Engineering Mathematics 2
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Lecture 44
Fourier Transform

So welcome back to lectures on Engineering Mathematics 2 and this is lecture number 44 on Fourier transform.

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So in this lecture, first we will go through the complex Fourier integral representation, so this is an exponential form of Fourier integral representation and then from this representation we will derive what is the so called Fourier transform.

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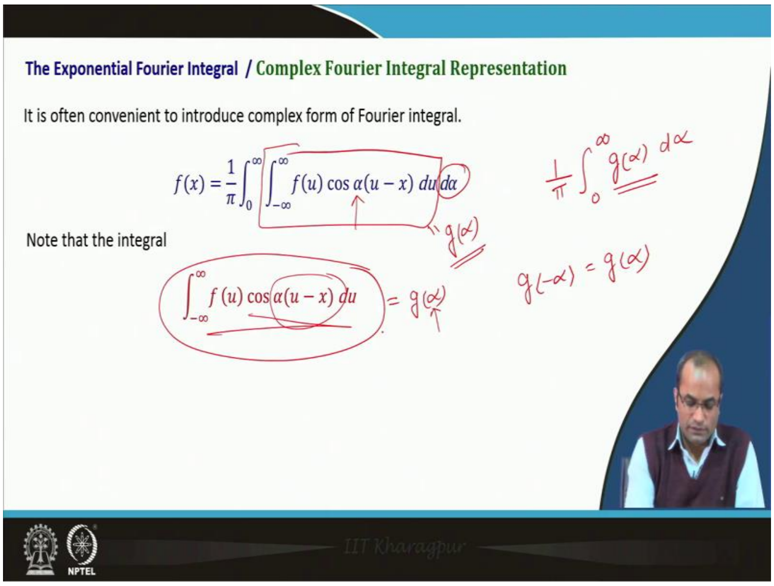

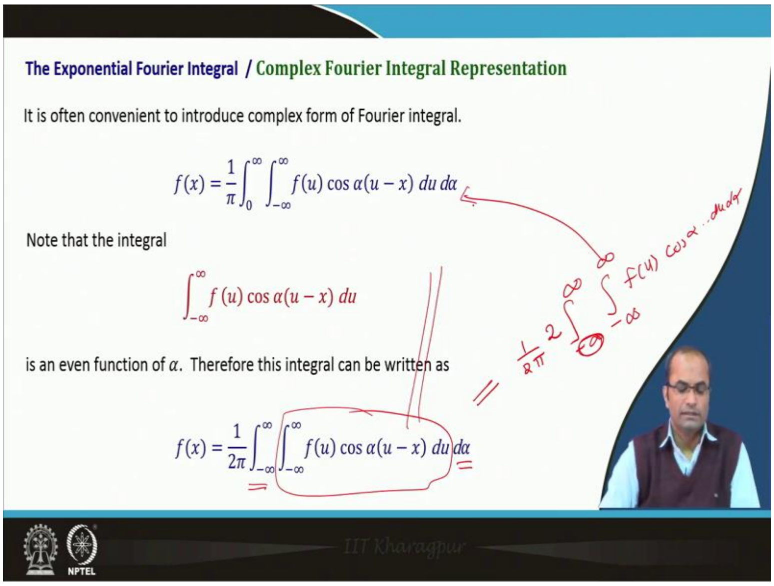

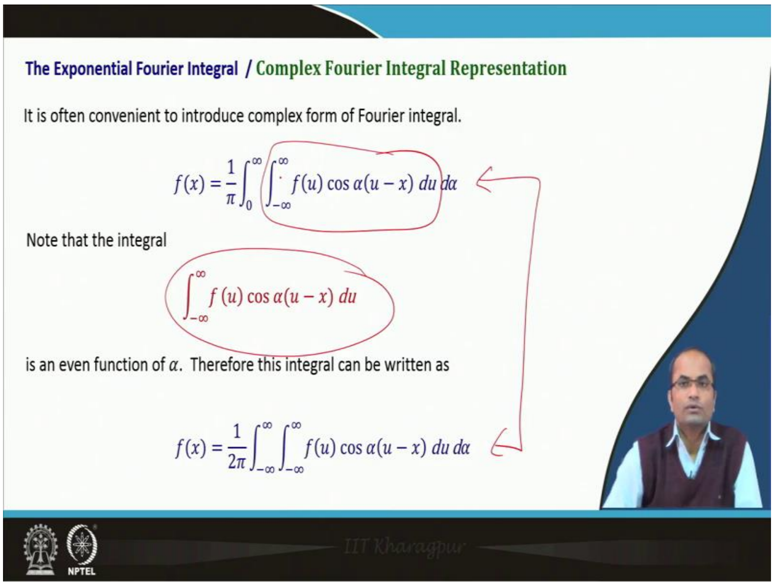

So just to begin with this complex Fourier integral representation, we will start with the Fourier integral representation which was given by this integral which has here fu and cos alpha u minus x du in its representation. What we should notice here that this integral, the inner integral, which is basically the integrant of the outer integral. Suppose this is g alpha, so the function of alpha we are treating it though we have u and u is anyway integrated.

We have x as well but we are considering this, the function of alpha because outer integral is over alpha. So basically, what we have? We have here 1 over pi and then this integral 0 to infinity, this g alpha and then we have d alpha. So now if we take a look on this g alpha, which is given by this one now, this is what we are denoting as g alpha, so in terms of this alpha, this function is an even function because g minus alpha is equal to g alpha since this alpha is sitting with this cos term and cos is an even function.

