Engineering Mathematics - II
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Lecture 41
Fourier Integral Representation of a Function

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So, welcome back and this is lecture number 41 on Fourier Integral Representation of a Function.

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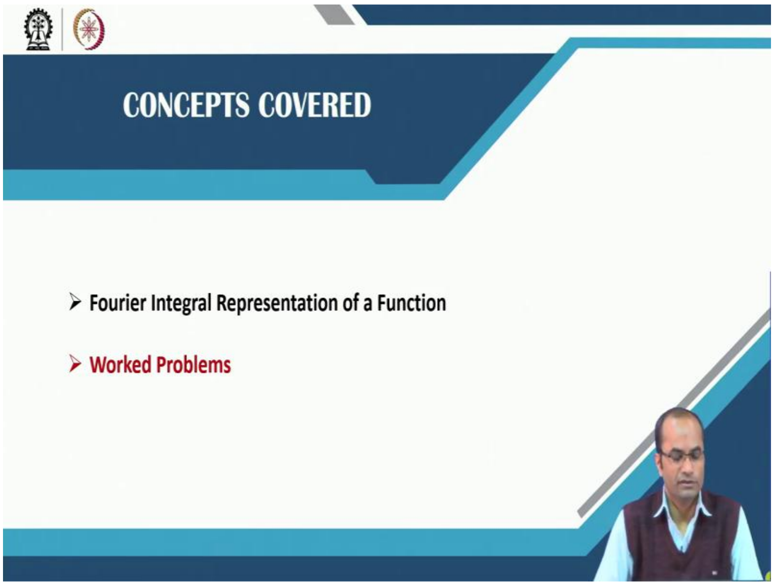

So, today we will discuss what is the Fourier Integral Representation that is analogous to what we have the Fourier series, but now instead of that series, that some we will have an

Integral representation of a given function and then some worked problems will be demonstrated for the Fourier Integral Representation.

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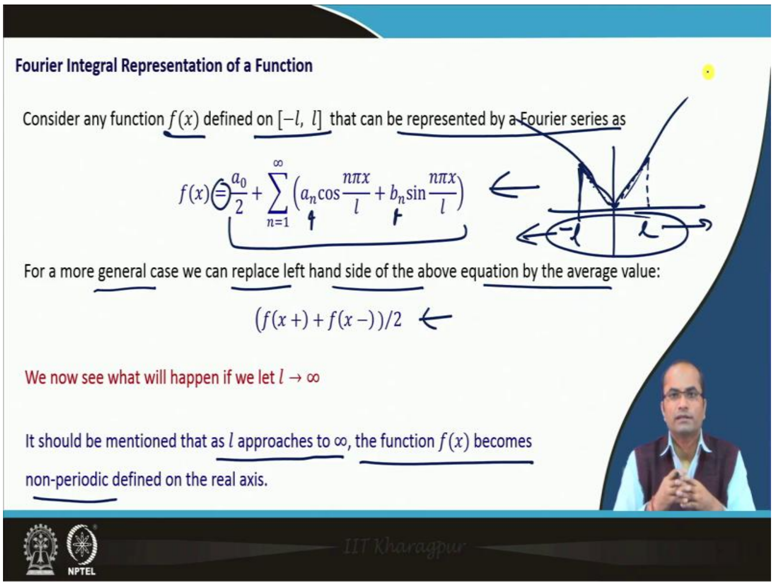

So, coming to the Fourier Integral Representation, we consider a function f x which is defined here on this interval minus l to l and that can be represented by the Fourier series, so that part we have already discussed and the corresponding Fourier series will be given as a naught by 2 and then we have these Fourier coefficients and cos and sin. And this equality we have already discussed the convergence in case of the continuity of the function and the existence of left and right derivatives we have indeed the equality of the series to the f x.

So, for a more general case, we can replace left hand side of the above equation by the average value that is also discussed so when we do not have the continuity at some points, in that case, the f x will be replaced by this average value and now the question is that what will happen if we let this l to infinity. If we increase this l, then what will happen to this representation which is the Fourier series representation of a function and that is the topic of this today's lecture.

So, it should be mentioned that as this l approaches to infinity, the function that is f x becomes non periodic because the function f may not be a periodic function, this is what we have discussed in the Fourier series. However, when a Fourier series converges, it converges to a periodic function.

So, the idea was that we take for instance a function minus l to l in this interval defined by for example this mode x or any other function and then we take this portion which is given in minus l to l and then write its Fourier series, then Fourier series will converge to the function whose period will be exactly defined by these two l and it will represent this function in this interval and it will have the periodic extension in the whole real axis.

But now the question is that if we let this l to infinity, then this function may not be a periodic function naturally, if may be periodic but it may not be periodic function and now we are talking about the whole real axis because if l goes to infinity, so we are covering this f x in the range minus infinity to plus infinity and the question is that what kind of representation now we will have instead of this Fourier series when l is very very large.

Because then there is no question of the convergence to a periodic function because this l is no more finite, so if l is finite and we are talking about the function which is given in minus l to l, then the Fourier series will converge to a periodic function and in one period, the function will be defined exactly the given function in the range minus l to l.

But if we are letting this l to infinity, then naturally this is not going to be periodic anymore even that representation will not converge to something which is periodic because initially itself we have taken this l from minus infinity to plus infinity the range of the function.

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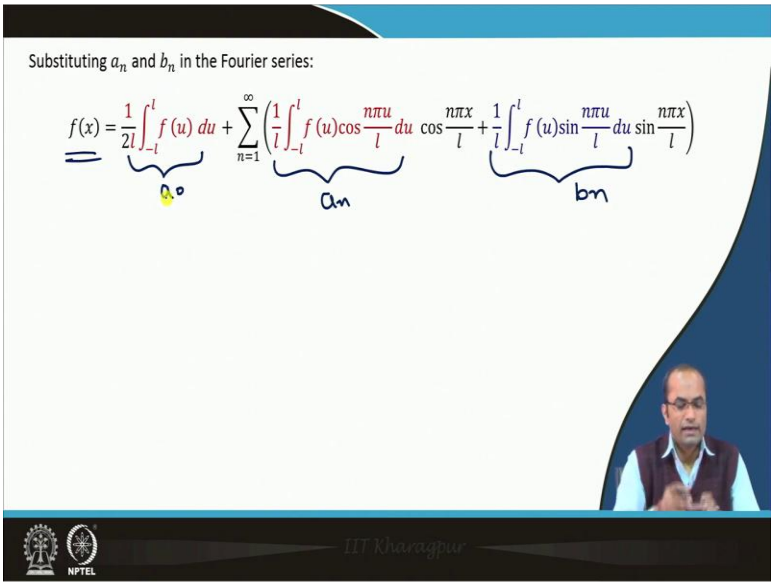

So, now if we substitute these coefficients in the Fourier series, that is a starting point now for going to this transition from the Fourier series to so called Fourier Integral Representation and finally we will introduce Fourier Transform. So, this lecture is actually the transition where we have from Fourier series to this Fourier integral and then Fourier integral we will represent the Fourier transform.

So, substituting the values of the a naught, so this is here a n and then here we have this b n. a n and this red portion here a 0. So, we have substituted, we have plugged these values in to the Fourier series and then we will proceed now, so this is equal to f x.

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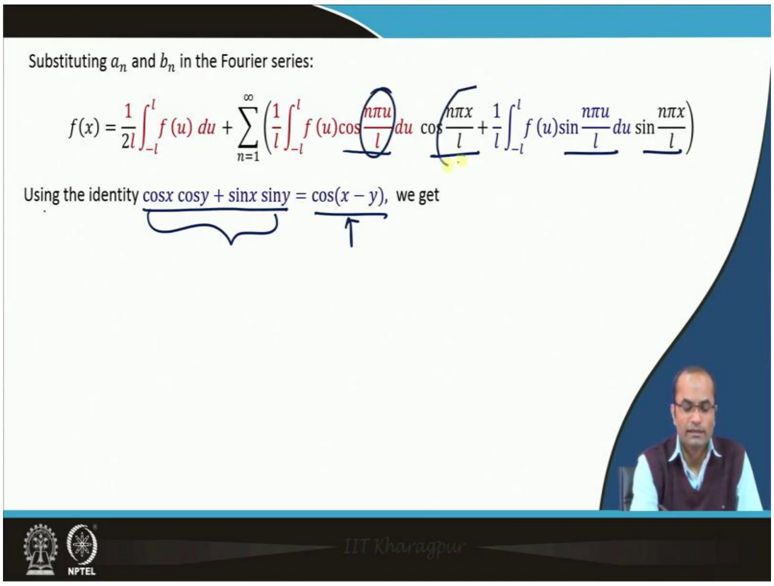

And then we can use this inequality that cos x cos y plus sin x sin y is equal to cos x minus y because here we have exactly this cos x and then here we have cos something y and then if we merge these two integral then we have here plus sin and then again sin. So, we have this structure and then this can be merged to have this cos and this x and minus this y term.

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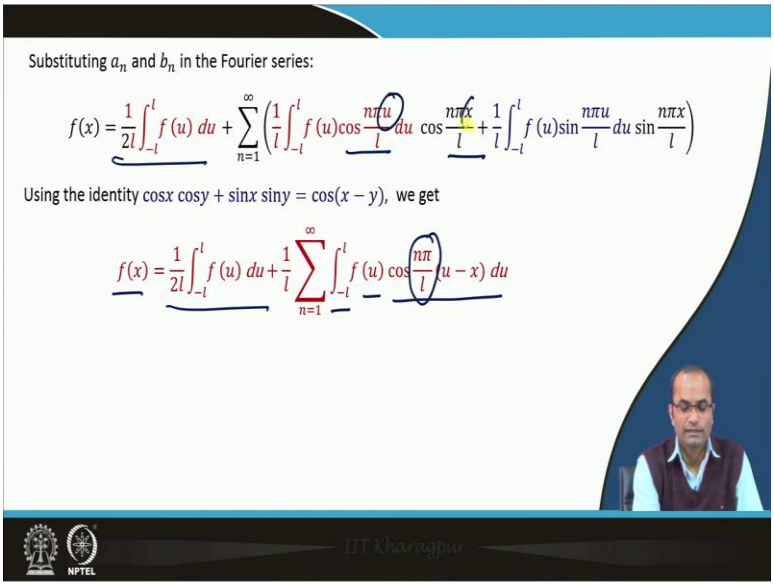

So, we have this f x is equal to **1** over 2l, the first term as it is and the second one is clubbed now in to this integral minus l to l, we have f u and cos this minus this one, so we have n pi over l as common and then u from here and minus x from there.

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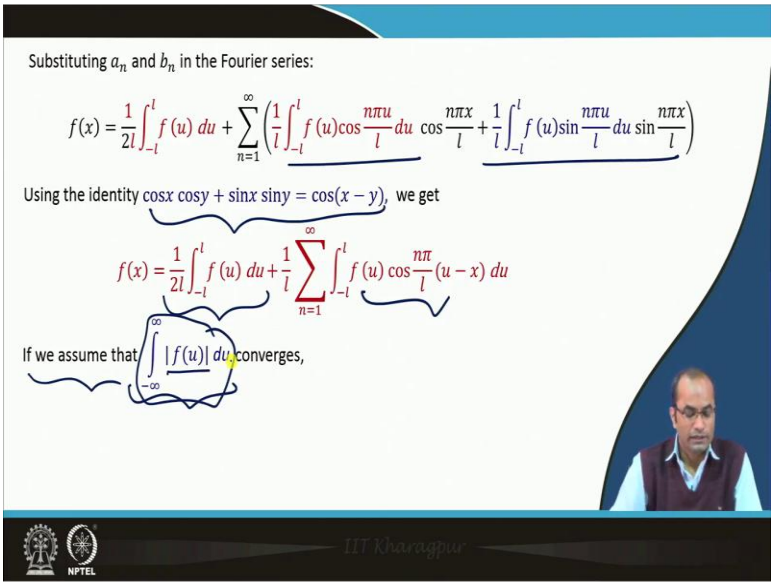

So, we have clubbed these two, we have joined these two integrals in to one and then using this trigonometric identity we have written this f u cos n pi over l and u minus x du. So, now if we assume that this fu du, so basically, we are talking about this first integral and we are assuming that if this integral converges indeed with the absolute value, then what will happen now?

