get now, sin square a alpha over alpha square d alpha is equal to this pi a and now for this question now which was asked is, for 0 to infinity and then sin square a alpha over alpha square we are getting pi by 2a and then this a is basically set to 1. So we can just remove this a from both the side by substituting this 1 and 1. So the value of this integral we are getting as pi by 2.

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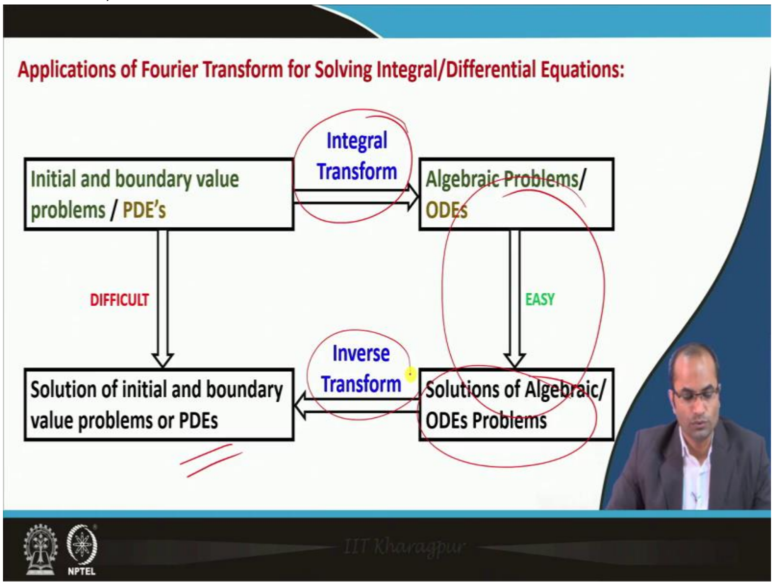

Now coming to certain applications of this integral, partial differential equation and many, at least two more lectures will come based on these applications, where we will be considering mainly the partial differential equations. But here just to give the idea or to begin with how these transforms in general indeed or in particular here this Fourier transform is used for solving the differential equations or integral equations.

So the idea is as follows: That, we have the problem which in the next lecture I will brief what do we mean by initial and boundary value problems. So we have some kind of differential equations here, either ordinary differential equations or PDEs or we have the integral equations.

So all these equations which normally if you want to find the solution of such differential equations, it appears to be difficult usually by many techniques, the direct techniques. But what we do usually or how this Fourier transform is applicable or coming into the picture, so if we take this integral transform of those problems, we get either algebraic problems.

So if our equation is like simple differential equation, we will get algebraic equation there or its integral equation, again we will get some kind of algebraic equation, and if we have the PDEs, for instance there, so their order will also be reduced, so we will get not only the order so one variable at least if there are two independent variables, one we will have here the differential, the differentiation over the other variable only, so one will be removed.

So you will get basically the ODEs which are much simpler to solve than PDEs. So this is the idea that once we take the integral transform, we get a problem which is in the different domain now, the transformed domain but they are much easier to solve so usually algebraic equation or ODEs, and then we can use this, so once we get the solution of those simple problems, we can use the inverse transform to get back to the solution of the original problem.

So we have to first apply the transform then somehow simplify it to get the solution in the transformed variable and then we have to take the inverse transform to get back to the original solution.

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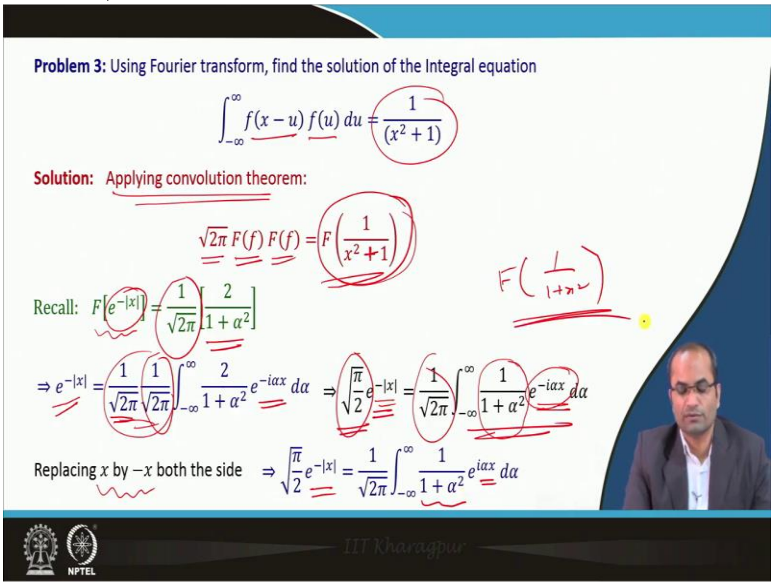