So this integrant here is an even function, so this is an even function of alpha and this function is being integrated over alpha from 0 to infinity. So now since this integrant is an even function of alpha, what we can do, we can rewrite this integral as 1 over 2 pi and instead of this integral 0 to infinity, we will make minus infinity to plus infinity and then minus infinity to plus infinity fu cos alpha u minus x du d alpha.

And to just compensate this because we have doubled the integral we have put 1 over 2 outside this integral or the other way around we can think as since this integrant is an even function, which is being integrated from minus infinity to plus infinity, then this can be written as 1 over 2 pi and then we have 2 times from minus infinity to plus infinity that will become 0 now.

So from 0 to infinity we have minus infinity to plus infinity fu and cos this alpha at delta du and d alpha, which is exactly this integral. So these two representation whether this or this one they are actually the same because we have used this property of the even function which was actually the integrant of this outer integral.

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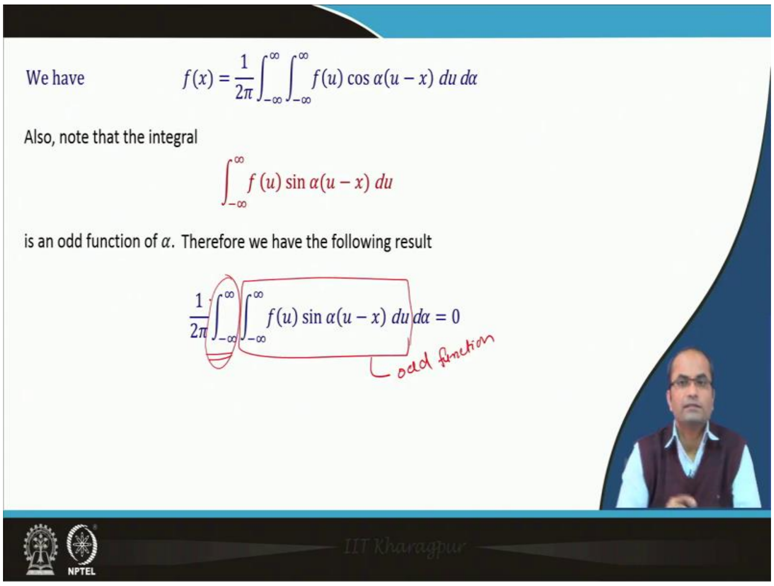

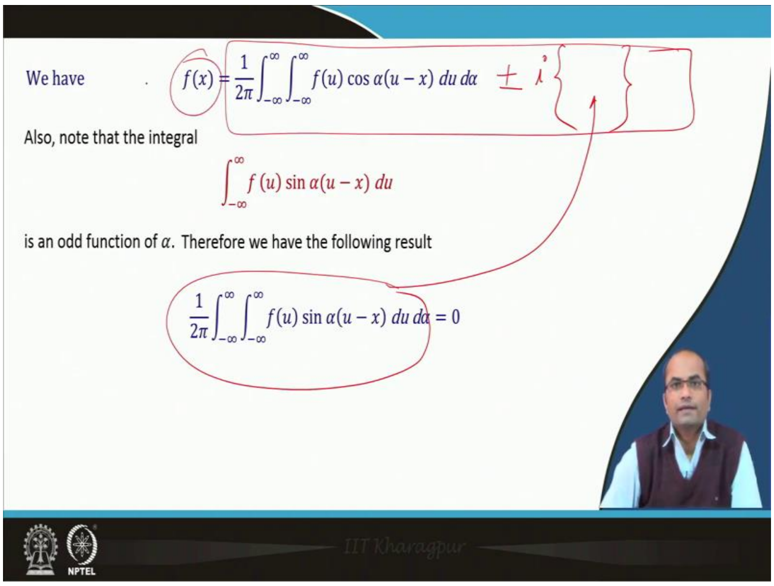

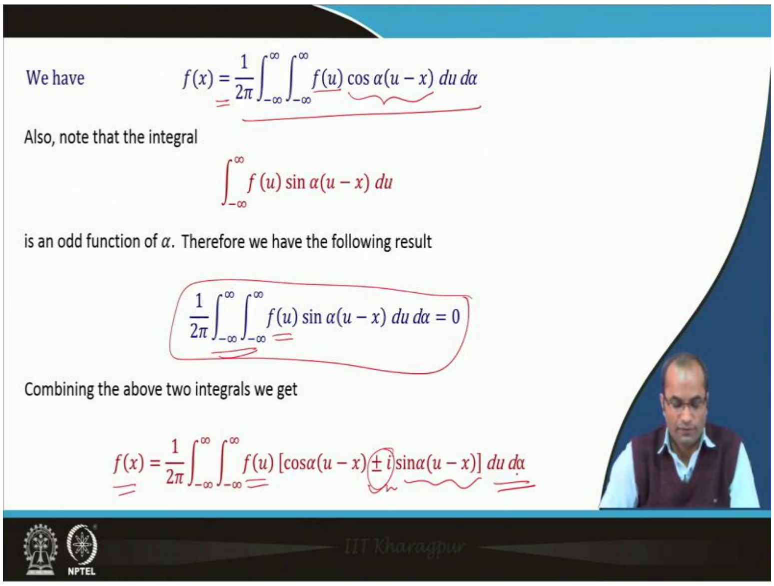

So for fx we now have this representation where the outer integral goes from minus infinity to plus infinity and also we should notice that this minus infinity to plus infinity fu sine alpha u minus x du because now instead of cos we have taken this sine and this alpha is sitting with sine and sine is an odd function so this here is an odd function and since this is an odd function if we consider this integral here now, minus infinity to plus infinity and exactly this same integrant which is an odd function.