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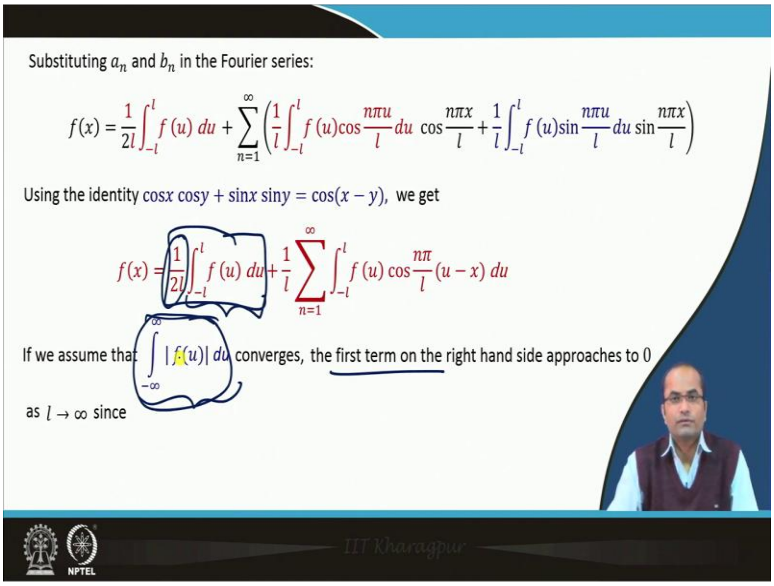

So with this assumption with this assumption that this integral converges we will see the first term on this right hand side, so this term here will approach to 0 as l approached to infinity because **1** over l is sitting outside and then this integral is bounded as a finite quantity because we have the convergence, the absolute convergence of that integral.

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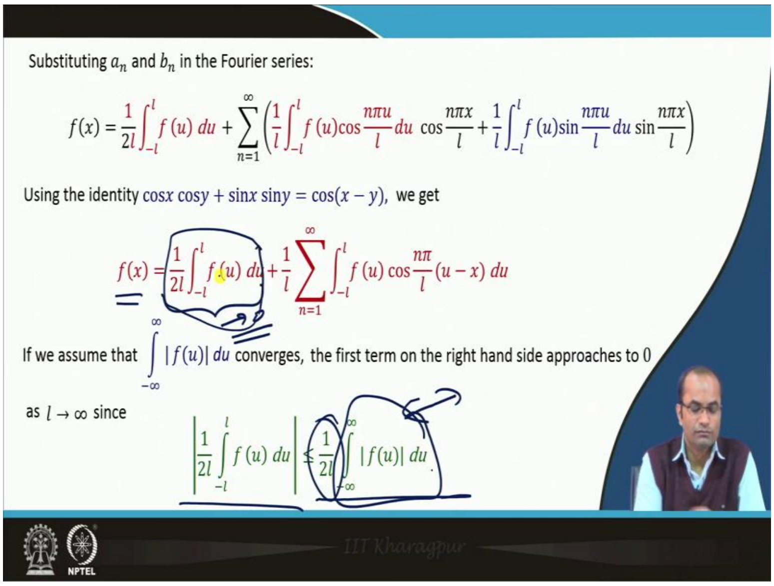

So, in that case. This will go to 0, because we have this relation now, this is even bigger and this convergence, so we have some value there and then **1** over l is sitting when l approaches to infinity, so something finite divided by l, l is approaching to infinity and this will go to 0. So this term here will tend to 0 as l approaches to infinity. So, in the limiting situation now, this f x will not have this term because this will vanish.

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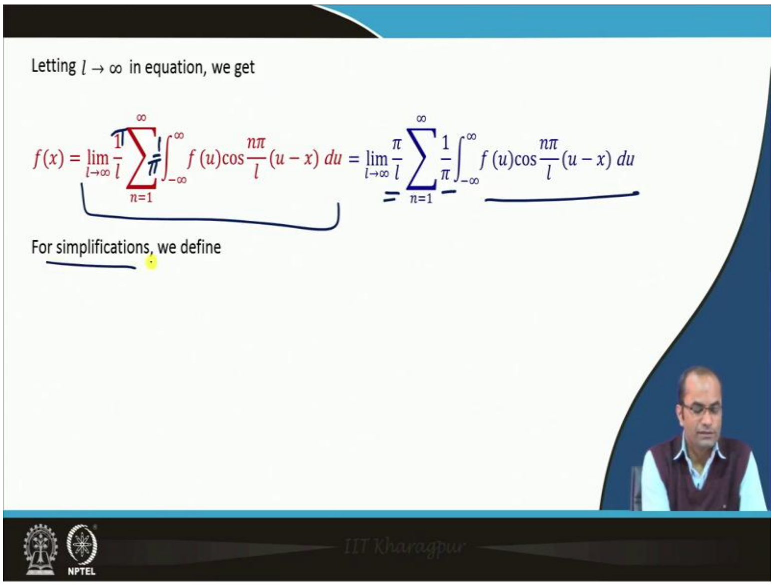

So, what we have when l approaches to infinity? We have only that a second term of that Fourier series basically which we can write down, so we can multiply here by pi and then we can divide by pi there, so this is exactly the expression here pi over l and then we have **1** over pi and minus infinity to infinity and exactly the same term.

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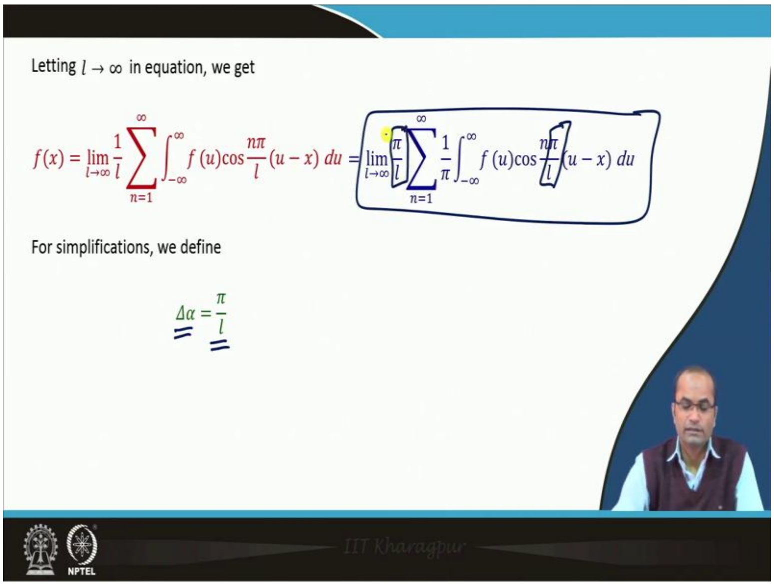

So, for simplifications now because this, by introducing some notations we can simplify this term, so the Delta alpha, if we denote by pi over l, so for instance this pi over l is sitting here and also this pi over l is sitting there.

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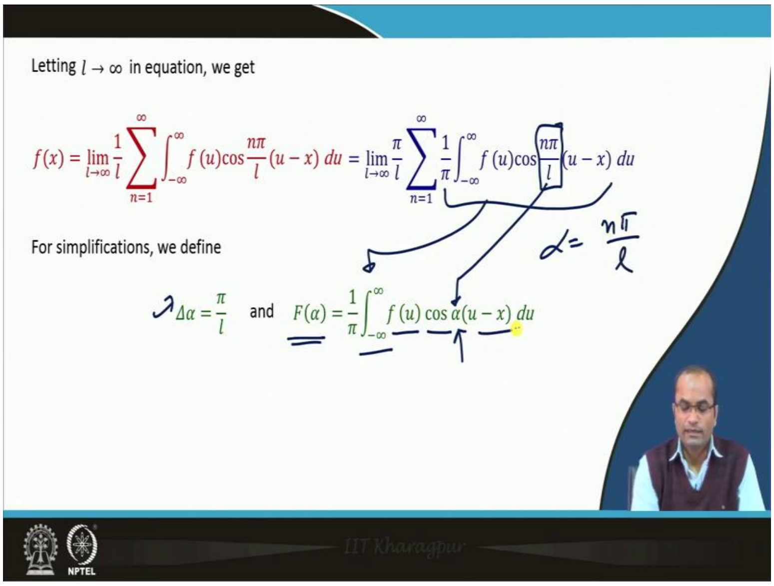

So, we can introduce this pi over l to a new a number Delta alpha and the F alpha, the big F we can introduce so that we can also simplify this integrand there, so this is exactly 1 over pi minus infinity to plus infinity, fu and cos and then we have introduced here this alpha instead of n pi over l. This is our alpha there, so alpha is n pi over l and then u minus x du.

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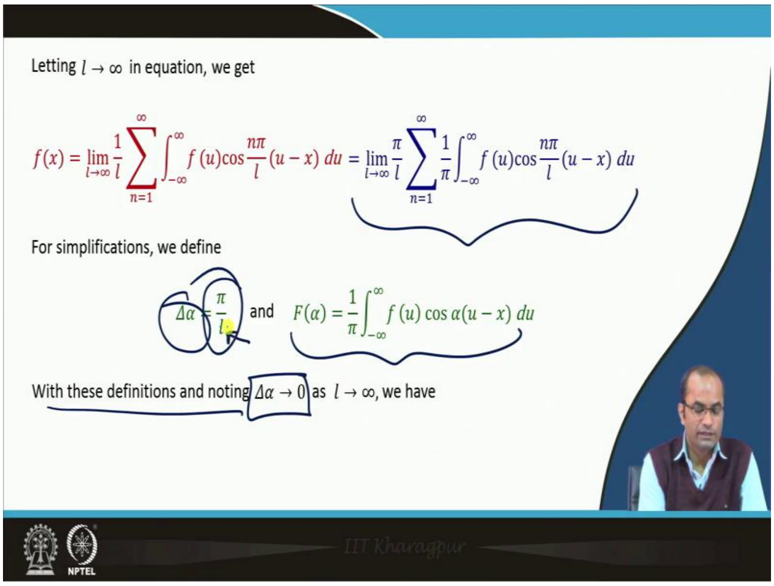

So, with this notation, we can now write down this integral, this Fourier series indeed in this form and then we also notice that when Delta l approaches to 0, that means this l approaches to infinity.

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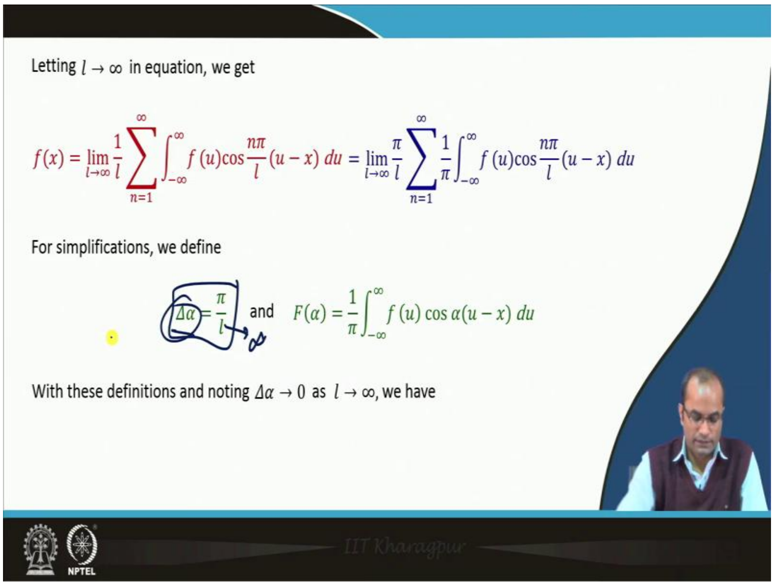

So, as l approaches to infinity, this Delta alpha will go to 0 with this relation. If this goes to infinity, then naturally Delta alpha will go to 0.

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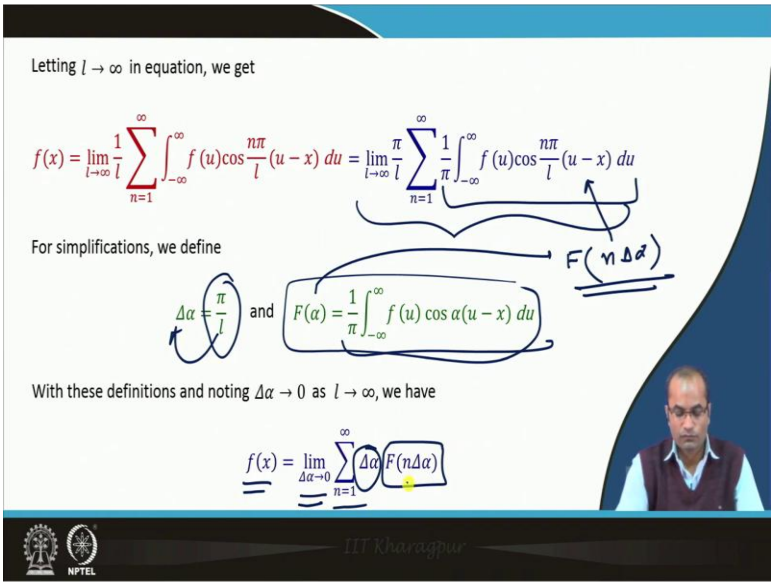

So, now this integral we can rewrite, so we have the f x and we have the limit instead of saying this l goes to infinity because l we will not see now in this tern here in this Fourier series, but we have changed this l to kind of this delta alpha. So, this delta alpha will now go to 0 instead of saying l goes to infinity and then we have this sum n **1** to infinity.

