Engineering Mathematics - II
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Lecture 40
Complex Form of Fourier Series

So, welcome back to lectures on Engineering Mathematics 2 and this is lecture number 40 on Complex Form of Fourier series.

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So, today in this lecture, we will discuss the, another form slightly different way of writing the fourier series and that is in the Complex Form. And, we will also look into what will be the form of this Parseval's identity for the Complex Form of Fourier Series.

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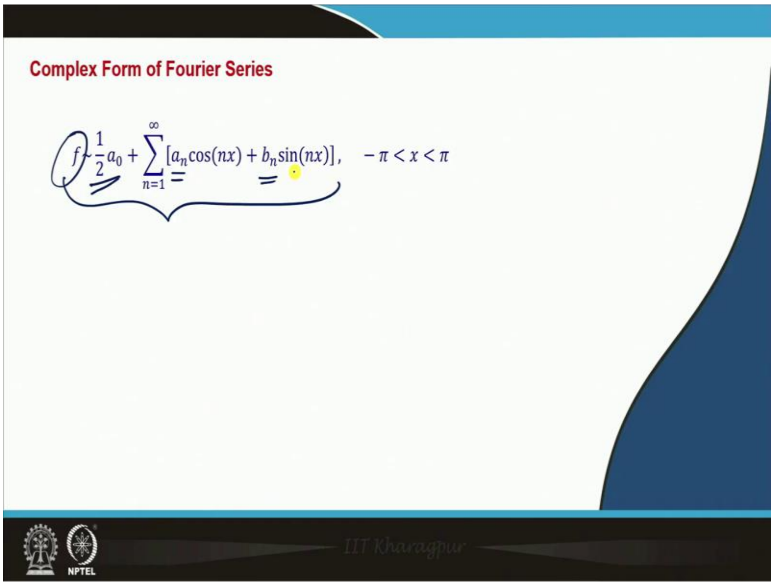

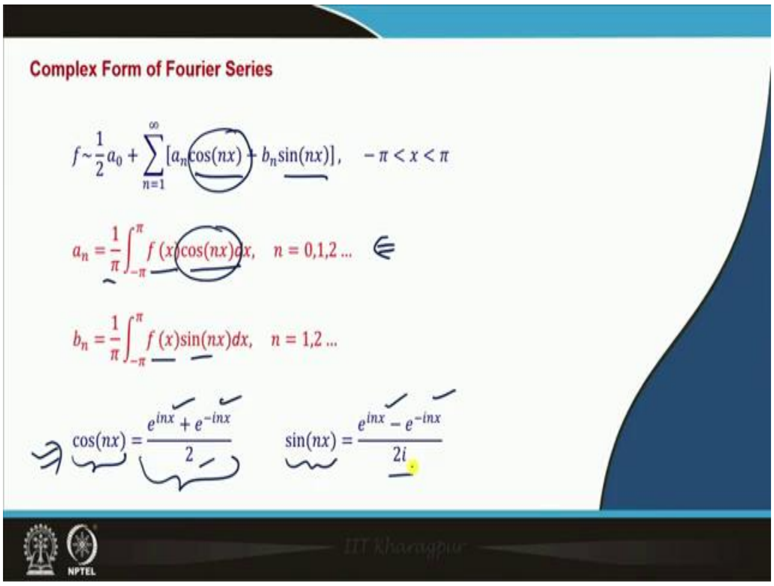

So, this was the series which we have already discussed. If f is an integrable or piecewise continuous function, then we can write its fourier series and these an’s and bn’s are called the fourier coefficients. So, these an, the expression for an we have discussed already is 1 over pi and then f x cos nx and for bn it was f xn sin nx dx for n 1, 2, 3 and so on. We have to use these Euler formula now, to convert into this so called complex form of the fourier series.

So, this cos nx, this appear for example, here in the fourier series also in the fourier coefficients, so this cos nx can be replaced with the help of this exponential formula e power inx and e power minus nx divided by 2. And similarly, for sin nx also, we have e power inx, e power minus inx and then divided by 2 times i.

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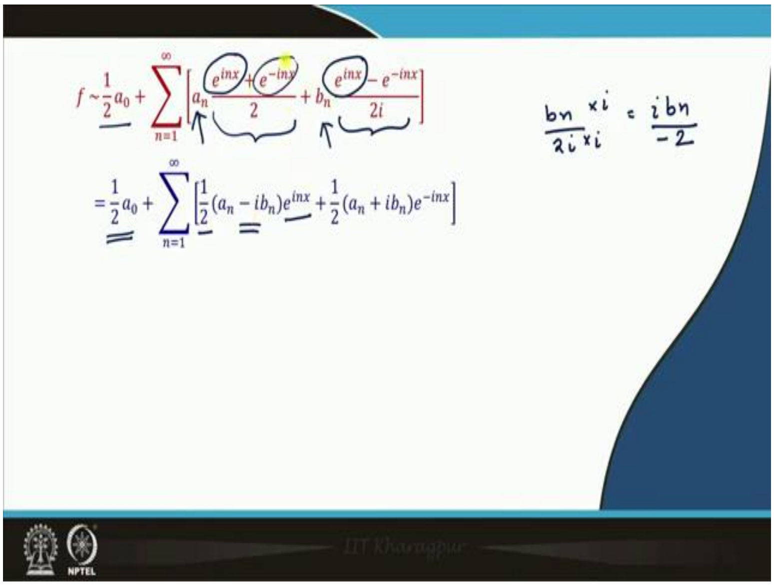