So since this is an odd function and our integral is from minus infinity to plus infinity, so using this property of the integral this will be 0. So we have now 2 integrals basically, one is equating to this fx and the other one is equating to 0, so if we add in this upper one plus or with minus whatever i times this integral here which has the value 0 so nothing will change still we have the sum here is equal to fx and this is what we will do now.

So we will combine the above 2 integral to get this one, so fx will be equal to the first part and plus the second one with multiplication of this i. So fu is common and the integrals are also the same, so we have taken this outside this fu and then we have cos alpha u minus x combined with plus minus i and then sine alpha times this u mins x and then we have du and d alpha.

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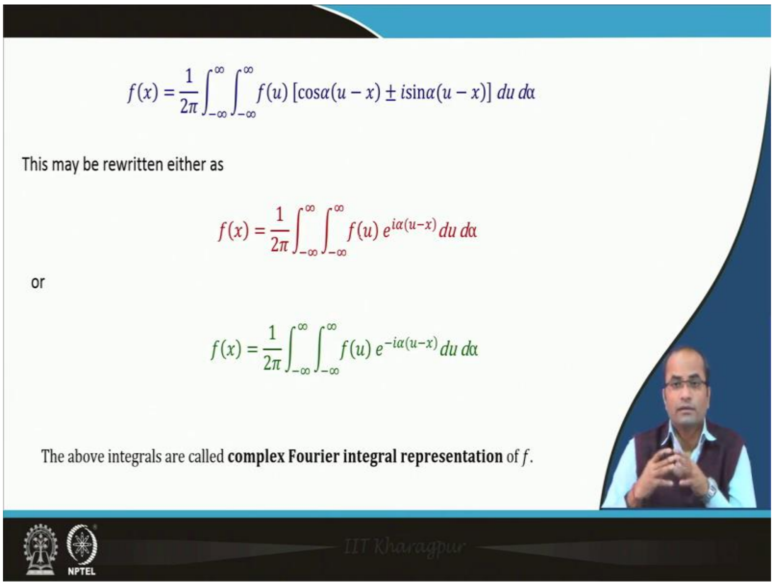

So having this representation now we will rewrite this again because we have this cos alpha and plus i sine alpha terms there which can be written in exponential form. So these integral we can write in 2 ways; one when we take the plus sine, so when we take the plus sine we will get the e power plus i alpha u minus x du d alpha or we can take this minus sine in there so if we take the minus sine, then we will get e power minus I alpha u minus x with the plus sign we will get here e power i alpha minus x term.

So with this these 2 representations either this one or we have this representation, these integrals representations are called complex Fourier integral representation of f because we have introduced this complex number there. So these are representations are called complex Fourier representation. What is the benefit of writing this is that will come in a minute when we introduce the Fourier transform.

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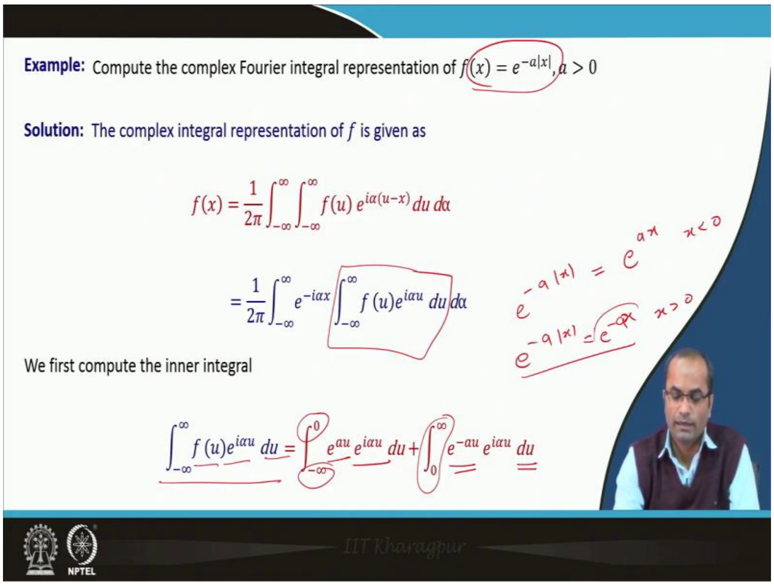

So, the first example where we will compute the complex Fourier integral representation of this fx which is equal to e power minus a modulus x or the absolute value of x and here a is taken as a positive number. So the solution now the complex integral representation of we know that we can write down this f in this form one over 2 pi, we have fu and we have taken with the plus sign there, so e power i alpha and u minus x du d alpha.

So we can substitute now for fu where the fx is given already there e power minus a and the absolute value of x. So here we have minus infinity to plus infinity and then e power minus i alpha x, so we have again splitted in to 2, so e power I alpha and x we can takeout, then this inner integral because this inner integral is over you, so we have fu and e power i alpha u and then the outer integral over the alpha.