The term here with this integral **1** over pi and so on, we have this f, indeed f and n delta alpha. F n delta alpha is precisely this integral, so f n delta alpha and then this pi over l we have delta alpha. And then this n goes from **1** to infinity.

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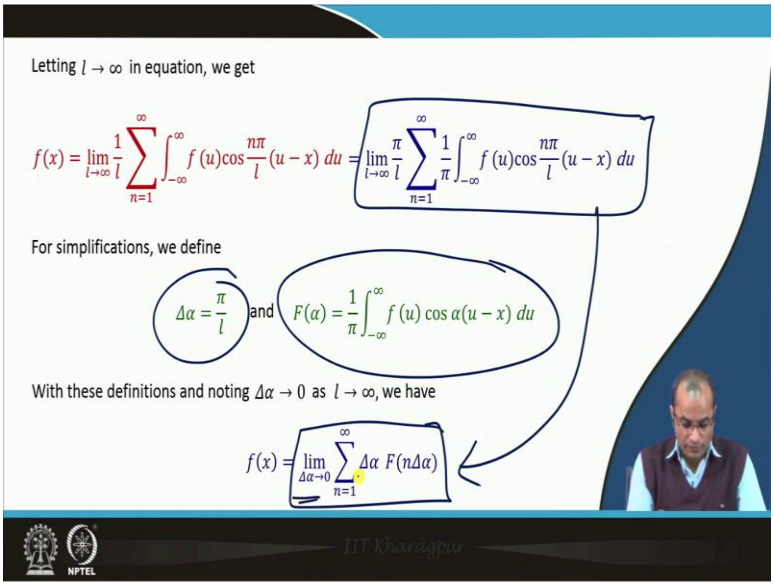

So, finally this integral here which was coming exactly from the Fourier series, letting this l to infinity, we have written in a more compact form introducing these two notations delta l and defining this function f alpha.

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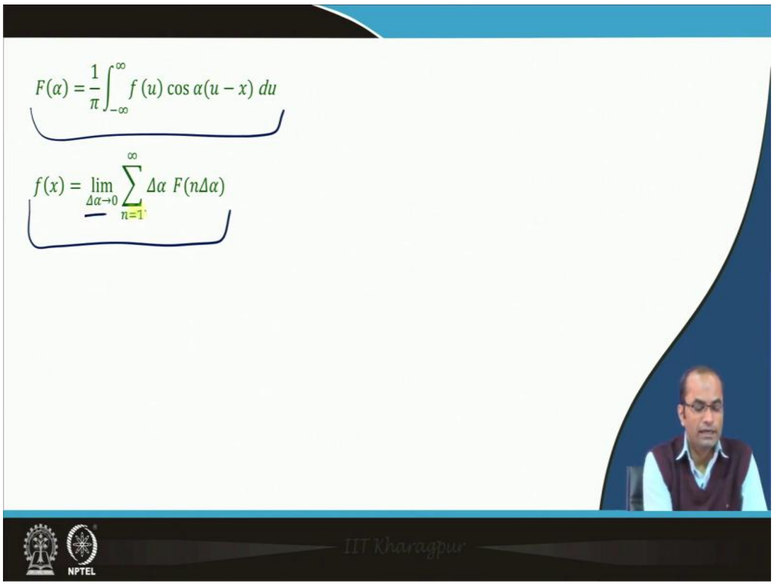

So, now we will move to this, so what we have? We have this f alpha as per the definition and the f x can be written now in terms of delta alpha and this function f which we have just defined

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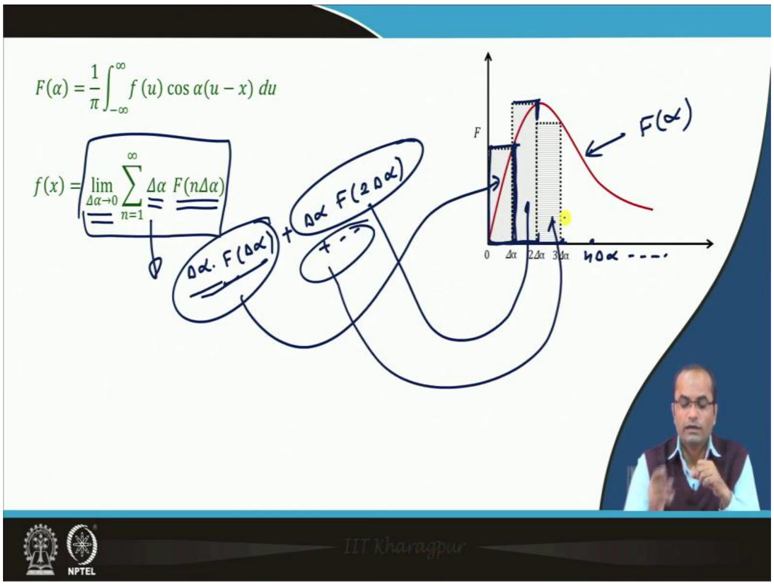

And now if we take a close look at this term here right hand side of this f x where we have the limit delta alpha goes to 0 and again delta alpha is sitting and then we have f, the function f evaluated or f n delta alpha. So, let us take a look at this picture, suppose that this is the function f alpha and now what is happening here.

So, if we have, if we divide this domain in to these small sections, so from here to here we have the delta alpha and then from here to here we have two times delta alpha, then three times, then four times delta alpha and so on and then take a look that what is happening in this summation, so we have for instance the first term is delta alpha into f delta alpha. Then we have again delta alpha in to f **2** times delta alpha and so on, this will continue.

So, what is this delta alpha? F delta alpha. So, the delta alpha is this distance and f delta alpha will be this height here, so this product the first product is nothing but the area of this rectangle here. So, we have this rectangle and that is the area defined by this product.

Similarly, the next term six again delta alpha but f is evaluated at **2** alpha that means this here and this product will give the area of the second rectangle and so on, the third term will be the area of the third rectangle, etc.

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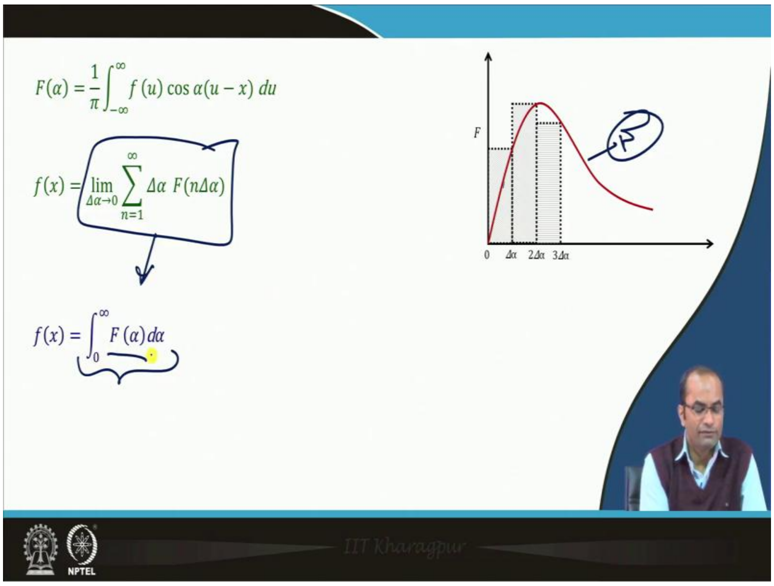

So, in the limiting situation when this delta alpha goes to 0, this area of these rectangles will converge to the area under this curve and which we know from the definition of the integrals that we can write down then in terms of the integral of this function f alpha d alpha. So, exactly as per the definition of the integral this limit, when it exist of course this will be this integral 0 to infinity f alpha and d alpha

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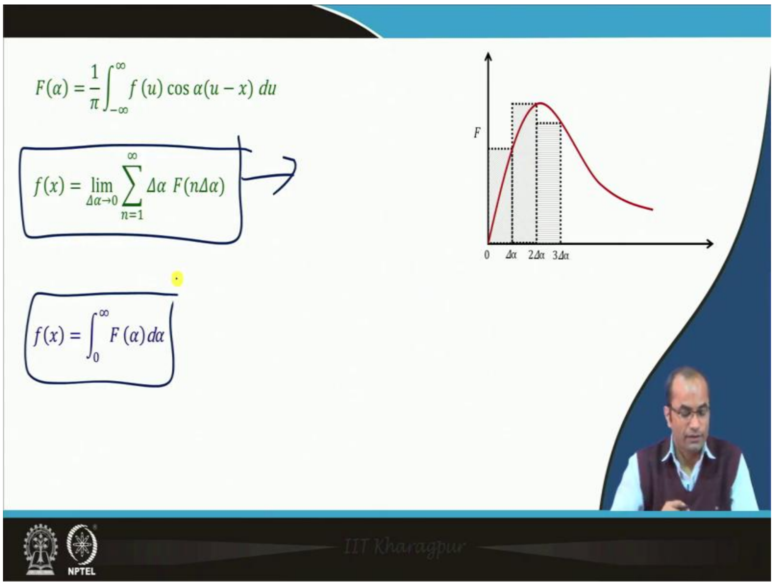

So, this is exactly the point where we have this Fourier series and but letting this l to infinity, we have another representation which is in terms of the integral now that 0 to infinity, f alpha, d alpha.

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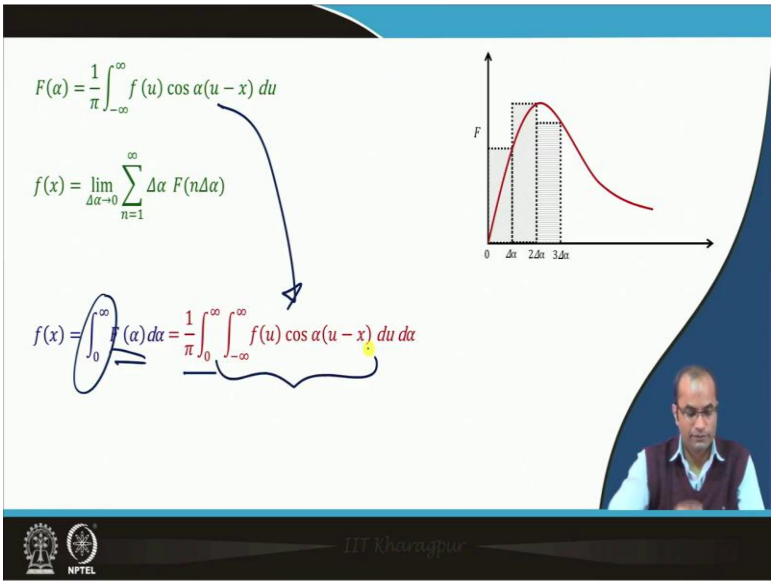

So, now again we can substitute this f alpha back to this integral so we have **1** over pi, 0 to infinity, the first integral and then for f alpha we have this integral.

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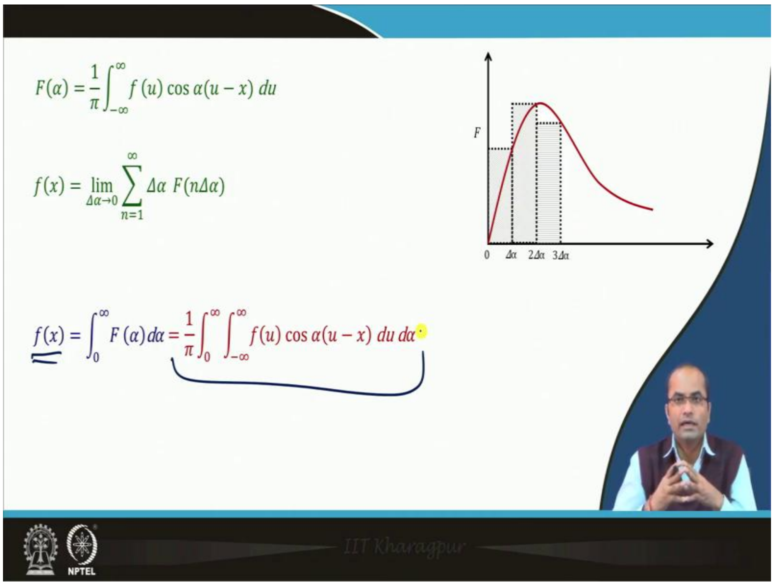

So, this is the representation of f x in terms of the integral. There are no more summations there or the discrete values of these coefficients which were denoted by a's and b's.