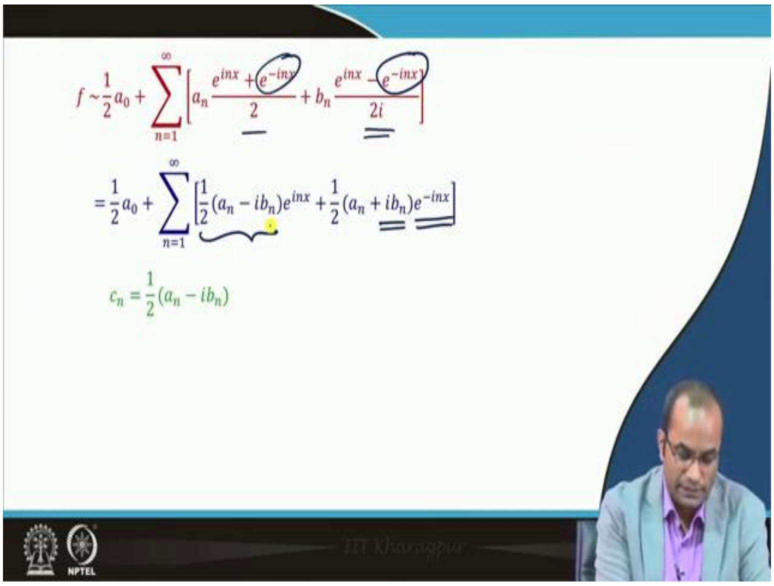

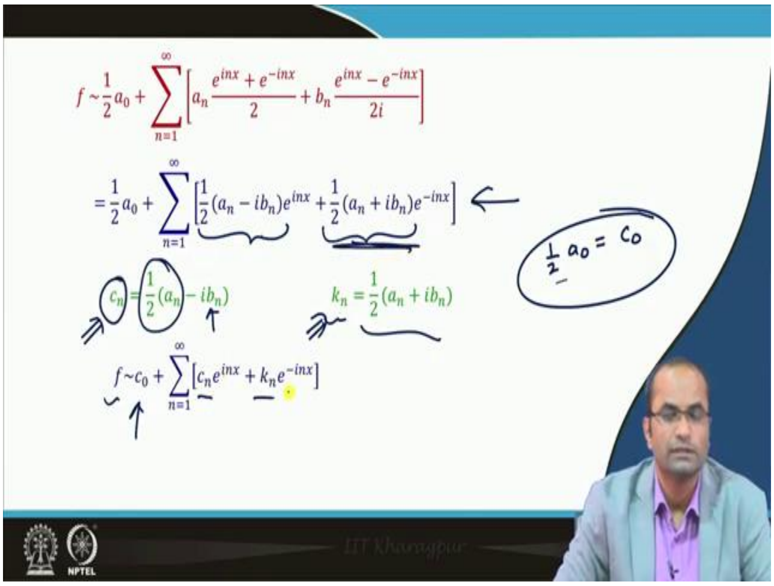

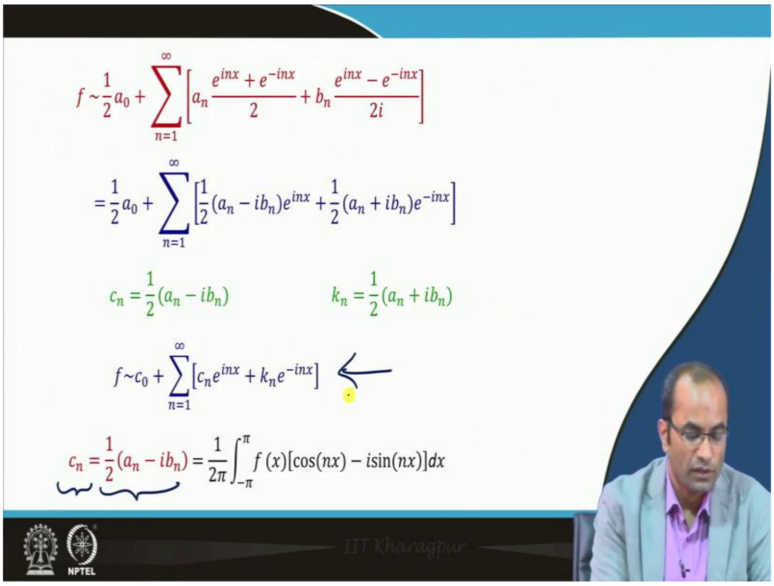

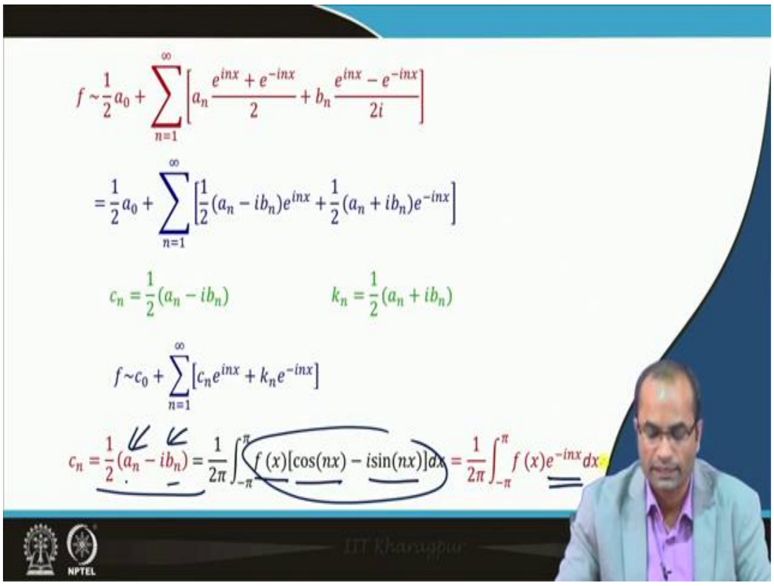

So, now we can replace in the fourier series. So, a by 2 and then this cos nx is replaced by this Euler formula, also this sin nx is replaced by e power inx minus e power inx divided by 2 i. So, we can now do some simplification because the e power inx is present in both the terms.

So, we can take this common e power inx. Also, the factor half will be common and then we have an from this term and when we multiply and divide by i, so this will become minus sin there but there will be i times bn because it was bn over 2i and when we multiply i both the places. So, that will be ibn and then 2 times i square and that is minus 2 there.