So the idea we will explain through this example for instance, a very simple one where we want to use, for instance this integral equation. So we have f x minus u f u du and the right hand side it is 1 over x square plus 1. So looking at this integral left hand side, this reminds us as the convolution integral. So this is basically the convolution integral and as soon as we observe that there is a convolution integral this Fourier transform serves the purpose to remove this integral.

So if we apply the Convolution Theorem then we will get here square root 2 pi the Fourier transform of this f, the Fourier transform of this f and then the right hand side we have the Fourier transform of 1 over x square and this plus 1.

So just, if we, because we need the Fourier transform now of 1 over x square plus 1 so to get this, if we recall that from the previous lecture, the Fourier transform of e power minus x was 2 over 1 plus alpha square and this multiplied by 1 over square root 2 pi. So e power minus x here, so if we take the inverse Fourier transform what we are getting 1 over square root 2 pi because of the inverse and then we have e power minus i alpha x d alpha.

So this 2 pi we can take to the other side, so we have here 2 pi and then this is also 2. So this 1 over square root 2 pi we will keep it and then square root 2 over pi can go to the other side, so we have square root pi by 2 e power minus x and then this integral.

So now the idea is that we want to get the Fourier transform of 1 over 1 plus x square. So here this form 1 over 1 plus x square is coming and then we will make this as the forward Fourier transform by, first we replace this x by minus x both the sides of the equation, so here it is anyway in the absolute value, so nothing will change. The right hand side will become i alpha x and that is exactly the idea now, that we can get the Fourier transform of this 1 over 1 plus x square.

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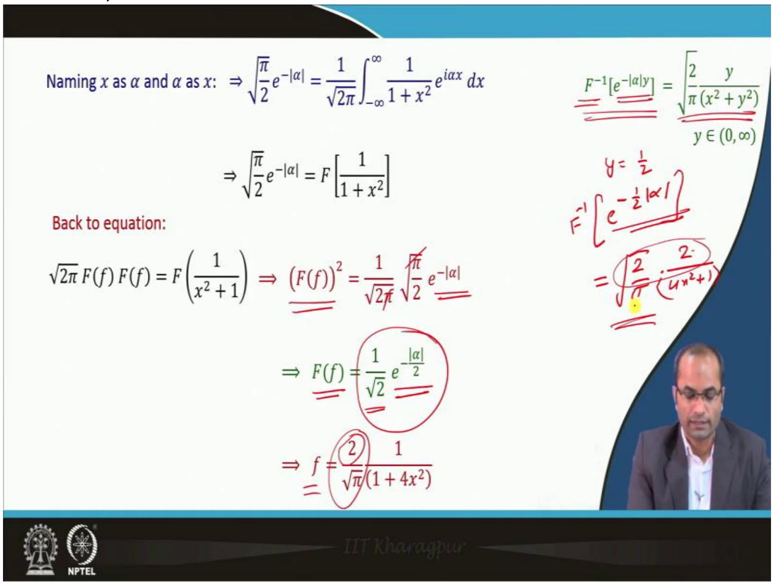