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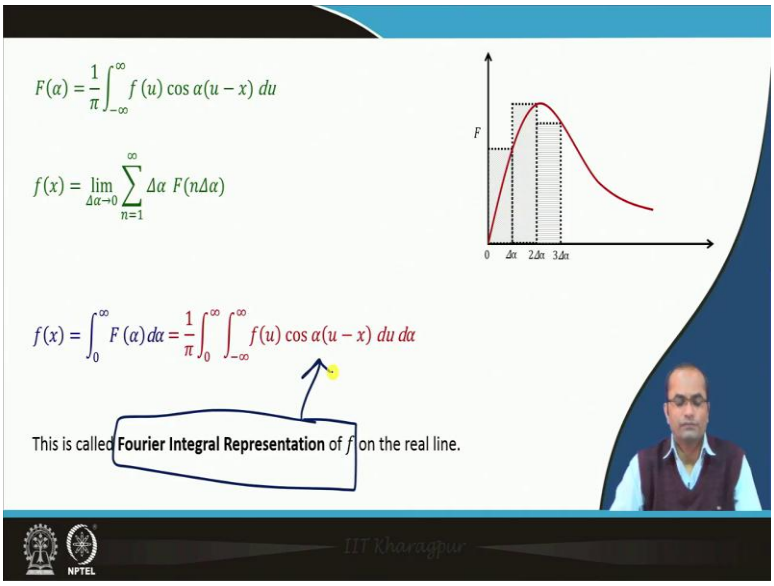

But we have this integral now and this is called the Fourier Integral Representation of f which is the transition from the Fourier series to this continuous setting now, where we have an integral representation instead of that discrete representation given in terms of the series.

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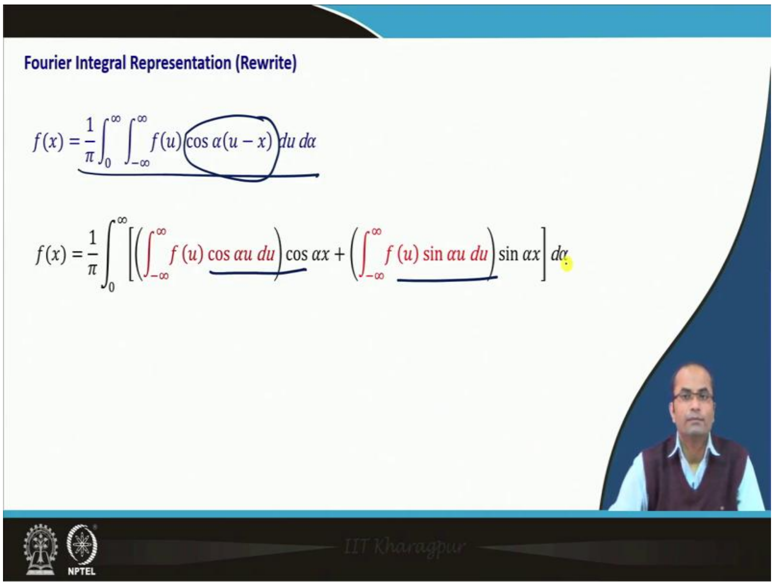

So, now we can rewrite this integral representation in a slightly better form, so this is the integral representation now and again we can use this identity cos a minus cos b to have again in a expanded form so we have the two terms then so two integrals, so the cos and cos and then sin and sin for this cos a minus b.

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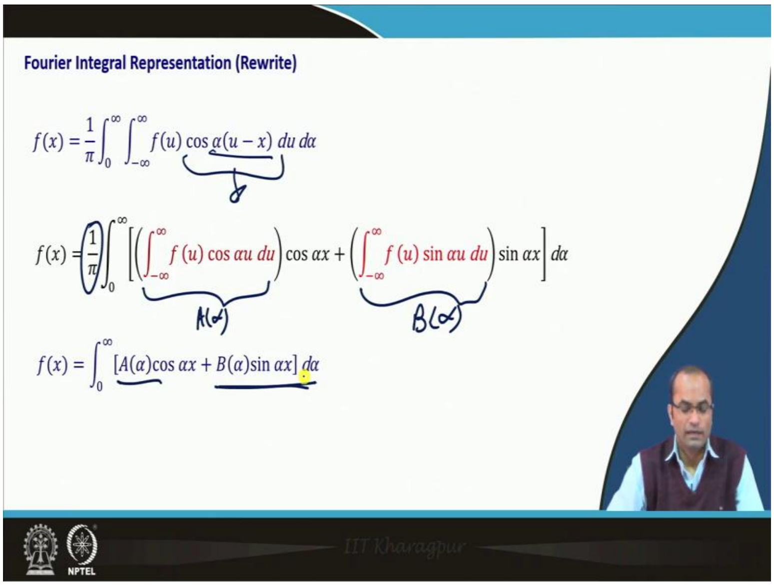

So, we have this integral now instead of this written in this expanded form for this cos a u minus cos, cos a u minus alpha x, alpha u minus alpha x. We have written now in terms of the cos and sin and then we have the integral over this u, so this integral if you note and we will define these terms with the another function then we can have a slightly compact form. So, if we say that this is a alpha with **1** over pi of course and if we call this b alpha, then we have this integral representation 0 to infinity a alpha cos alpha x and then b alpha sin alpha x d alpha which is already there.

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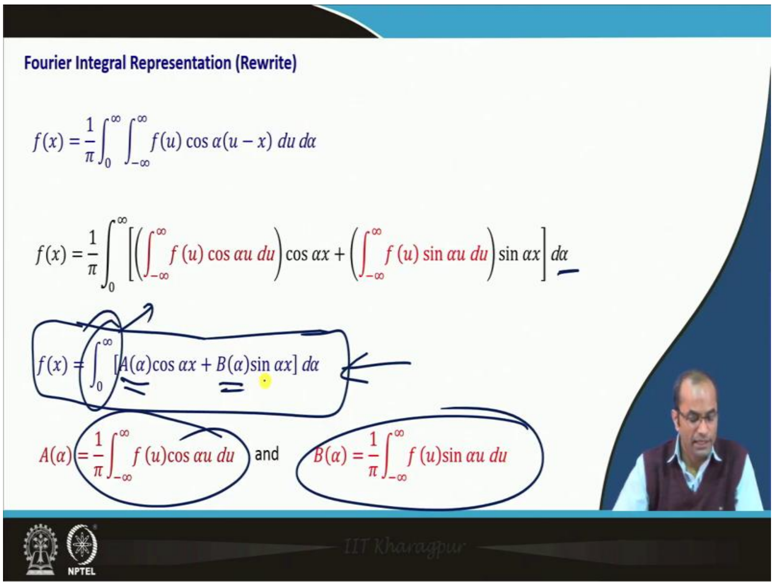

So, having this integral representation where this a alpha is given by this integral, b alpha will be given by this integral and then we have this f x which is given by this integral here. So, if we note, if we take a closer look at this integral representation, then we will realise that this is similar to what we have for the Fourier series and in the Fourier series we have something here in the form of the summation and there were Fourier coefficients a n and b n.

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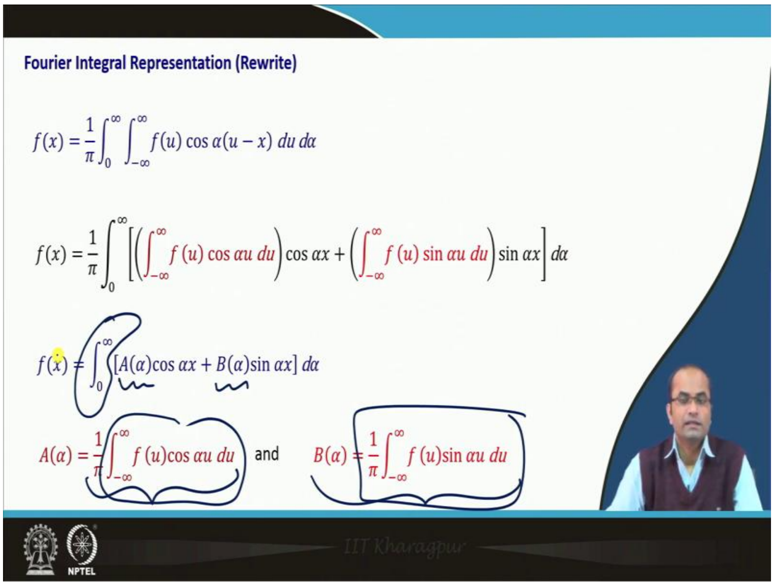

In this case we have instead of those coefficients again we have a and b. These coefficients which are defined again in terms of integral whereas earlier also it was in terms of these a n and b n were defined terms of the integral, but this summation here for the Fourier series is replaced by this integral.

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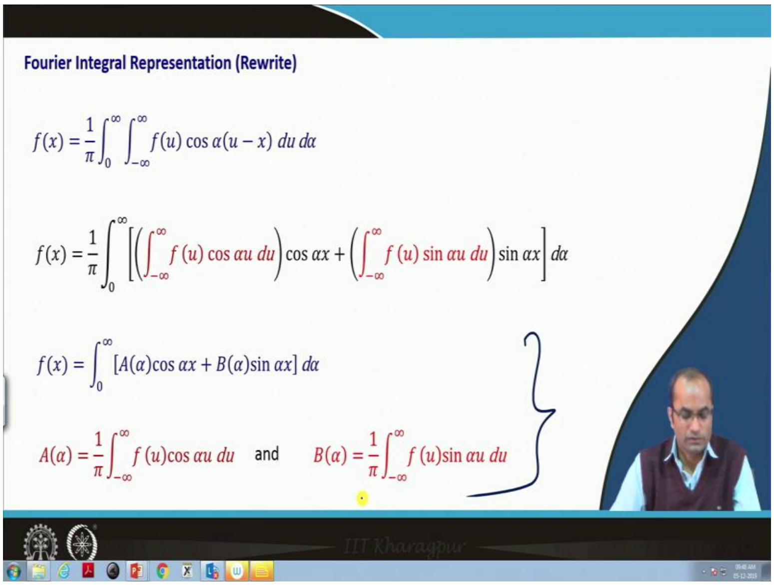

So, it is analogous to what we have the representation in terms of the Fourier series but now, we have integral representation when we let this period of this l, the period of the function which was denoted by l and if we let that l to infinity then we are ending up with such a representation which is called integral representation of the function.

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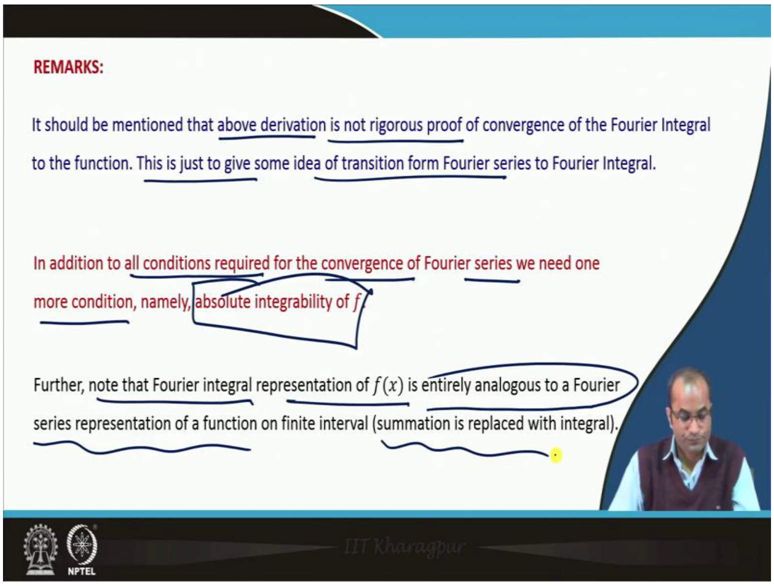

Well, so we have some remarks here, so it should be mentioned that above derivation is not the rigorous proof of the convergence of the Fourier integral. We have just gone through assuming smoothness on the function because we have written f x equal to the Fourier series and then we have done some kind of convergence results which was not very rigorous proof in terms of the mathematics but it was just to give the idea that how this transition is taking place from the Fourier series to Fourier transform.

So, this is just to give you some idea about this transition and in addition, all conditions required for the convergence of the Fourier series, we have also included their absolute integrability of f because with this absolute integrability of f, you were able to set the first term to 0 when l approaches to infinity.

And not that this Fourier representation of f s is entirely analogous to the Fourier series representation of the function which was defined in finite interval and now we are talking about this infinite interval and there the summation is mainly is replaced by the integrals otherwise the structure of the integral representation is just analogous to what we have for the Fourier series representation.

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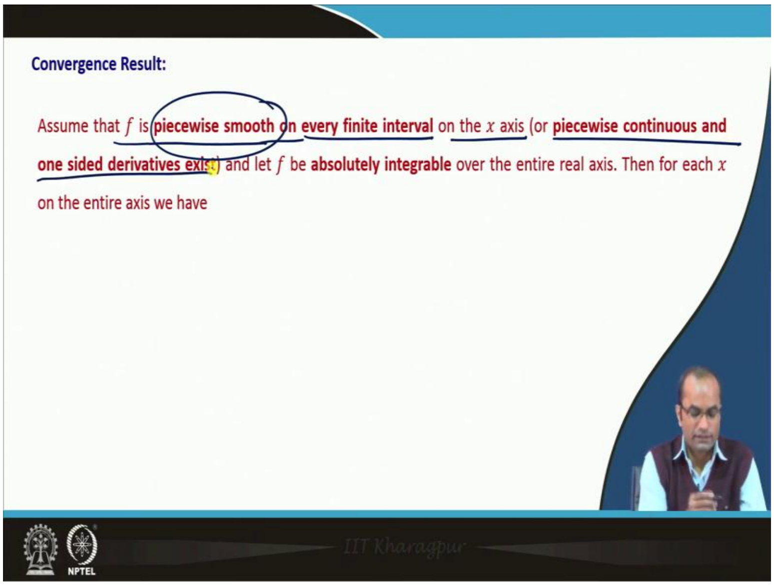

Well, just to summarise this result again in terms of this convergence theorem, so let this F be piecewise smooth, so that is sufficient now, so we are talking about piecewise smooth again on every interval on the x axis or we can replace by piecewise continuous and one sided derivative. So, similar conditions what we have discussed for the Fourier series.