So, that minus 2 is considered here. And similarly, for the second one, so we have e power inx e power inx is present at these 2 places. So, e power inx, we will take common again and then half factor. And then similar setting can be done here, but this time there was a minus sign, so it will come automatically as the plus sign. So, we have the form in the complex numbers.

So, if we set here for this first complex number half a n minus ibn, a new number, new name cn, and for the second, this combination of the coefficients, if we call it kn as half an plus ibn. So, having these 2 or new numbers kn and the cn, we can now rewrite this fourier series as follows.

So, the fourier series of f will be for half a naught, half a naught, we have defined as c naught, which is also coming directly from this definition, where the b0 will be set to 0 naturally because b0 is 0, we start b with b1 and so on. So, here again the c0 is half a0, which is replaced here by this c0, the second this is cn, and then the third place here we have this kn.

So, that is the complex form of this fourier series with having these coefficients the cn, half an minus ibn, which is equal to, so we will further simplify the final expression for c1, cn and also then again, some more simplifications will take place for this form of fourier series. So, we have the cn, half an minus bn and we know the an and we know bn. So, that was in terms of this f x cos nx and the other one bn was this f x sin nx.

So, by substituting these an and bn here, we have a more simplified form f x cos nx minus i sin nx. And this cos nx minus i sin inx, we have written in terms of again this exponential function e power i nx with this integral minus pi to pi f x e power inx dx.

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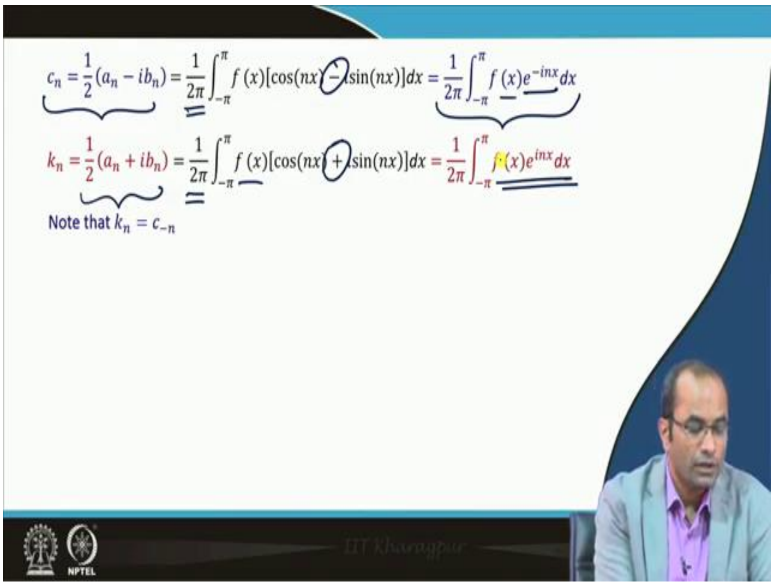