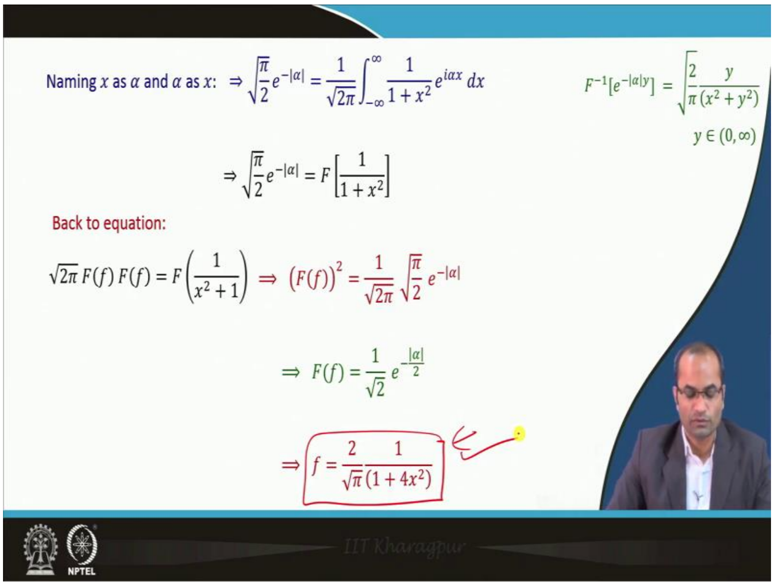

So again, we can replace x by alpha and alpha by x so we have this equation here and the right hand side exactly gives us the Fourier Transform of 1 over 1 plus x square. So the Fourier transform of 1 over 1 plus x square is pi by 2 square root and e power minus the modulus alpha. So back to the equation now, our equation was square root 2 pi and the Fourier transform of f and the Fourier transform of f, the right hand side was the Fourier transform of 1 over 1 plus x square, and here we have this whole square.

So Fourier transform of f whole square is equal to 1 over square root 2 pi and then we can use this calculation that the Fourier transform of 1 over 1 plus x square is square root 2 pi e power minus this absolute value of alpha, and then we can remove this square. So we have the Fourier transform of f as 1 over square root 2 and then square root pi already gets canceled, so we have e power and then here also we have to have half power that means minus absolute value of alpha divided by 2.

So now we recall this result which was already proved in the previous lecture that the Fourier inverse of this e power minus absolute value of alpha and then some parameter y sitting there then this is y over x square plus y.

So if we take here y is equal to half for instance, we are getting exactly Fourier inverse of e power minus half absolute value of alpha is equal to square root 2 over pi and then y is half there, so this 4 will go so we have 2 over 4 x square and plus 1. So having this now we can use this inverse transform there and we can get this f as 1 over 4 x square plus 1 with this factor 2 over pi.

So the inverse here will be used for square root 2 over pi and then 2, so we have 2 square root 2 there and the square root 2 gets cancelled so we have 2 and then square root pi. So this is the f now which was given in the integral form that was the integral equation and we could extract out of this integral equation as f, 2 over square root pi 1 over 1 plus x square.

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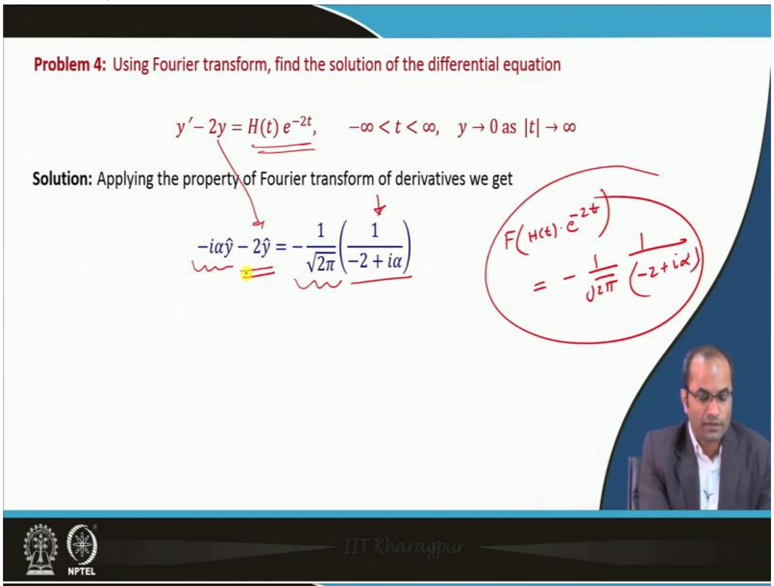