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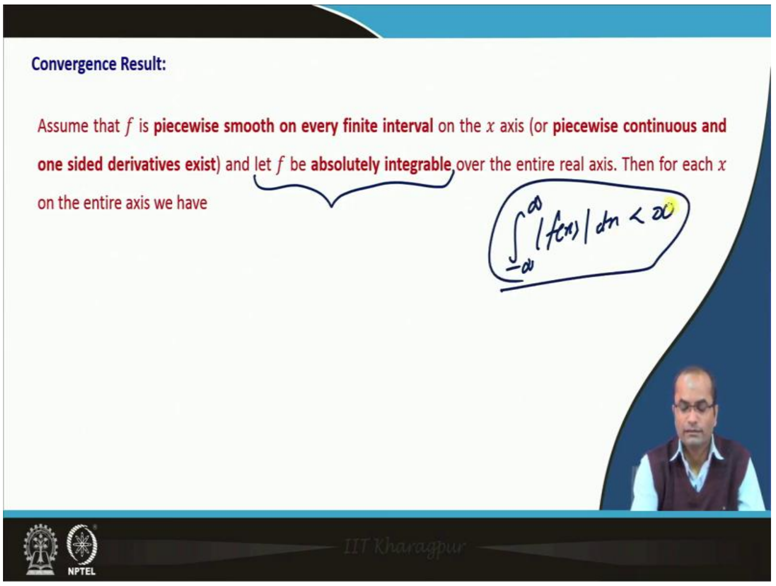

And on the top, we have an additional condition here that f be absolutely integrable that means the integral here is minus infinity to plus infinity of this f x d x that is finite that converges

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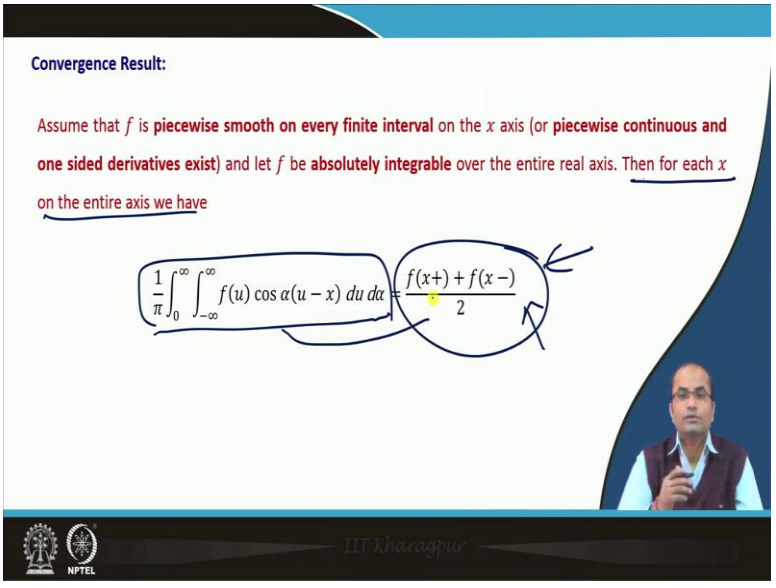

So, with that condition, we have then for each x on for the entire axis, we have the following result that this integral representation of the function f will be equal to again the average value so because that is the same as the Fourier series that was the starting point here in our derivation, we have taken this f x but in the more general setting we could have taken this average value and finally this integral will converge to this average value.

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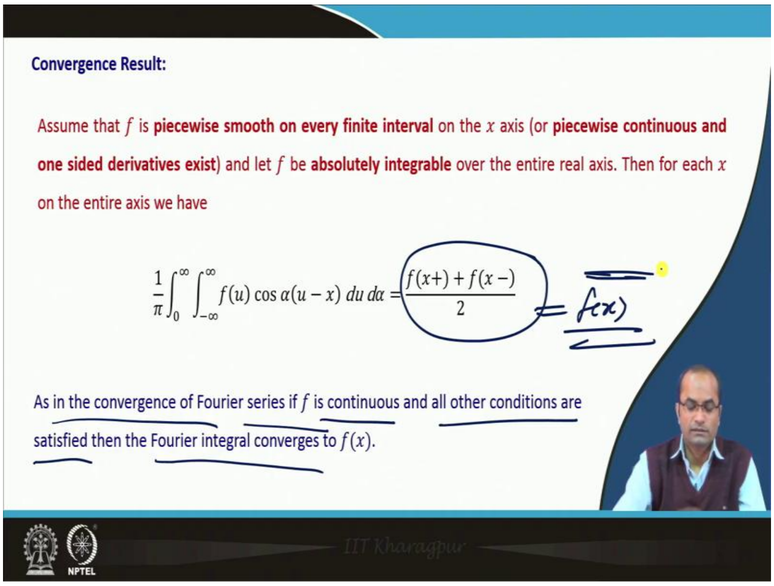

So, in the convergence of the Fourier series if f is continuous and all other conditions are met, then the Fourier integral converges to f x, so obviously if f is continuous at some point x there, then this will be equal to this f x for that point x. So, that is again similar to what we have discussed for the Fourier series.

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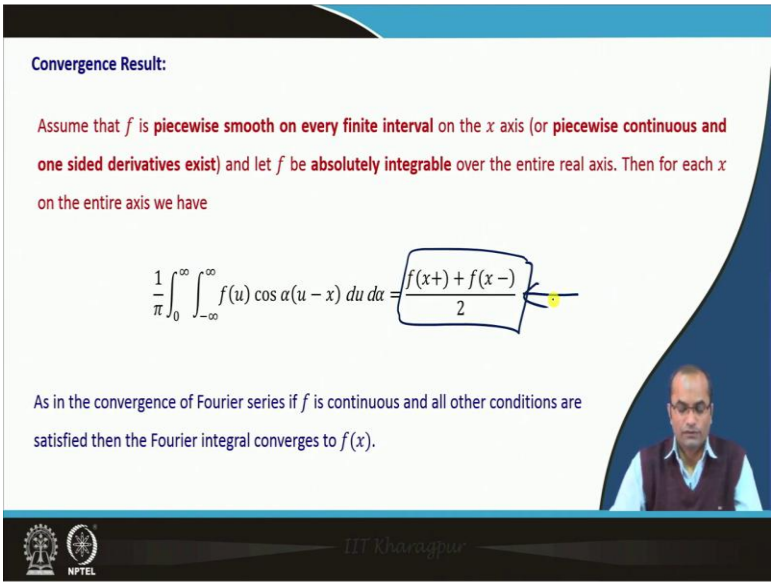

So, if the function is not continuous, then we will take this average value, if the function is continuous, this will become the function value itself.

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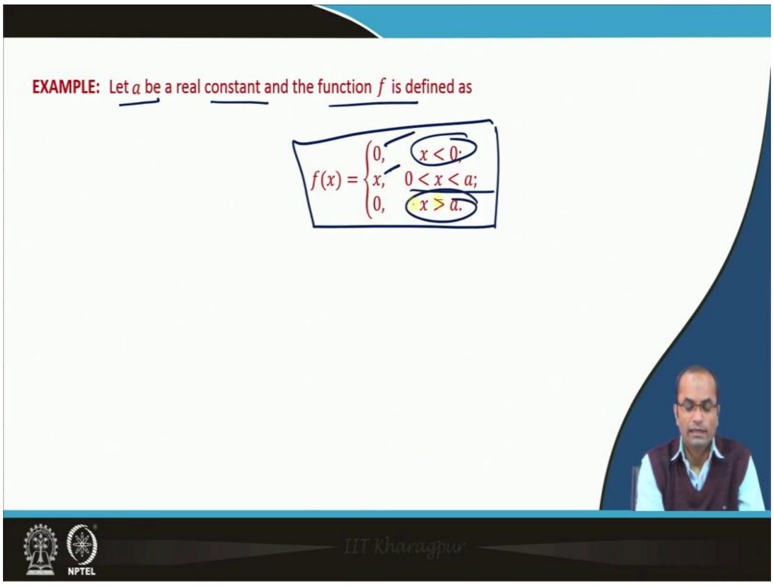

Okay, lest go through some examples now. So, we let a be a real number, a real constant and the function f is defined by this function here, where x is less than 0, the value is 0, 0 to a only, this value is x and then x greater than a again the value is 0.

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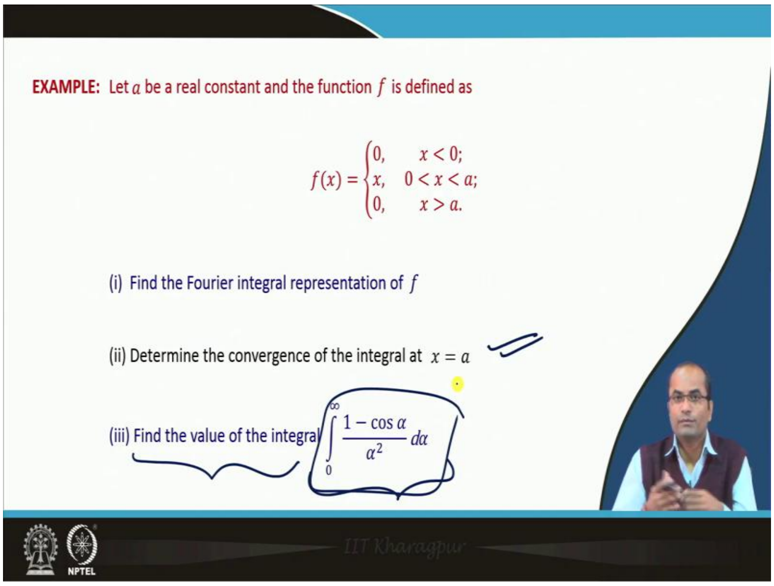

So, now we want to find the Fourier integral representation of f and we want to determine the convergence of this integral representation at x is equal to a and finally we want to find the value of this integral. So, like earlier in case of the Fourier series, we have noticed that we could get the sum of the series using that convergence result of the Fourier series and now with the convergence result of this Fourier Integral representation, we will be able to find some integrals which may be very difficult to evaluate without using this convergence result.

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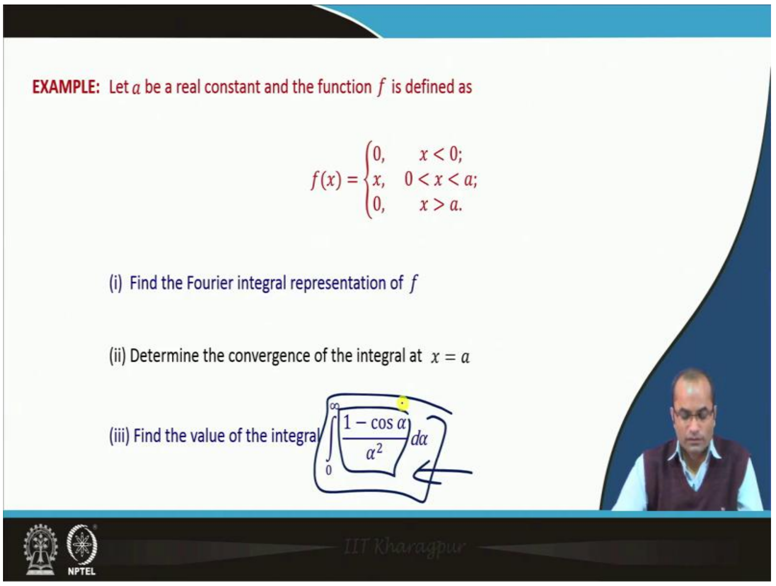

So, here also we will see that how to get the value of this integral 1 minus cos alpha over alpha square and as you notice here that this integral is not a trivial integral to get the value but using this Fourier integral representation and using this convergence theorem, this can easily be obtained.