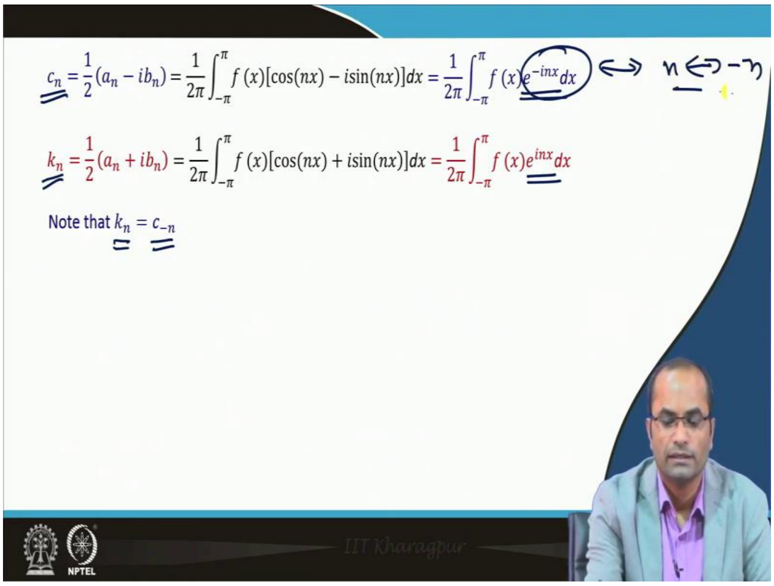
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$n \Leftrightarrow -n$
$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos(nx) + i\sin(nx)]dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$
Note that $k_n = c_{-n}$
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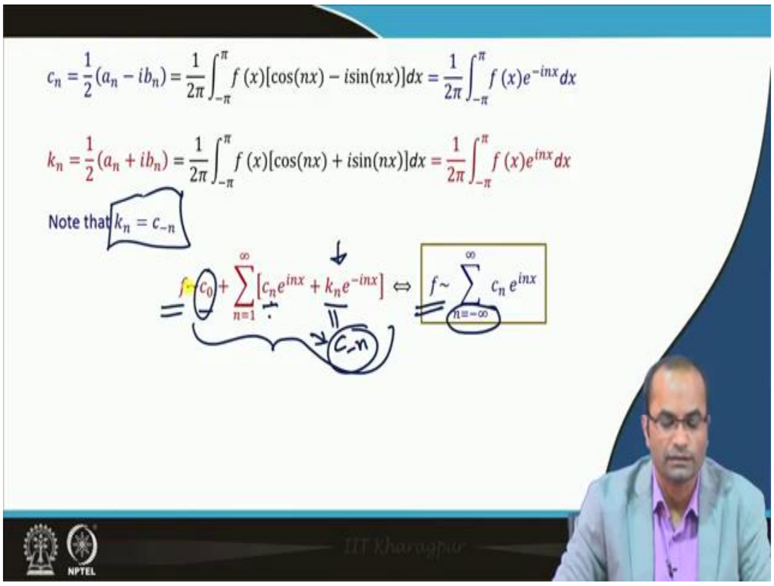
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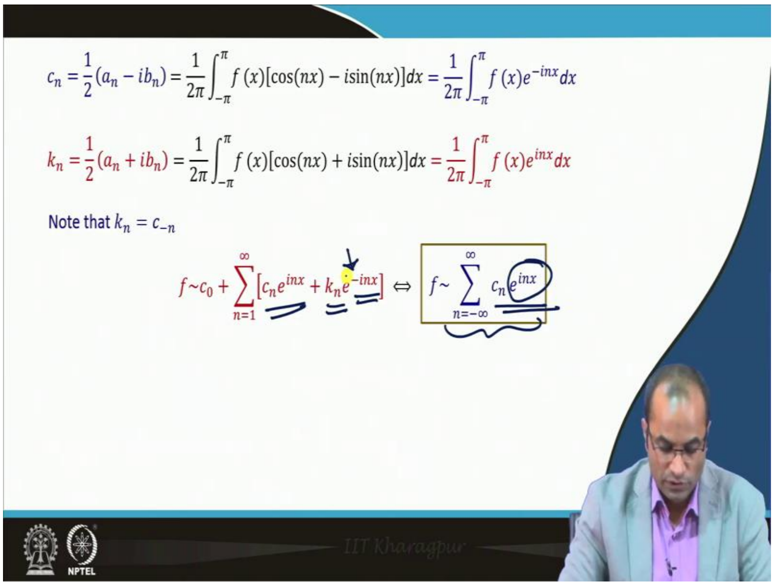
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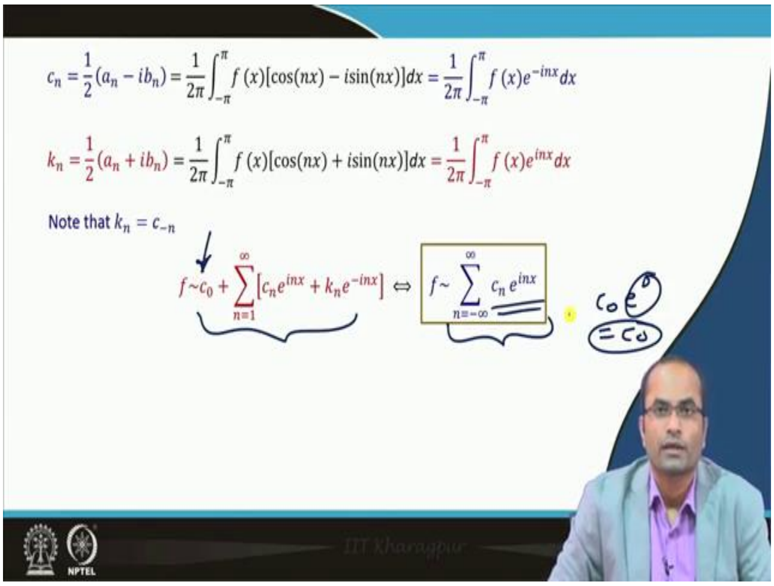
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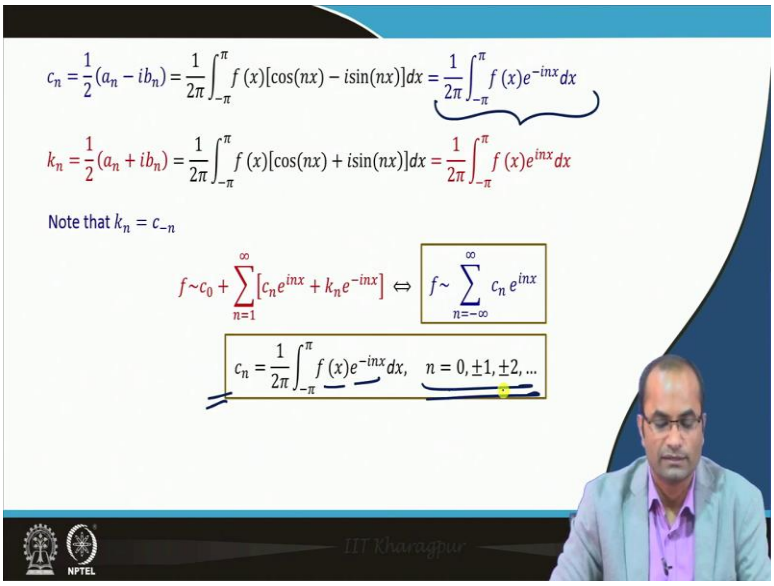
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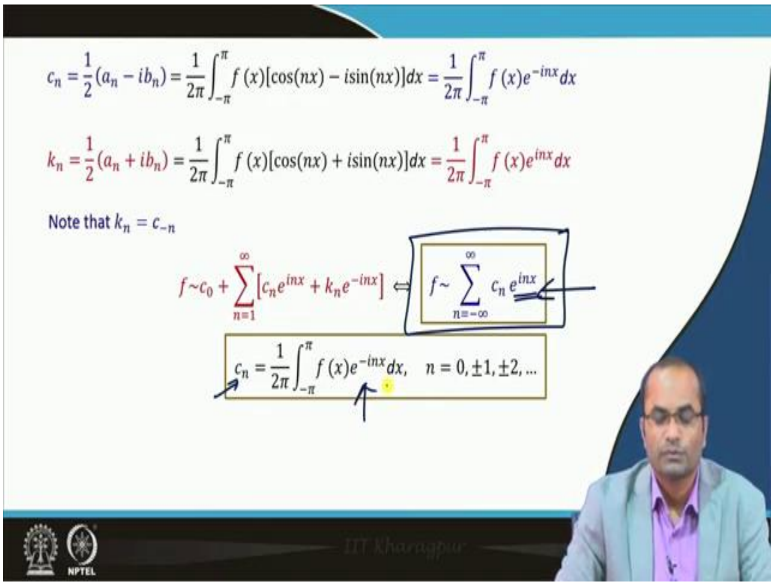
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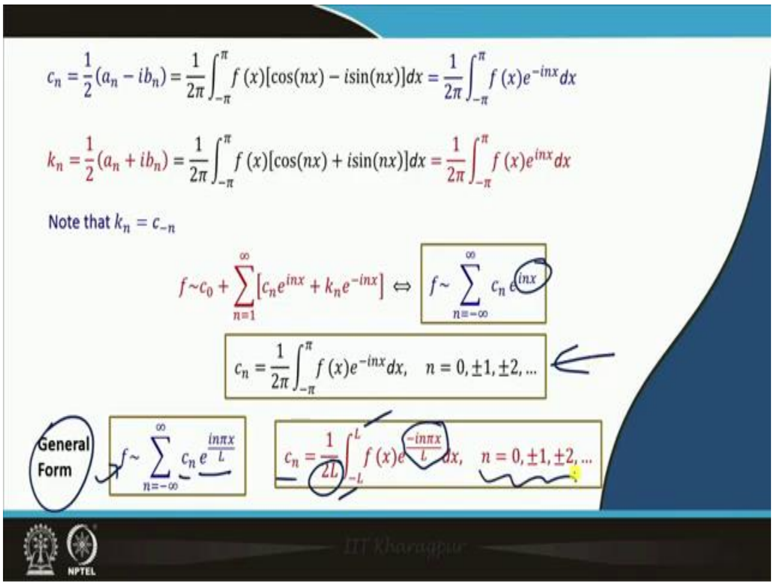