So having this now we can compute this inner integral first and then we can put that value here and that will be the complex Fourier representation of f. So the out, inner integral with this fu e power i alpha u du, since this fx is given by e power minus a absolute value of x, so we can break this integral minus infinity to plus infinity in to 2 parts, one will go from minus infinity to 0, the other one will go from 0 to infinity.

So when u is in the rage minus infinity to 0, we will have this e power ax, so e power au in the range minus infinity to 0 and then the term e power I alpha u when we are in the range 0 to infinity for u, the exponential e power minus this x will become e power minus x ax for positive values of x and e power minus a absolute value x will become e power ax if x is negative and if x is positive it will be e power minus ax, so that we have used here.

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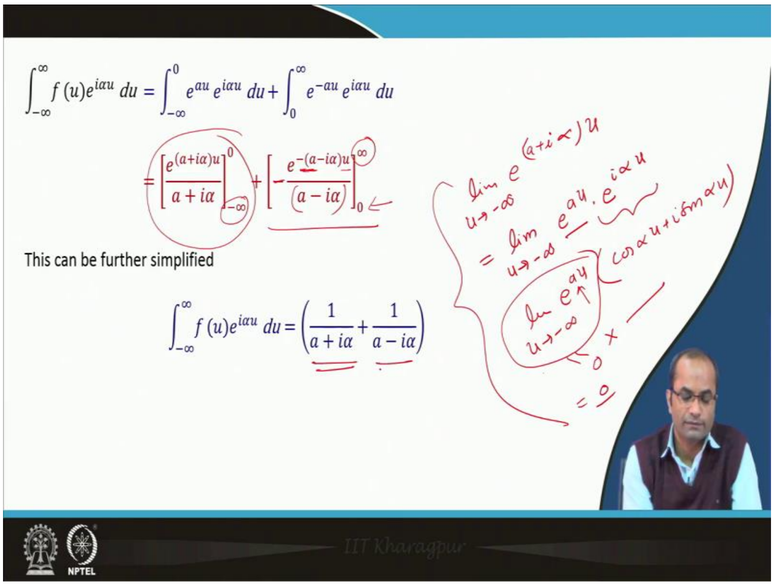

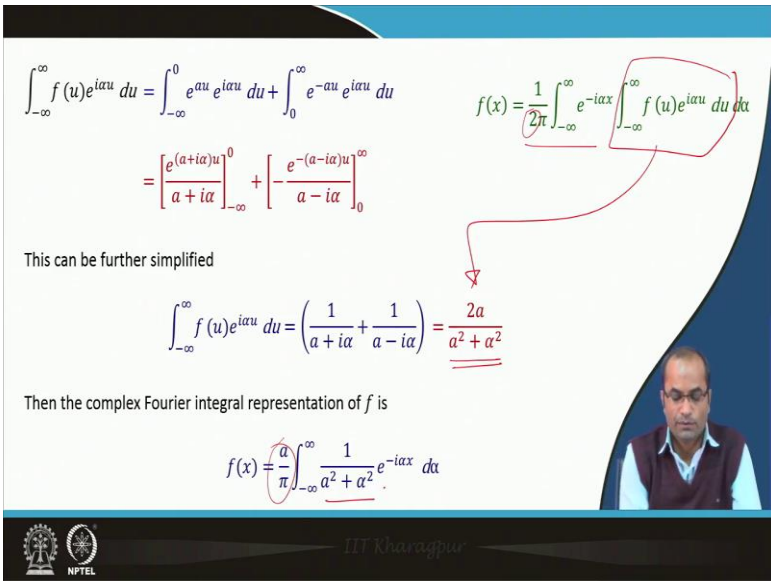

And now we can integrate this so we have these 2 integrals and the first one so we can combine because its exponential function so e power a plus alpha u and when integrated so we have a plus i alpha and e power a plus i alpha u and the limits we have to substitute from minus infinity to plus infinity. In the second case we have e power minus a and minus i alpha u and therefore, this a minus i alpha will be there in the denominator term and when we have integrated so there will be a minus sign.

So this limit again from 0 to infinity, so if we look at this first one when we put u 0, so we have e power 0 so this will be 1 over 1 plus i alpha and when we take this limit minus infinity, so we have e power a plus this i alpha and this u and the limit of this u is going to minus infinity. So what do we have? We have the limit u going to minus infinity e power au and then multiplied by e power i and alpha u there.

So this e power i alpha u we have just the cos alpha u plus i sine alpha u, so that is a bounded quantity and then together with this a power eu and the we have limit u going to minus infinity. So because of this u going to minus infinity, e power au and a is a positive number, so this will go to 0 and then we have here something finite so this is everything is 0. So when we substitute minus infinity this will be 0.

And in the second integral the other way around it is happening when we first substitute, we led this u to infinity, in that case we have already minus sign there so we have again e power minus au and u is now going to infinity so the same situation will happen there when u is going to infinity this will be 0 and only when u goes to 0, it will survive.

So this can be simplified from the first we have 1 over a plus i alpha, from the second we will get 1 over a minus i alpha which can be again simplified to give this 2 a over a square plus this alpha square. And then the complex integral representation of f which was given in this form we have already evaluated this integral the value is 2 a over a square plus this alpha square so we can substitute that, so this 2 will get cancelled with this 2 pi.