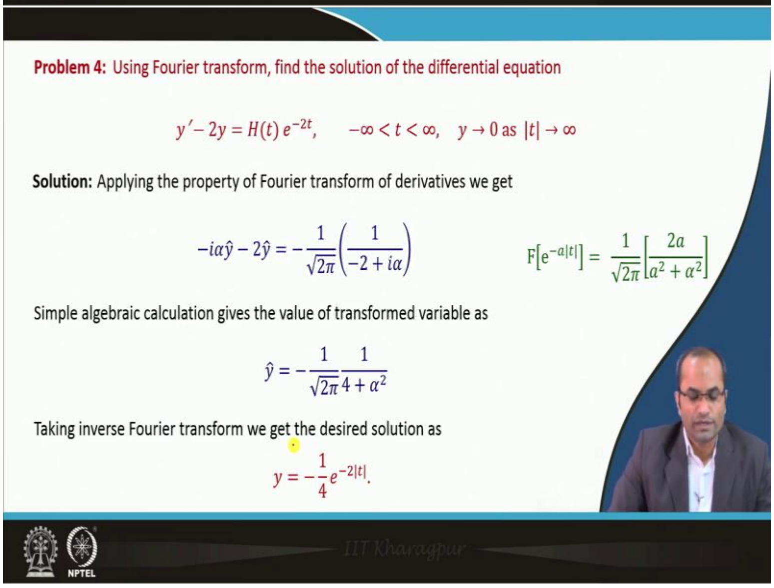

So using this Fourier transform we will find the solution of the differential equation this y prime minus 2 y and there is a Heaviside function sitting there e power minus 2t and then we have the range here minus infinity to infinity for t and y goes to 0 at both the ends whether t goes to plus infinity or t goes to minus infinity. So this is the differential equation, the ordinary differential equation.

The idea if we have the different differential equation, the idea will remain the same so we are just demonstrating here on this one problem. The major application which we will discuss in this lecture that is for solving the partial differential equation that to with associated conditions that we call boundary conditions or initial conditions. So that, there will be two devoted lectures on that. So on these differential equations here the idea will be demonstrated and we can use many other differential equations.

So in any case what we have to do, we have to apply the Fourier transform on this given differential equation so applying the Fourier transform of the derivative so here this is the idea behind Fourier transform because you remember that the derivative theorem for Fourier transform was that it can remove the derivative once we apply the Fourier transform to the derivative of a function.

So if we apply this Fourier transform here, what will happen? So i alpha and y hat, y hat is the Fourier transform of y, minus this 2, the Fourier transform of y is this y hat and the right hand side this also we have done before, so this will be coming as minus square root 2 pi and 1 over minus 2 plus i alpha that is the Fourier Transform. So Fourier transform of this Heaviside function t with e power minus 2t this was also done in previous lecture, this is minus 1 over square root 2 pi and then 1 over minus 2 plus i alpha.

So that is used here. So we have applied the Fourier transform to both the sides of the equation, the left hand side we got everything free from the derivative, the right hand side we got also the Fourier transform. So everything is in terms of alpha or y hat. So y hat is also a function of alpha. So this simple algebraic calculation now what we can do here y prime is y hat is common, so we take now, so the next step is to get y hat.

So what is y hat now in terms of anything, everything else means the alpha. So that we will do now, so solving this we have the y hat equal to this minus 1 over square root 2 pi and 1 over 4 plus alpha square because from this side also we have 2 plus i alpha. Here we have this 2 minus i alpha. If we merge this minus sign there, so we have this 4 plus alpha square as y hat.

So once we have the expression for y hat, we have to go for the final step taking the Fourier inverse. So we can take the Fourier inverse here and just to recall again that the Fourier inverse of e power minus a t was given by this formula and now we want to get this Fourier inverse of this, so Fourier inverse of 4 plus alpha square that means this a has to be supplied here as 2, so this will give us now 1 over 4 that will be coming because of that factor there.

So 4 is there so this 1 over 4 will come with the minus sign because minus sign is there and e power minus this 2 absolute value of t, so that is the solution which is the solution of this given differential equation.