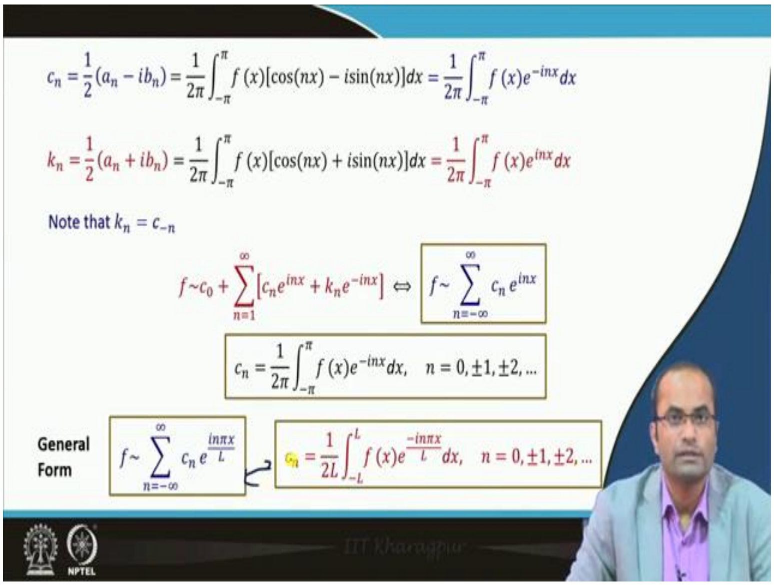

So, we have the cn with half of an minus ibn and final expression in terms of f x, we have 1 over 2 this f x e power minus inx. Similarly, we can do for kn, and kn is half an plus bn where half the similar to this term we have 1 over 2 pi in that f x, this sign will be replaced by the plus sign now that is the only difference and finally, we will have in the integrand here f x e power inx for this k nx.

What we should also realize now that for cn, the expression has e power inx and for kn, it is e power inx so the only difference is with the sign So, we can say that the kn is equal to c minus n, if we replace this n here by minus n, if we replace n by minus n, then we will get exactly kn there.

So, we have this relation between kn and cn. So, having this now again, we are getting back to the fourier series, so the fourier series is c naught, then cn and then kn here. So, the kn can be replaced by the c minus n, then we have f can be represented in this, in a compact form having this summation from minus infinity to plus infinity. Because this cn is, kn is nothing but c minus n. So, in this form, when n is 1, we have c1, but at the same time we have here c minus 1, then we have c2 then is this place we have c minus 2. So, all these plus and minus are appearing together here and the c0 is already there in the front.

So, in this compact form, we have all c's with negative integers and also for positive integers. So, c minus n and c plus n and e power, this inx will also take care automatically for the negative values, we have the for c also we will have the negative values and automatically c power i when n is a negative number this will be compensated.

So, this form is exactly same as written in this compact form which is n equal to minus infinity to plus infinity cn e power inx, so when n is 0, we have here c0 e power 0 which is 1, so we have just c0 for n equal to 0 and that term will be also incorporated in this infinite sum.

So, what is cn now, the cn is 1 over 2 pi f x e power minus inx given here and n is now 0, plus minus 1, plus minus 2, et cetera. So, this is the compact form of the fourier series, much better than having those cos and sin, it is written in a very compact form cn e power inx, and then the cn will be given by this integral. So, we do not have 2 coefficients now only 1 coefficient is serving the purpose.

So, we can write a general form, general in the sense when we have the interval from minus L to plus L for instance. So, there would be slight difference, so f x, so the function f can be written down in this fourier series, cn and e power instead of inx, we have in and pi x over L as usual we have done before, for this general form. And the cn will have now instead of pi, we have L there minus L to plus L, f x and again, there will be slight change here instead of inx, we have in pi x over Ldx.

And when L is pi, this will reduce to the standard form. So that is the case always we have, so here n varies from 0 to infinity. So, this is the general form which we will use in one of the examples soon.