So we have a over pi outside and then 1 over a square plus alpha square e power minus i alpha x d alpha. So this is the complex representation of this Fourier integral and now we can move further to introduce what is the Fourier transform.

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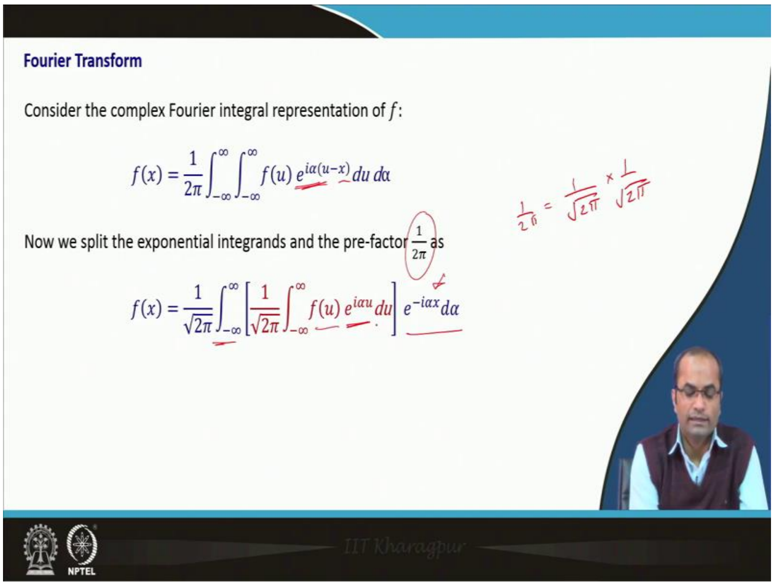

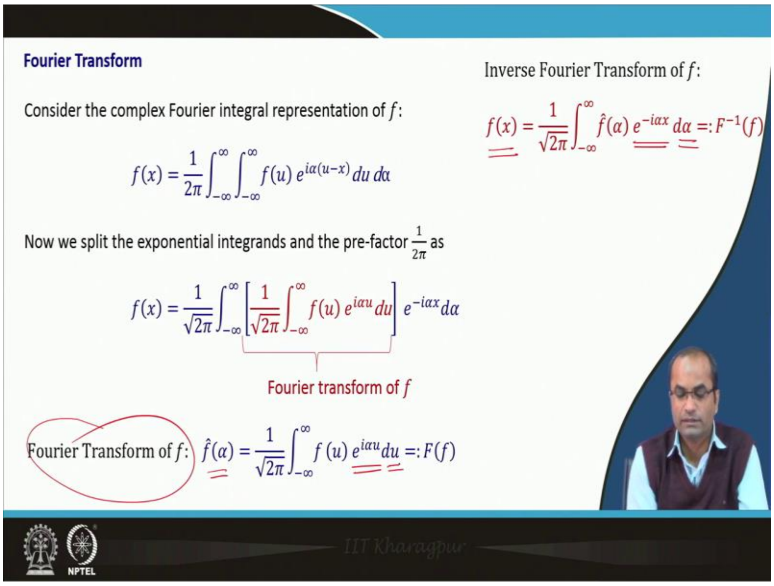

In the previous lecture we have already introduced the so-called Fourier cosine transform and Fourier sine transform and now we will introduce the Fourier transform. So here we have to consider now the complex Fourier integral representation which we have just discussed which is fx equals to this 1 over 2 pi and written. We have taken again this positive sign but 1 can work with the negative sign as well.

So we have minus infinity to infinity, here also minus infinity to plus infinity and then we have fu e power i alpha u minus x du d alpha. So here we have to split, I mean we need to split this 1 over square root, 1 over 2 pi into 2, so 1 over 2 pi we will write as 1 over square root 2 pi multiplied by 1 over square root 2 pi and then 1 we will keep with the inner integral and 1 with the outer integral as follows.

So one over square root 2 pi with the outer 1 which is over this e power i alpha x with the negative sign and then e power i alpha u is the inner one with fu and then du. So now we can introduce this Fourier transform so this quantity here in this bracket is called the Fourier transform of f and then, so let us just introduce here with this notation f hat alpha, the inner integral which is 1 over square root 2 pi fu e power i alpha u du.

Because this is a function of alpha now, so we are naming it as f hat alpha and the notation we use will be used as a big F, so F means Fourier transform of this f is given by this formula 1 over square root 2 pi minus infinity to plus infinity and fu e power i alpha u du. Now coming back to the inverse Fourier transform because we have introduced here this inner integral as the Fourier transform and suppose this is a function of alpha f hat alpha.

So given this Fourier transform and if we can perform this integral, the outer integral 1 over square root 2 pi minus infinity to infinity f hat alpha and then e power minus i alpha x d alpha, then we will get back to the function and that is what we call the inverse Fourier transform of f, which is given by this Formula here that fx is equal to 1 over square root 2 pi minus infinity to plus infinity.

Then we have this f hat alpha e power minus i alpha x d alpha and the notation we use here that is the inverse of this f operated on this f so inverse transform, inverse Fourier transform of f is given by this Formula here. So what is the difference basically? In the forward formula or in the formula of Fourier transform, we have e power i alpha you, whereas here we have with minus sine alpha x and this is integrated over alpha of course to get this function of x and then here to get the function of alpha it is integrated over you.