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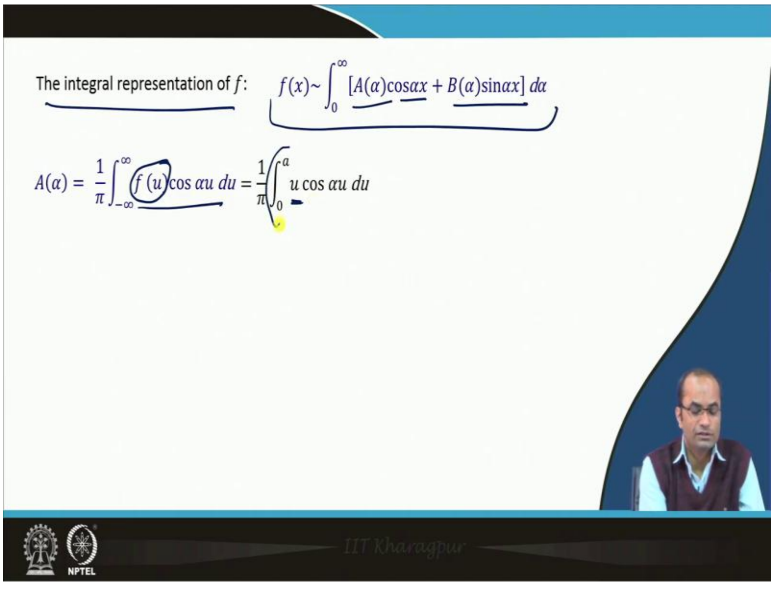

So, the integral representation of this f will be given by this term here, a alpha cos alpha x and b alpha sin alpha x where a alpha is given by this integral and this we can evaluate now, so the fu is u in the range 0 to a.

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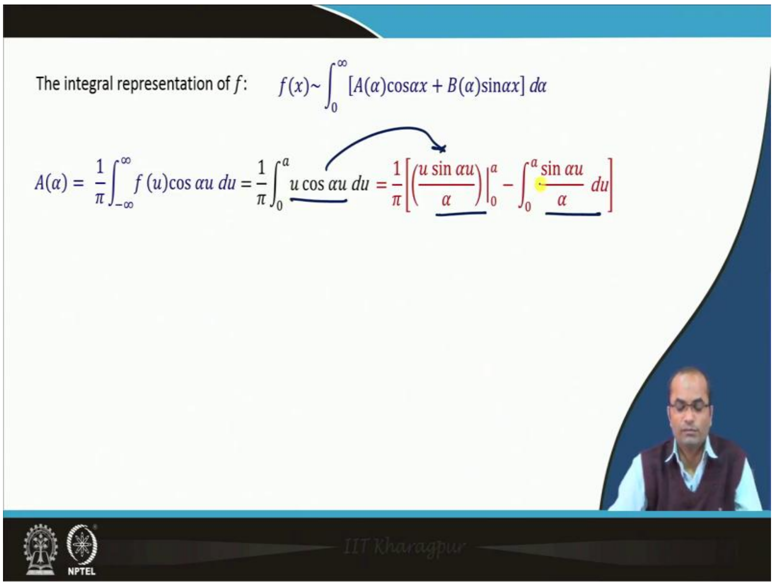

So, this integral deduces to 0 to a with the u cos alpha u and that can now be integrated easily, so we have u and then the integral of this cos is sin alpha u over alpha and again here sin alpha, u over alpha and this u the derivative will be **1**.

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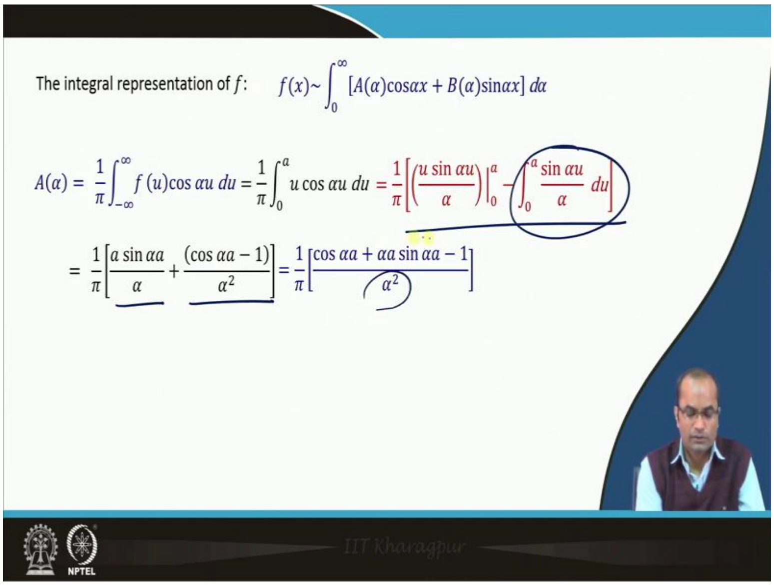

So, integration by parts can easily be done here and then substituting these limits and again doing this integral there, we ended up with a sin alpha a and then here we will have this cos alpha a minus **1** and over this alpha square. So, then we have here **1** over pi, coming from here this can be just merged to have this alpha square there

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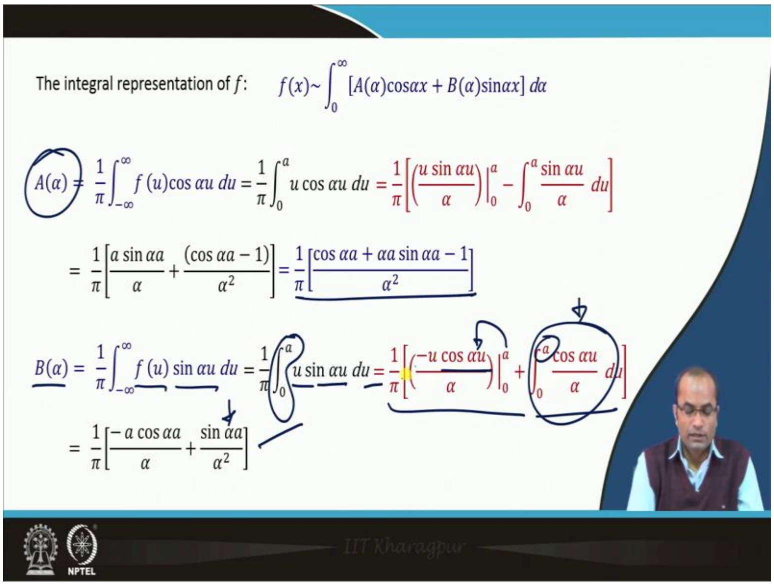

So, this is the term for this a alpha, this coefficient and then similarly we can compute this b alpha with this formula f u sin alpha u where this f u is again u in this range 0 to a and we have sin alpha u and then du. So, this can be again integrated with the help of the idea of this integration by parts and then here we have this cos alpha u this time, so when we put u, so we have cos alpha a and then cos 0 will be **1** so we have finally this will be sin and then in both the cases that will be computed at 0 that will be 0, but at a we will have here also sin alpha a which is coming already there.

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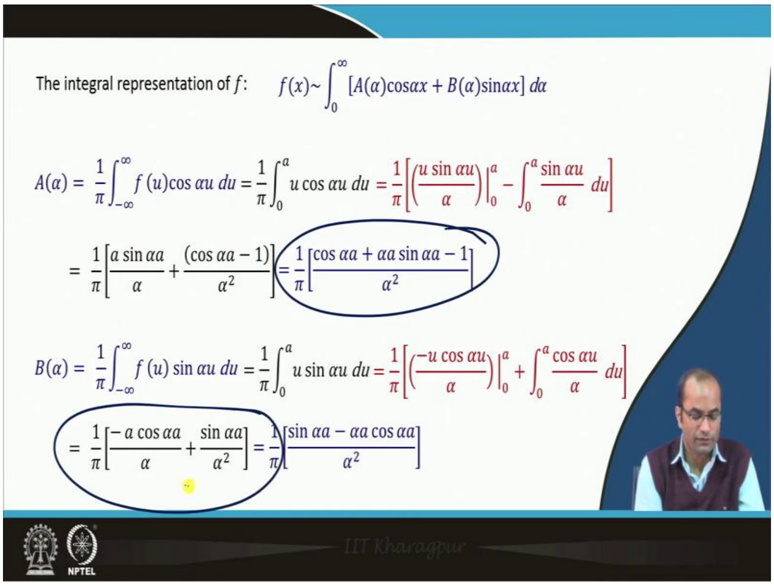

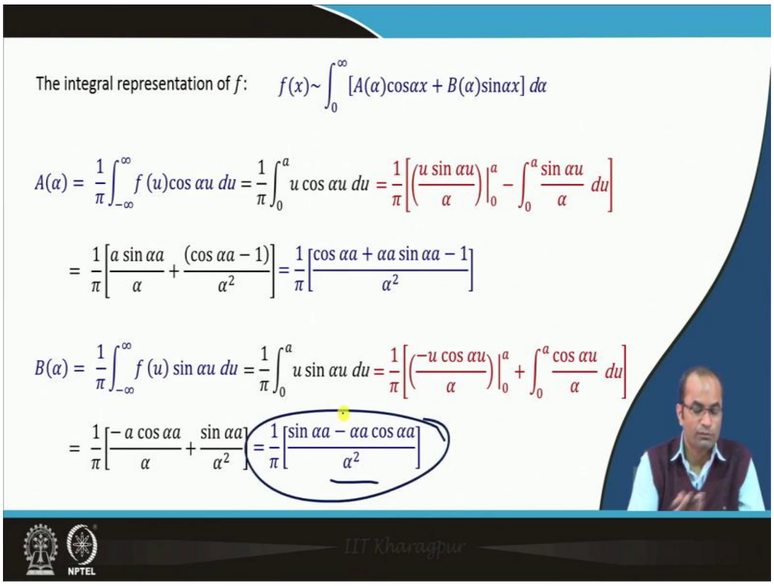

So, we have the expression for a alpha and then we also have for b alpha which can be further simplified to have this alpha square, so both the coefficients are computed.

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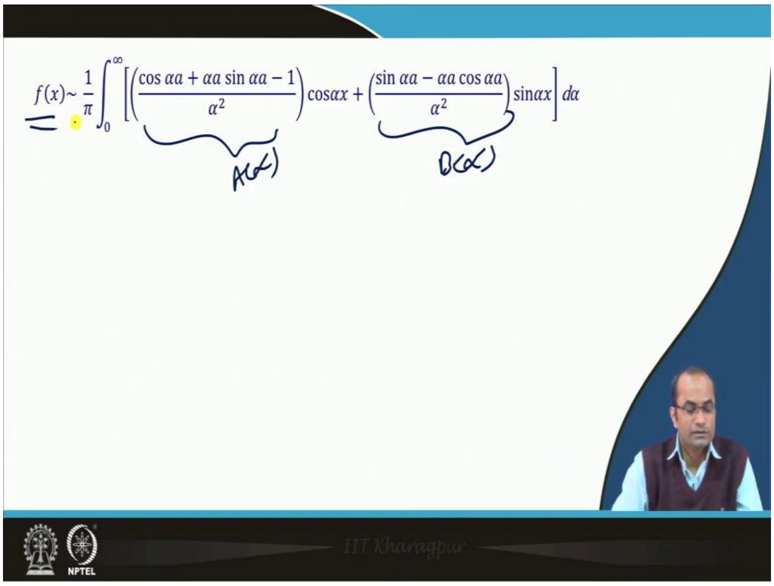

And then we can substitute this a alpha and this is b alpha and this is a alpha in this f x in this integral representation.

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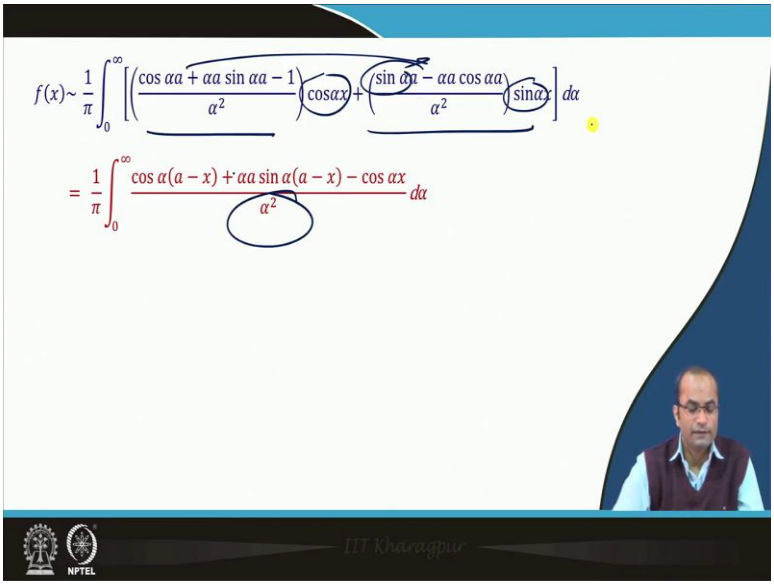

So, that can now be simplified, so this alpha square and then we can just write down this with the help of the trigonometric identity because here again we have cos a cos b, then sin a sin b, and here sin a this cos b and cos a sin b, etc.