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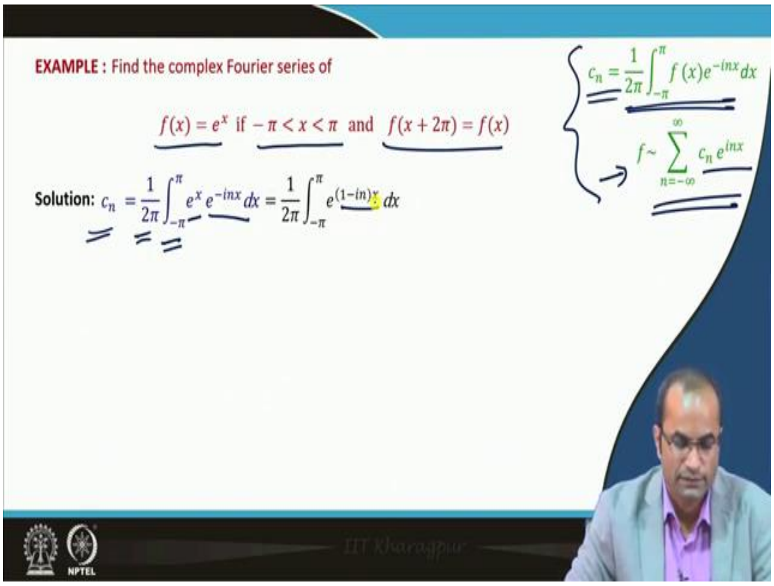

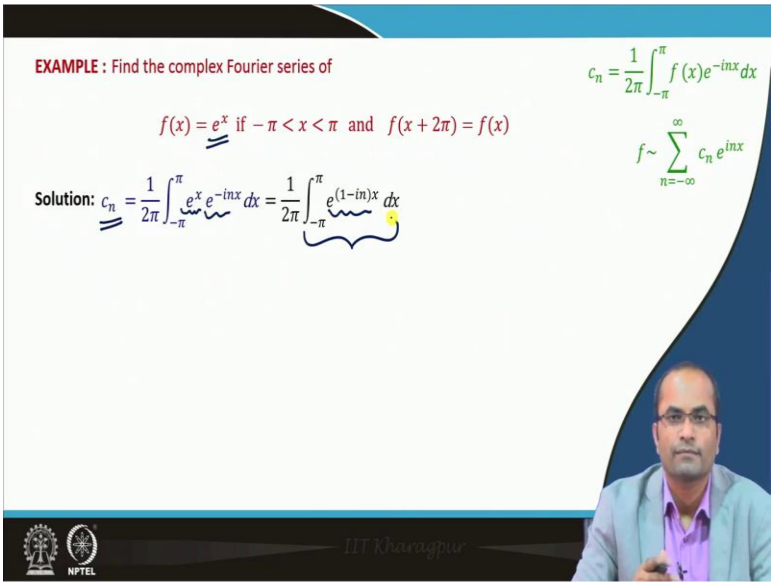

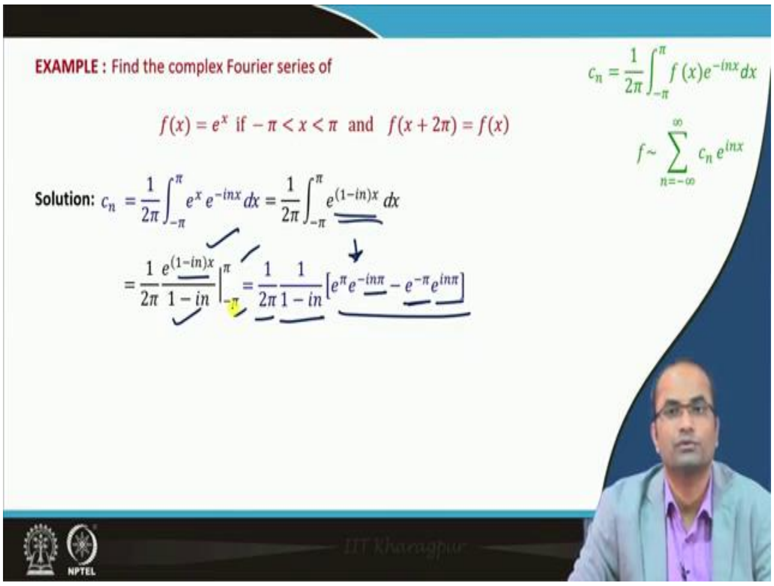

EXAMPLE : Find the complex Fourier series of

$$f(x) = e^x \text{ if } -\pi < x < \pi \text{ and } f(x+2\pi) = f(x)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \qquad f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Solution: $c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} e^{(1-in)x}\,dx$

$$= \frac{1}{2\pi}\frac{e^{(1-in)x}}{1-in}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\frac{1}{1-in}\left[e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right]$$

$e^{\pm in\pi} = \cos n\pi \pm i\sin n\pi = (-1)^n$, since $\sin n\pi = 0$

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$$= \frac{1}{\pi}\frac{1+in}{(1-in)(1+in)}(-1)^n\sinh\pi$$

$\sinh\pi = \dfrac{e^{\pi} - e^{-\pi}}{2}$

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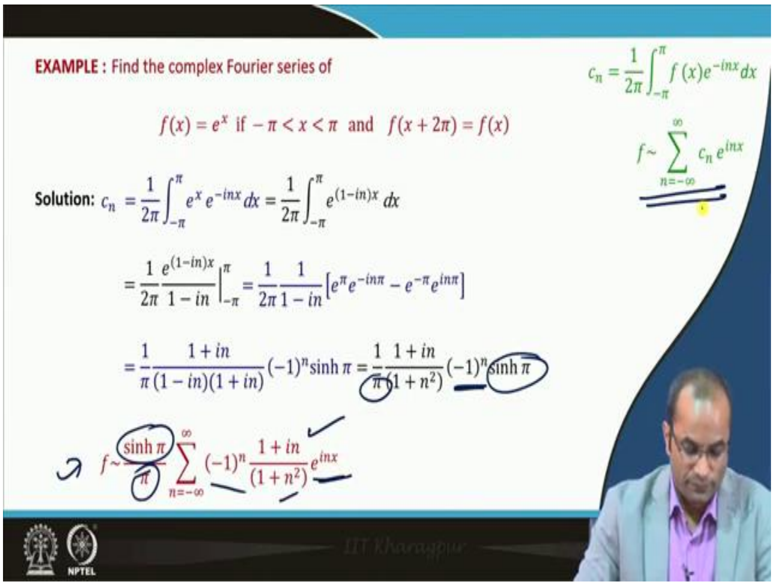

So, if you want to find, for example, this complex fourier series of this function f x is equal to exponential x and x lies between minus pi and pi. And then it is periodic, so we can say f x plus 2 pi is equal to f x. Well we know already, that the complex form of fourier series is given by this term here and minus infinity to plus infinity cn e power inx and for cn, we have the formula in terms of again exponential functions.