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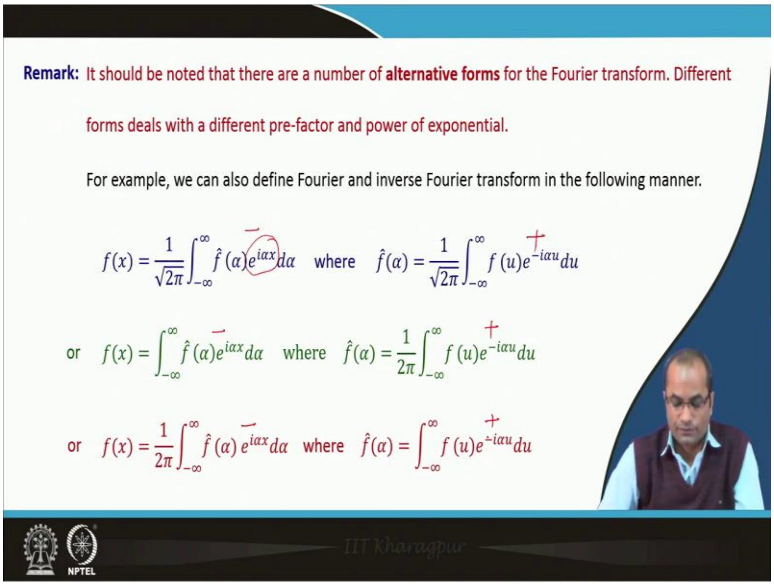

So now just a remark it should be noted that there are number of alternative forms because during the derivation we have seen that we can split in various ways because there was a factor there and in this process of coming to this complex form we have this plus i or minus i so that can also be considered at minus I, though we have taken here just the plus i that form.

So there are many alternate forms of this Fourier transforms and different forms deal with different pre-factor and this power of exponential. So there are various forms for instance this is what we have use here for the inverse Fourier transform and this is the Fourier transform then multiplied here by e power i alpha x and then we can have with the minus sine there to get this hat alpha or we can have basically the one factor there instead of this one.

And then we can have the complete factor 1 over 2 pi sitting with this Fourier transform or this factor may go with the inverse Fourier transform and then there will be no factor with the Fourier transform. So these are the for instance various forms which are available in the literature, so one variation is with respect to these factors and the other variation will be with respect to the exponent.

So here we have this plus sign, but we can take minus there then we can take plus there here also we can work with minus and then its counterpart will have the plus sign here minus and then here we have the plus sign so there are various combinations various forms which are available in the literature.

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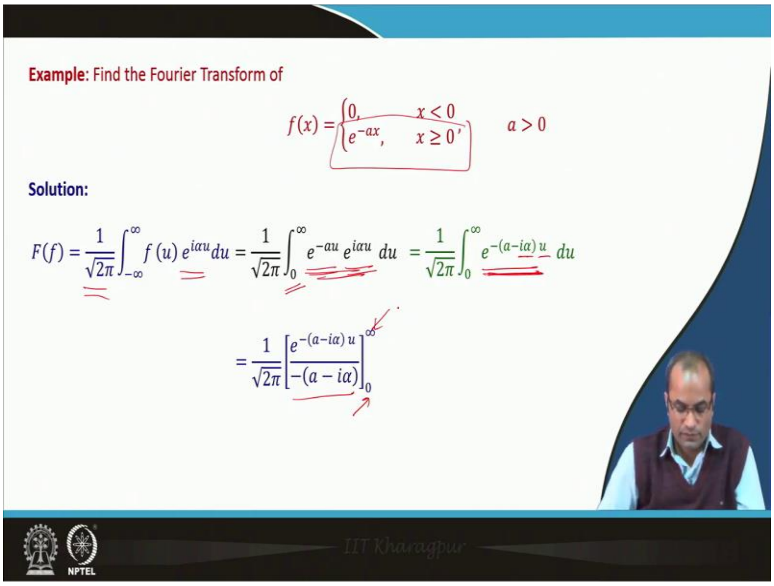