So again just to repeat, first we have to apply the Fourier transform to the given differential equation and then if it is an ordinary differential equation we will get just an algebraic equation which can be solved to get y hat, and once we have this transformed, the Laplace transformed here we can go for the inverse transform to get back to y.

So given any differential equation whether it is first order or it is second order differential equation, these steps will remain the same. We have to first take the Fourier transform and then simplify it to get this y hat and then we have to go for the inverse Fourier transform and that will solve the differential equation.

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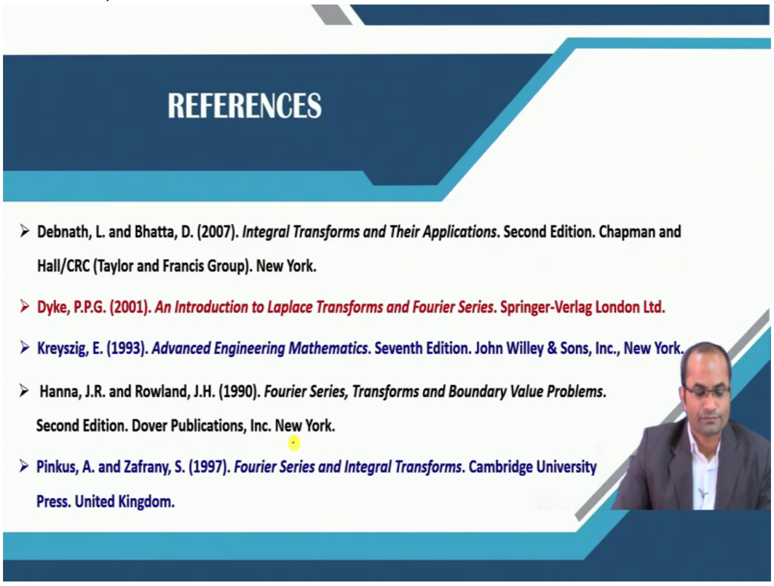

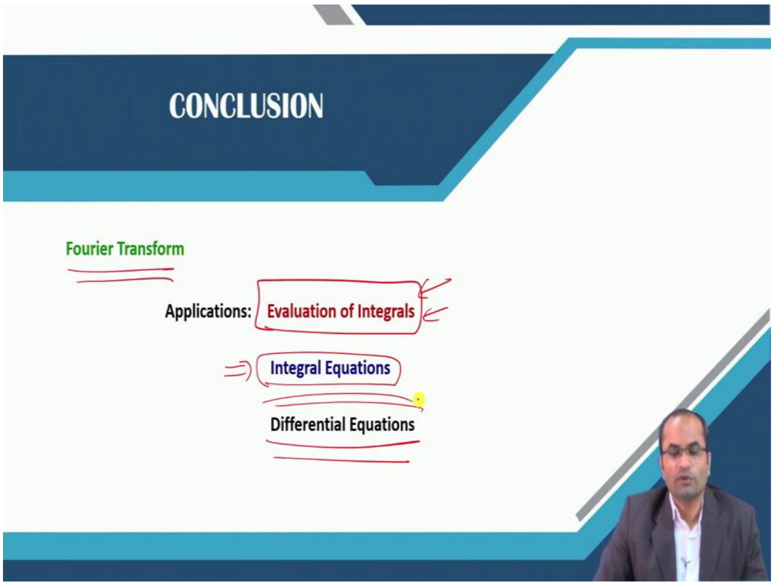

So these are the references we have used for preparing this lecture. And just to conclude in today's lecture, we have discussed various applications of this Fourier transform, in particular how to use it for evaluation of integrals and the idea was that we will apply Fourier transform of a given function and then taking its inverse transform, we will get several interesting integrals which we have also demonstrated in this lecture.

Also, the idea can be extended for solving the integral equations and mainly the property of the convolution is very much useful and we have demonstrated with the help of one example. And then, we have very shortly explained here that how to apply for a given differential equation so we have to, if it is ordinary differential equation just after the application of Fourier transform, we will get an algebraic equation that can be solved for this y prime and later on we can take actually the derivative of this y prime, sorry the inverse Fourier transform of this y prime to get back to the solution y. So that is all for this lecture and I thank you for your attention.