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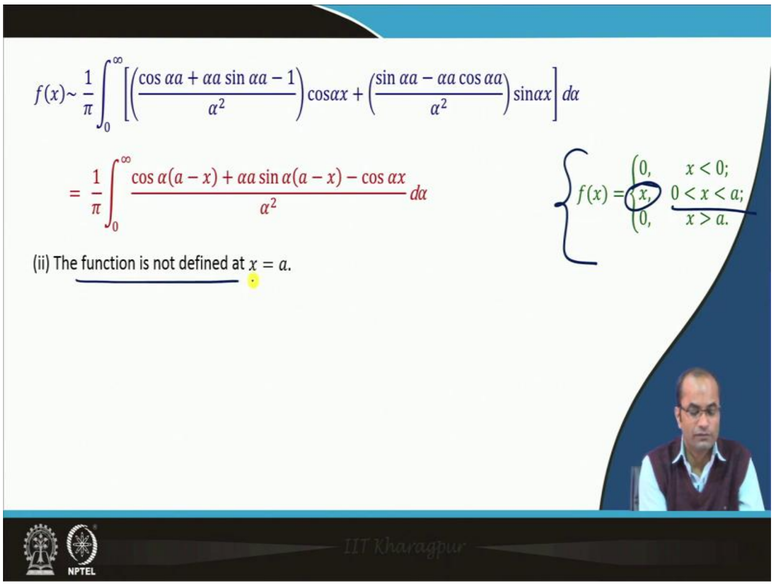

So, that can be simplified to give this expression for this f x and you just recall that f x was defined in this way so in 0 to 1, the value was x, otherwise it was 0, so we are precisely getting now the function is not defined here at x is equal to a

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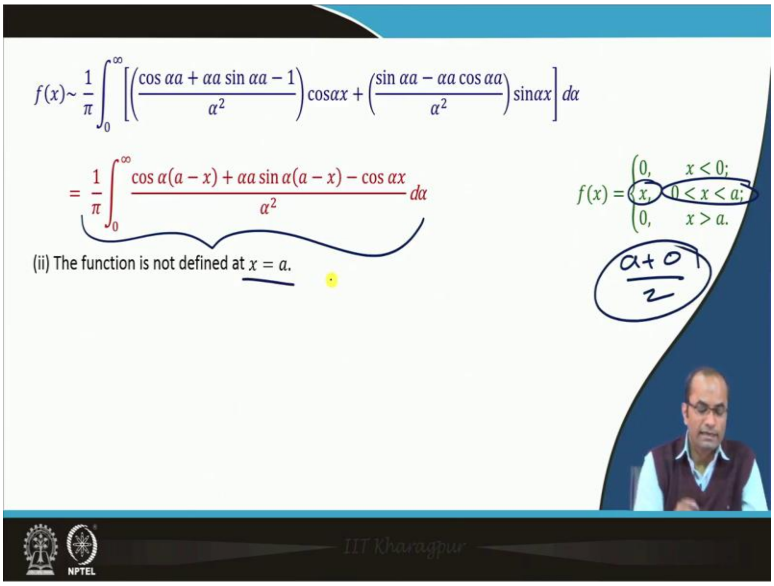

So, we should note that the function is not defined at x is equal to a, however when the question is about the convergence of this Fourier integral, then that will converge to the average value of the limits so at a we can take the average value and that can be like from the left side the limit is a and then the right side it is 0 by 2. So, this Fourier integral at x is equal to a will converge to this a by 2.

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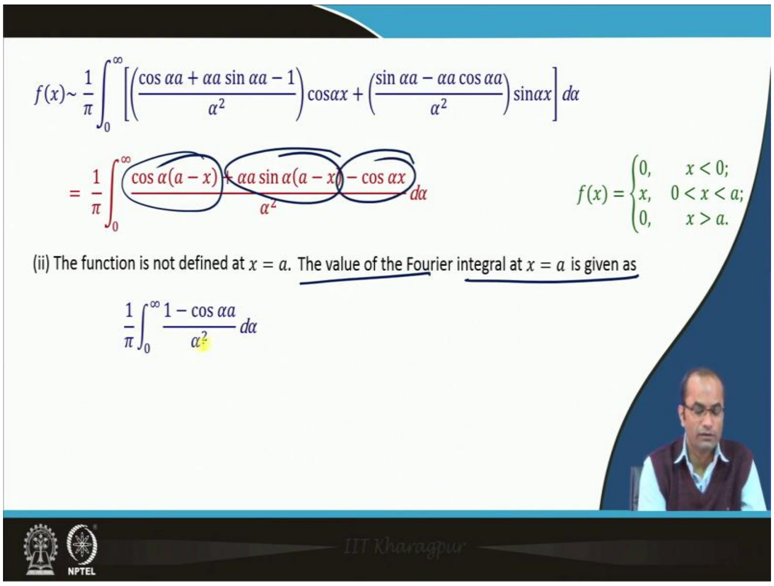

So, the value of this Fourier integral at x is equal to a is given as, so x is equal to a if we put there, this is going to be **1** there then this will vanish and then we have cos alpha over alpha square.

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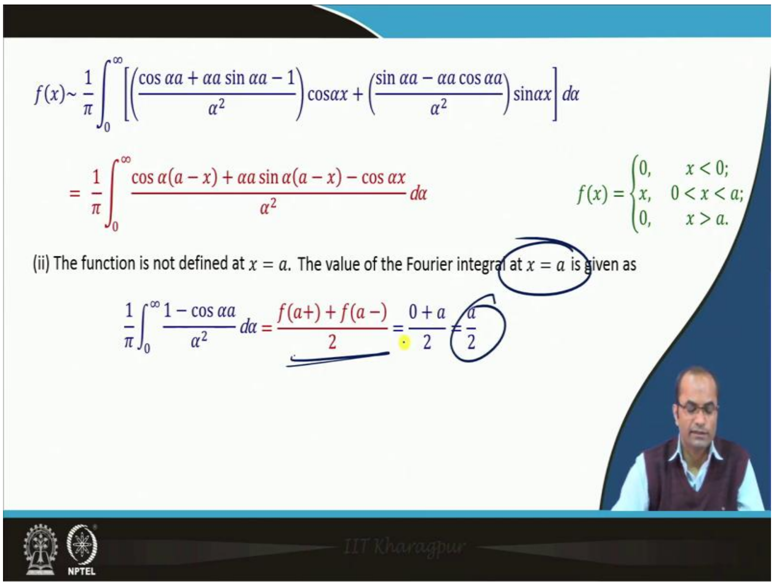

So, this is at x is equal to a, x is equal to a we have this Fourier integral and this will converge to this average value which is a by 2.

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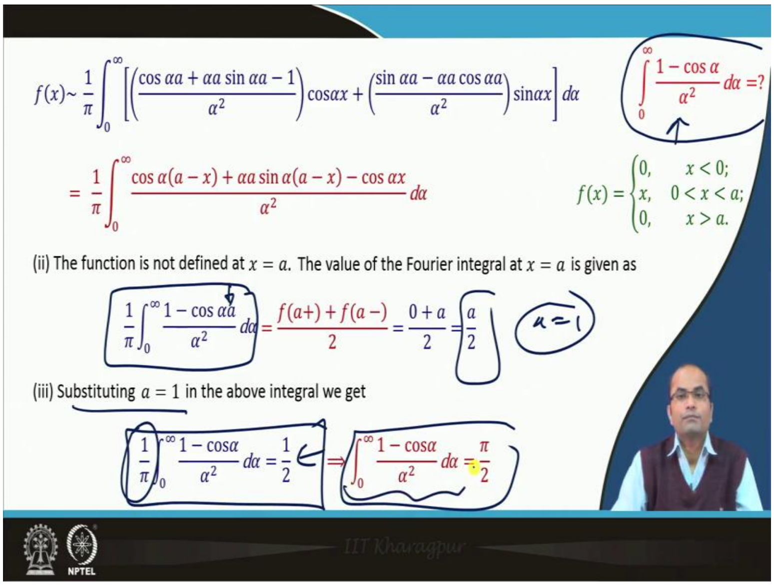

So we have here that the value of this integral is a by 2 for any a now. And we want to get the value of this integral which is 1 minus cos alpha over alpha square d alpha and then we can notice that if we put this a is equal to 1 there, we are exactly getting this integral which was asked in the question.

So, substituting this a equal to 1 what we get? We get this desired value of this integral and the value is just half. So, we got this pi by 2, so there was a 1 by pi factor there, so the value of this integral is pi by 2.

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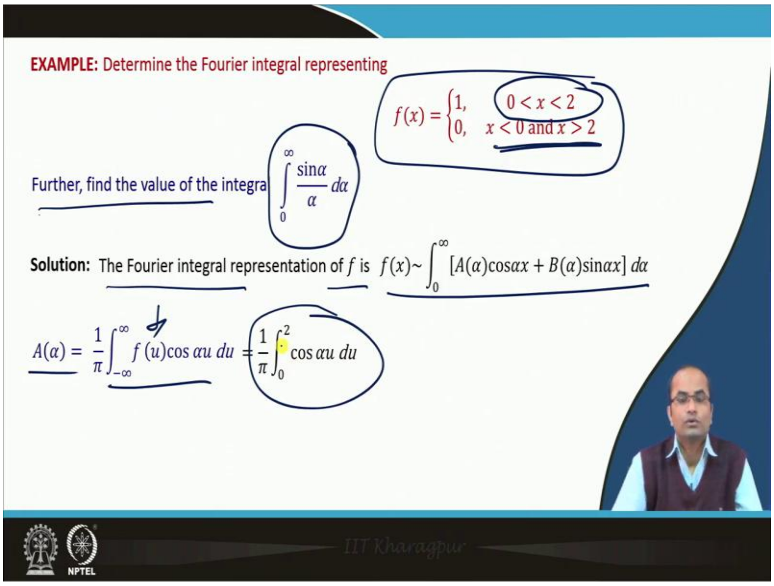

The another example where we will determine the Fourier Integral of this function which is defined in 0 to 2 as 1 and then in the rest when x is less than 0 or x is greater than 2, the value is 0. So, we want to, further we want to find the value of this integral again the convergence result will be used to get the value of this integral. The Fourier Integral Representation of f again will be given by this formula where the a alpha we can compute by substituting this fu which is defined as **1** only in the range 0 to 2.

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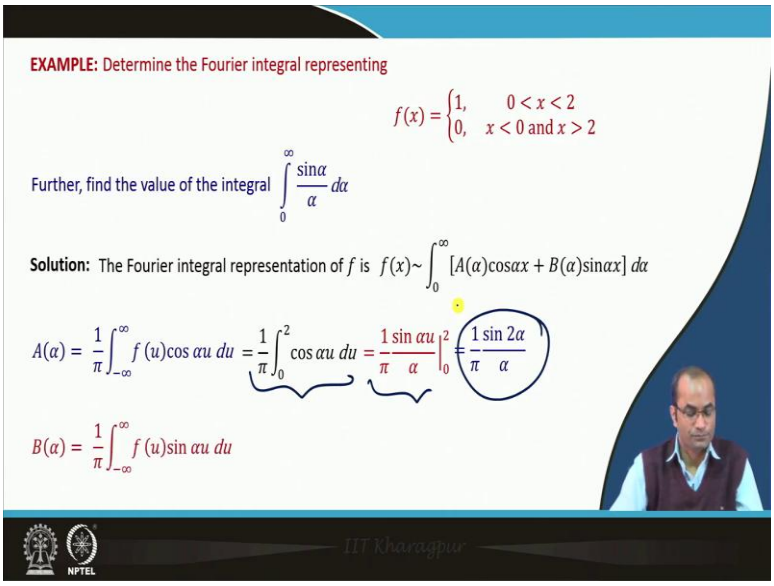

So, we can easily perform this integral and we will get the value 1 over pi sin 2 alpha over alpha.

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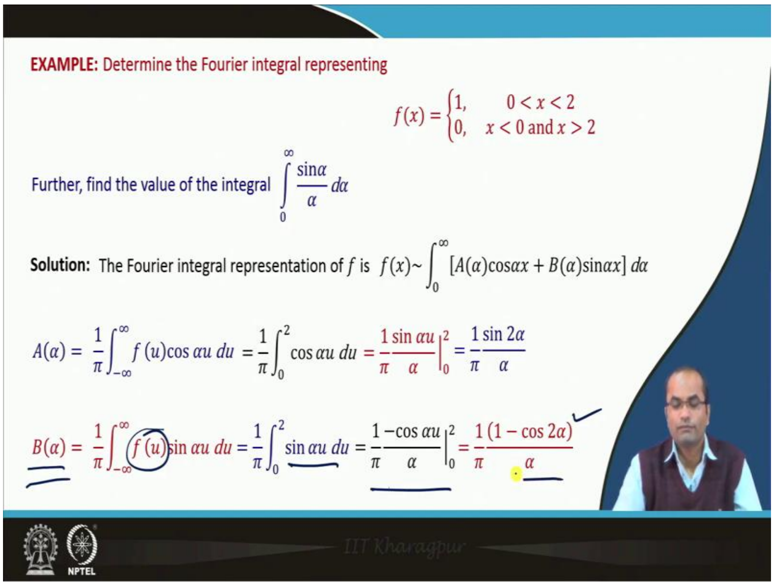

Second for b alpha again we can use this function to have sin, then again cos and then we can find this value as 1minus cos 2 alpha over alpha, 1 over pi.