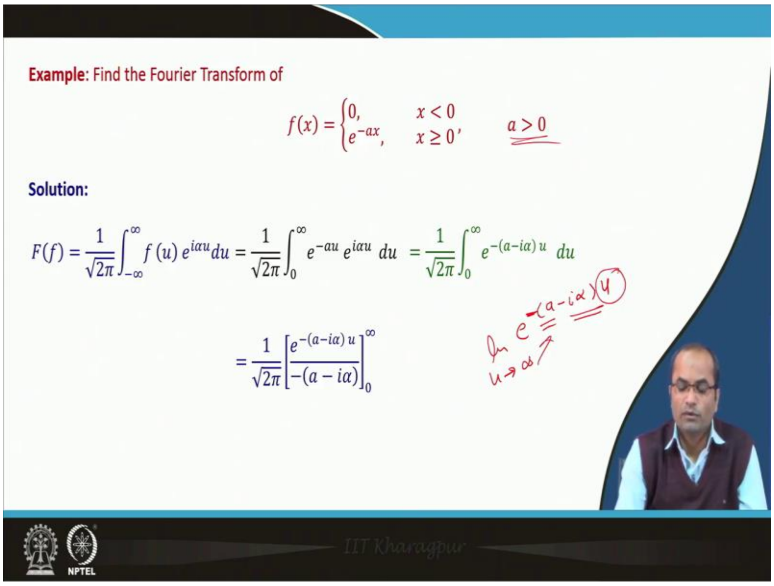

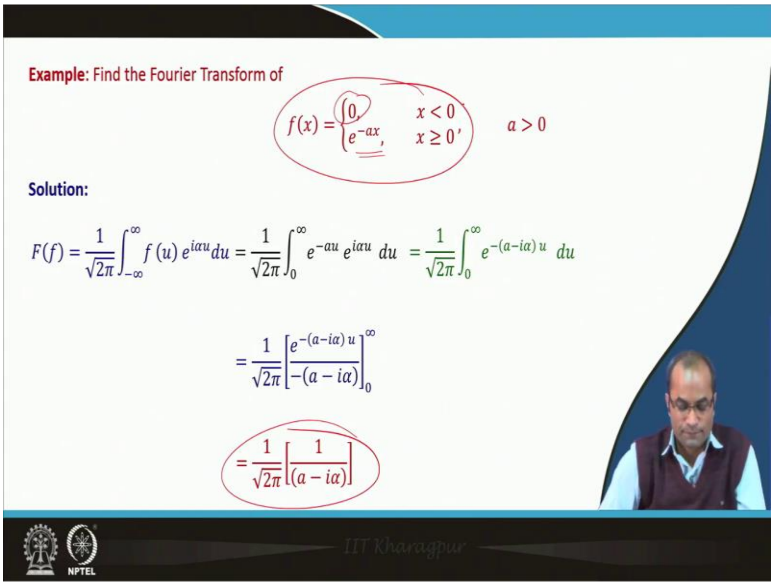

Okay, so just 2 examples based on this Fourier transform so we want to find the Fourier transform for instance of this function, which is defined as 0 for x negative and when x positive it is defined as e power minus ax and a is a positive number. So we take the Fourier transform, this definition with the plus sign so fu e power i alpha u du and 1 over square root 2 pi, so this is easy to remember that this 1 over square root 2 pi is sitting with both inverse and the forward transform.

And then we start with this plus sign first and then the inverse we take the minus 1, so this will be the notation used in this lecture. So here we have 1 over square root 2 pi with this exponential plus i alpha u and then we can use the definition of fx, which is just given for the plus x so we have then 0 to infinity e power minus au and then e power i alpha u du.

So these can be combined now since we are working with the exponential function, so here the integrant is with minus a and then mins alpha i with u and then we have this du there. So this can be now integrated, this exponential function to give this minus a minus i alpha in the denominator and then we have these limits 0 and infinity, so we have to see that what is happening to this e power minus a and minus i alpha u when this u goes to infinity.

And this is exactly what we have discussed also in the previous lecture because we have with minus a and a we have taken as positive so with this u, this will go to 0 because the term sitting with at e power i alpha u that is bounded term finite term, so we have e power minus au and u goes to infinity so this will go to 0 and when we are at this 0 there, then that term will survive.

So what we will get 1 over square root 2 pi and when we substitute the 0 there so we have the 1 over a minus i alpha. And then this negative will become compensated because we have this minus and then we put this limit. So this is the Fourier transform of this function e power minus ax which is just defined for positive values of x and it is 0 in the negative half.

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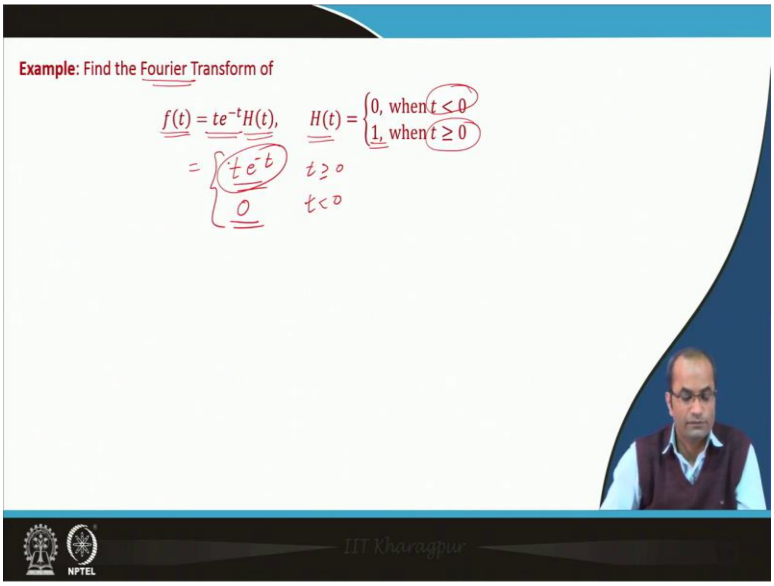