So, cn if you want to compute now 1 over 2 pi and then integral minus pi to pi, then f x, we have e power x, e power minus inx. And then these exponential functions are merged here 1 minus in into x.

So, there is an advantage here, for instance, in this case, if we are writing the complex series for this exponential function, evaluation of this cn is easy because we have exponential and we have again exponential. So, the exponents are added there. So, we have 1 minus n into x and then we can easily integrate it, this is much easier than having those cos an's sin and then computing an's and bn's. So, now the integral of this is also easy. So, we have 1 minus n and divided by this 1 minus n and then this x will vary, so the lower limit pi and the upper limit we have pi, the lower limit is minus pi.

So, here again, we have 1 over 2 pi and then 1 over 1 minus n and here we have substituted the pi for x. So, we have e power pi and e power minus inx. And then for the minus pi, here we have e power minus pi and e power inx with the plus sign now, because x is substituted with minus pi.

So, having this now e power, inx, at these 2 places, so e power inx plus or minus we will see in both the cases in pi this is cos n pi and plus, so either here minus then plus minus, so plus minus and i sin n pi, sin n pi is 0 and this cos n pi is minus 1 power n. So, e power whether plus or minus in pi is nothing but minus 1 power n. So, we have here minus 1 power n and at this place also we have minus one power n, minus 1 power n which we can take a common now. And then we have, e power pi minus e power minus pi with this 2, this has written as sin hyperbolic pi.

So, sine hyperbolic pi, as per the definition is e power pi minus e power minus pi and divide by 2. For the cos we have the plus sign there.

So, this exponential e power pi minus e power minus pi divided by 2 is replaced with sin hyperbolic minus 1 power n is for the c power inx. And then the rest, we have multiplied here 1 plus in and also divided by 1 plus in and then, so 1 over pi we have 1 over pi 1 plus in and then here 1 minus in, 1 plus in. So, 1 minus i square n square that is 1 plus n square and minus 1 power n and then we have sin hyperbolic pi.

So, then we can substitute in the fourier series, this coefficient cn, so then sin hyperbolic pi is a constant term, pi is also then outside, then we have minus 1 power n and then 1 plus in divided by 1 plus n square and then e power inx, as a result of this fourier series. So, what we have observed that writing this fourier series in complex form has its, sometimes it is simple to compute the fourier coefficients for example and this form has also direct applications in the control theory also in other area of electrical engineering.

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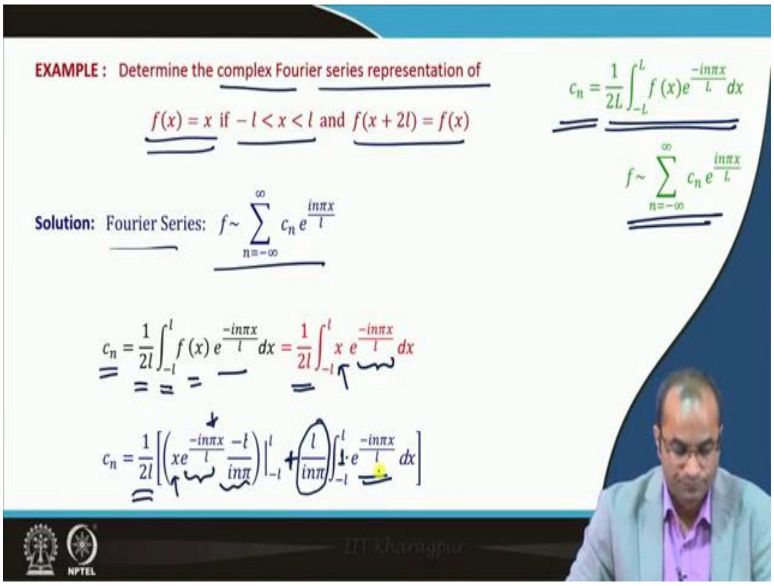