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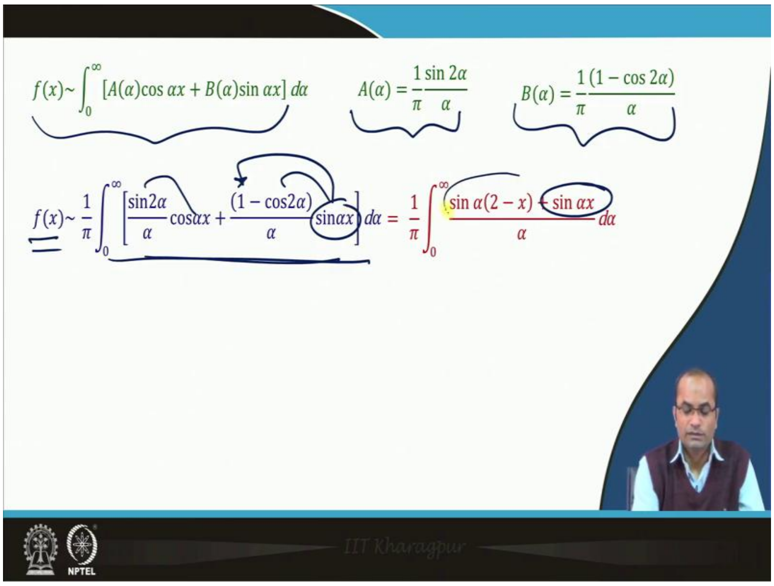

So, this was the Fourier Integral Representation, we have computed a alpha, we have computed b alpha that means this Fourier Integral representation by substituting this a alpha and b alpha into this Fourier Integral Representation we will get this value which again can be simplified, so the sin alpha term is a single term which will remain there and then we have a sin a cos b and then there is cos a sin b term which can be merged in to this sin a minus b type of identity.

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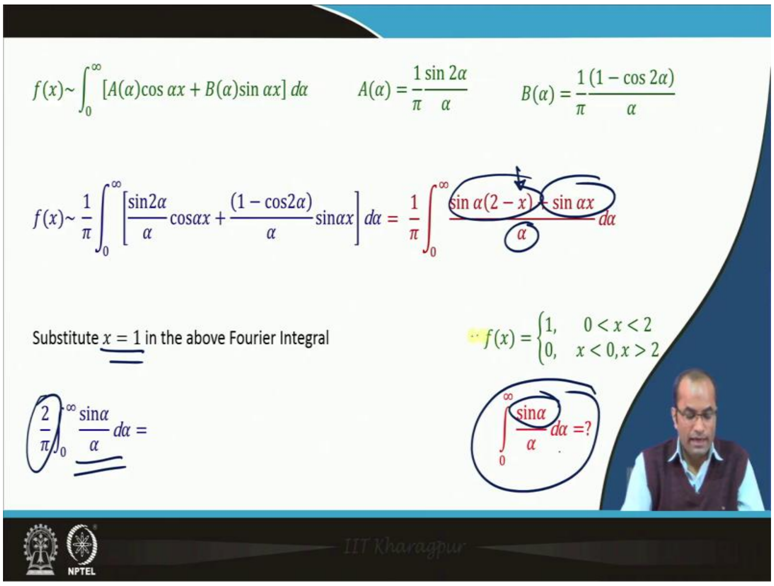

So, here we have now the compact form sin alpha 2 minus x sin alpha x over alpha d alpha that is the integral representation of the given function. And in the question this was asked that what is the value of sin alpha over alpha d alpha, so we have already here alpha and see if we put x is equal to 1 for instance, this will be sin alpha, this will be sin alpha, so we can get this desired integral just by substituting this x is equal to 1 in this above integral. So, if you put x is equal to, that will be **2** over pi and sin alpha over alpha d alpha.

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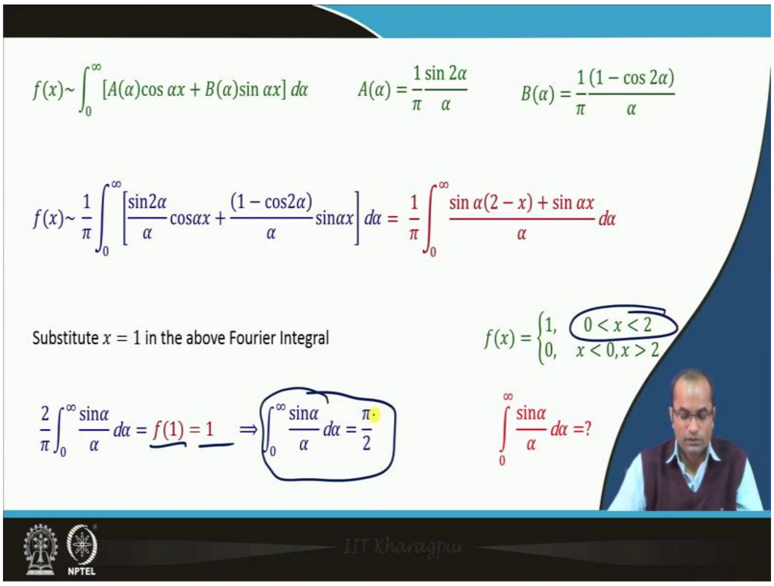

And now the question is that to what value this integral will converge because at x is equal to 1, the function is continuous there, so at x is equal to **1**, the function value is 1, so this integral will converge to the function value, the function is continuous there, so there will be no averaging done there, so we have f1 which is exactly 1 there, so this integral sin alpha over alpha d alpha, it will be pi by 2.

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Well, so there are the references we have used for preparing this lecture.

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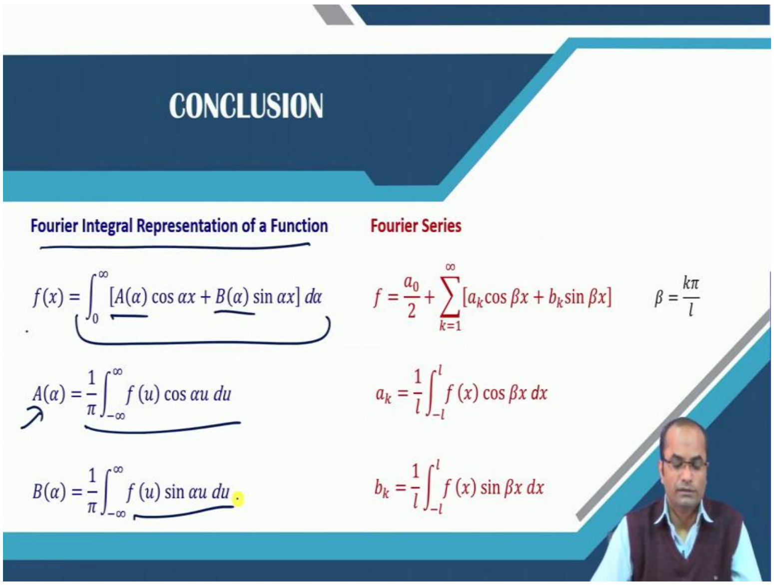

And just to conclude that this Fourier Integral Representation of a function is given by this integral where a alpha and b alpha are these coefficients computed using this integral and b alpha can be computed with this integral.

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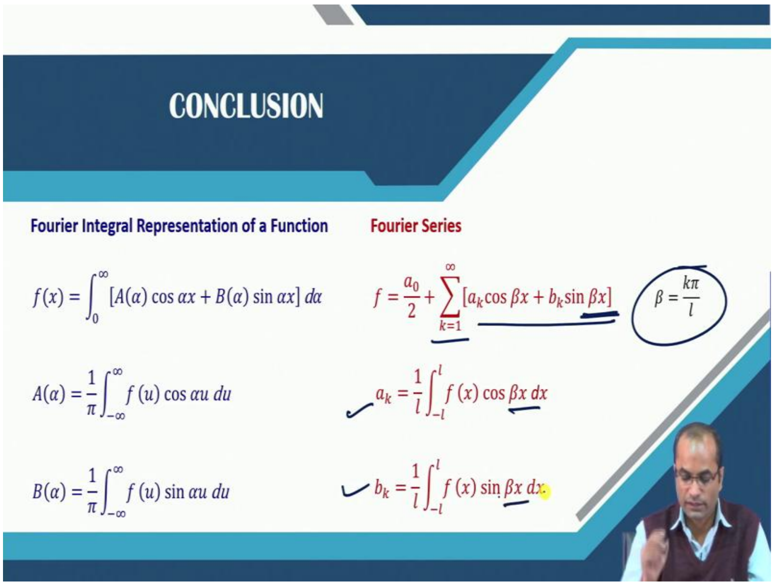

What is interesting to note down that if we look at the Fourier series which was defined as the sum from 1 to infinity with a k and this b k, the coefficients a k and b k were defined as **1** over l minus l2l and minus l2l, etc. Where this beta we have just replaced for k pi over l, otherwise it was cos k pi x over l sin k pi x over l and here also k pi x over l at this place also it was k pi x over l. So, we can just relate this.

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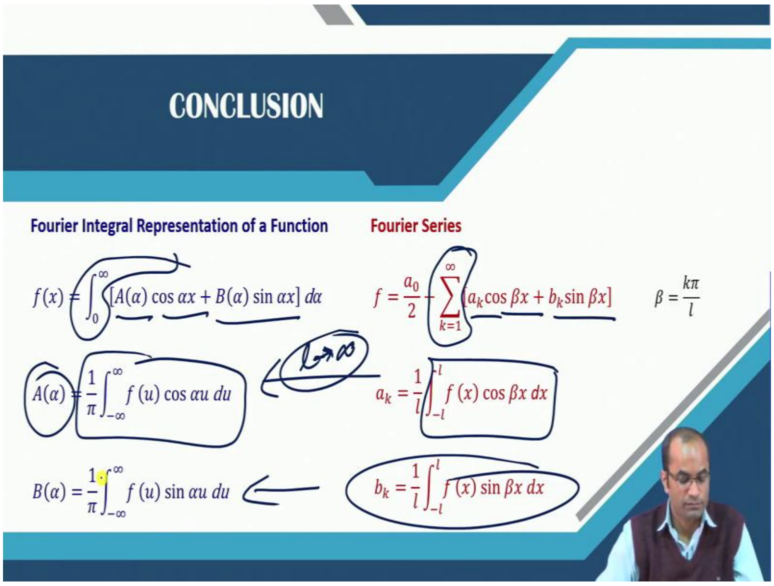

So, here we have this integral 0 to infinity, here we have this summation from 1 to infinity. We have a k, here also we have a then cos, then cos, here also b sin and then we have there also b sin and this a alpha is going to be computed with this minus infinity to plus infinity though here it was minus l to l and obviously in the process here we let this l2 infinity, I mean that was the whole idea of this transition and here also we have a similar result that these minus l to plus l is just replaced by the minus infinity to plus infinity.

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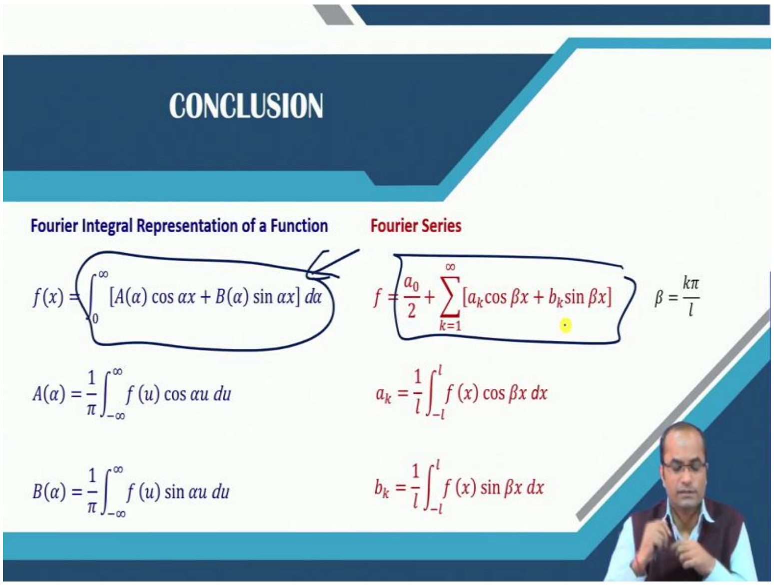

So, both the representations are analogous but remember that this Fourier series if it converge, it will converge to a periodic function whereas this integral will may not converge to a periodic function because if the given function is not periodic, we can write down its integral representation but for the Fourier series we should have a function defined in a finite interval and then we can write its Fourier series and that Fourier series will converge to a periodic function defined in each period to the function which was used for writing the Fourier series.

So, there is the difference between the two and this is the transition period where we will move to in next lectures about the what is the so called Fourier transform. Well so that is all for this lecture and I thank you for your attention.