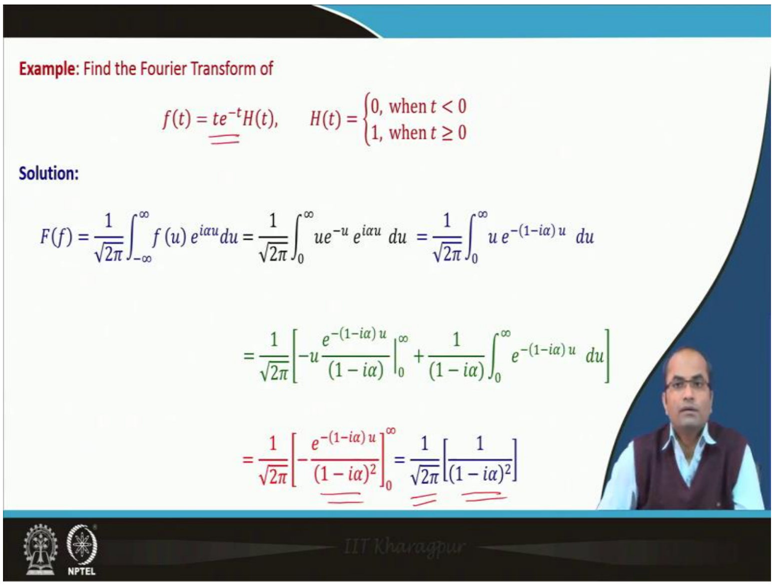

Well so this is another example where we will compute this Fourier transform of this function ft, te power minus t and this Heaviside function is sitting with. This Heaviside function means when t is greater than equal to 0, the value is 1 otherwise the value is 0. So gain this function ft is defined for the positive value of t non negative values of t and 0 otherwise. So for t a negative the value is 0.

So it i again similar to what we have done before but now it is a te power minus t so the function is different. So coming back to the solution now so the Fourier transform of this f given by this formula then we can use it because this is defined for positive value, so we have this integral from 0 to infinity and this fu is now ue power minus u and e power i alpha u du term. So then we can combine the exponentials there, so still we have u there and then exponential e power 1 minus i alpha u and this integral over du.

So this we can now because the u is there and the exponential is there, so we have to integrate this by parts. So integrating by parts the u as it is and then the integral of the second one, so we have e power minus i alpha with u and then this in the denominator as 1 minus i alpha plus we have again this 1 minus i alpha because this will be integrated and then the differentiation of u that will become 1, so we have this integral now.

Coming back to this first one when we have this infinity, so again we have this 1 minus u and then e power i alpha u. So the same situation this is a cos alpha u plus i sine alpha u and then together with e power minus u and if we take the limit as u approaches to infinity since it is e power minus u the negative exponential, this will go to 0 and this is something bounded sitting here so everything will go to 0.

So this when we take the limit as u approaches to infinity, this term will vanish and also when we take u approaches to infinity because the u is also sitting there so that term will also vanish. So this term will no longer be there, then we have 1 over 1 minus i alpha and if we integrate this so we will get again one more such factor 1 mins i alpha square and here e power minus 1 minus i alpha u with the negative sign and then 0 to infinity.

So again when we are talking about this limit as u approaches to infinity, this will go to 0 and then the only term which we will receive now when u approaches to 0. So here u approaches to 0, this will be 1, e power 0 will be 1 and then minus sign because we are anyway taking this when this limit is approaching to 0, so that minus will be compensated and we have 1 over square root 2 pi 1 over 1 minus i alpha whole square. So this is the Fourier transform of te power minus t and the Heaviside function t.

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So these are the references which we have used to prepare this lecture.

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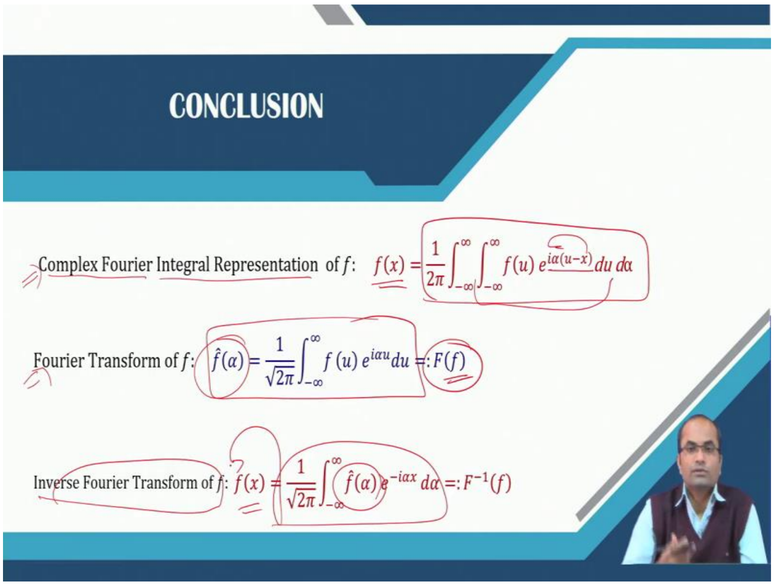

And just to conclude todays in this lecture we have studies the complex integral representation of f which is given in this form i alpha u minus x and there were other forms as well where we can take for instance here the minus sign in front of this i. So and then from this representation we have introduced what is the so called Fourier transform of f and this inner integral with respect to u taking this i alpha x out.

We have this representation which is we call Fourier transform of f denoted by normally this f hat alpha because this is a function of alpha and the notation we use big F of this small f so before your transform of f. The inverse Fourier transform can exactly be defined with the help of this complex integral representation and that will be exactly this fx and then we can substitute this f hat alpha there and get back to f again, so that is what we call the inverse Fourier transform and this notation will be used for the inverse Fourier transform.

So that is all for this lecture and I thank you for your attention.