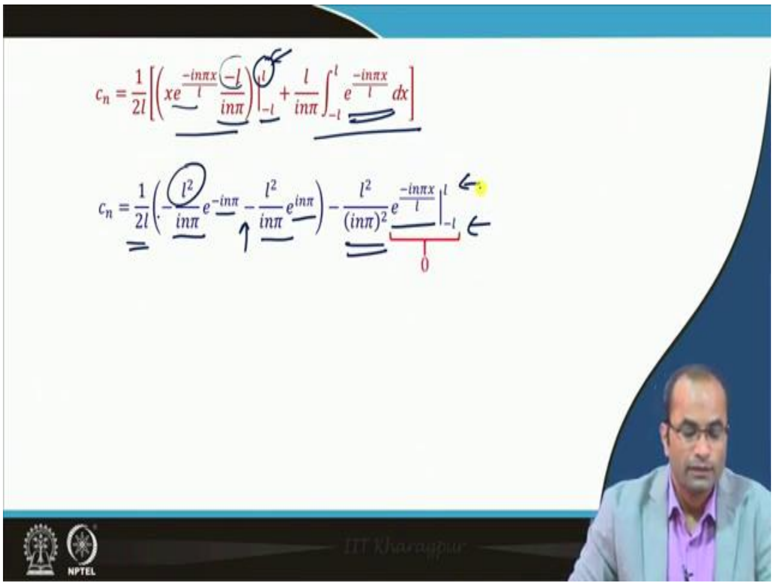
$c_n = \frac{1}{2l}\left[\left(xe^{\frac{-in\pi x}{l}}\frac{-l}{in\pi}\right)\Big|_{-l}^{l} + \frac{l}{in\pi}\int_{-l}^{l} e^{\frac{-in\pi x}{l}}dx\right]$
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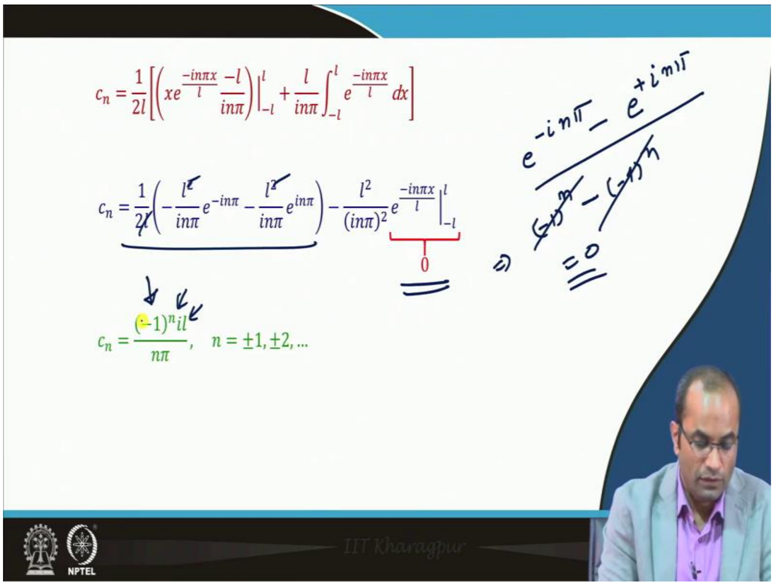
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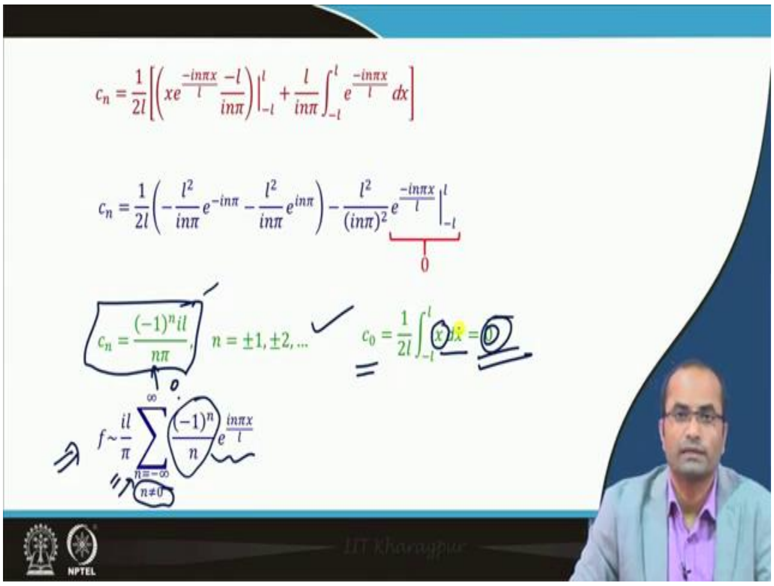
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$c_n = \frac{(-1)^n il}{n\pi}, \quad n = \pm1, \pm2, \ldots$
$c_0 = \frac{1}{2l}\int_{-l}^{l} x\,dx = 0$
$f \sim \frac{il}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{(-1)^n}{n}e^{\frac{in\pi x}{l}}$
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So, we have the, another example where we want to determine the complex fourier series representation for this f x is equal to x and it is defined in minus l to l and again it is 2l periodic. So, the same thing now we have the cn this is more the general formula and then we have the fourier series written in this complex form.

So, the fourier series we have this then we can compute a cn by 1 over 2l minus l to l, f x and then e power in pi x over l. So, 1 over 2l that is already there f x is replaced with x and then e power in pi x over l and then we can integrate by parts again. So, we have 1 over 2l, and then this x as it is the integration of the exponential function e power in pi x, and then this factor will be just 1 over that factor. So, it is reverse here minus l over in pi.

Similarly, the differentiation of x will be 1 and then again, this integration will come with this factor l over i in pi with the minus sign and therefore, it has become plus here and then we have e power minus in pi x over l.

So, we need to again differentiate here once more so, let us proceed. So, this is the first term then we have the second term, out of the first term we have to substitute first l there. So, we have l and then there is another l here. So, l, l will be l square with the minus sign, we have in pi and then exponential minus in pi, because this l will get cancelled with this l there, when we are substituting the positive value of l. Then we have minus l, again the same thing the x will be minus l, so and then there is a minus already so, this minus sign will be there. And then we have again here in pi and e power with the plus in pi.

Here again we have to integrate. So, e power minus in pi x over l and then they will be factor l over in pi, which and as a result this has become now the square and again, we have to take care the limits from minus l to plus l.

And this term is 0, that is the fact here because when we put this l there, we have e power minus in pi. And then with minus sign we have e power minus in pi with plus sign now, because this minus l will make it plus, so, we have these 2 term and the first one is minus 1 power n and the second one is minus 1 power n, the same thing, but with the minus sign. So, this gets cancel and therefore, we have this value 0 and we can simplify now, the first one.

So, 1 over 2l, so this l will get cancelled with this l, the i we can multiply and divide, so minus i square, so both term will become positive and we have i there. We have this one, l there and the e power in pi or e power plus in pi both are minus 1 power n.

So, we have in the very compact form this coefficient here cn as minus 1 power n and il divided by n pi and n varies as plus minus 1, plus minus 2, et cetera. So, here the c0, we can separately compute because this expression is not valid for n equal to 0, so c0 is 1 over 2l minus l to l and f x dx. So, this is in odd function minus l to l, so this will be 0, so we have c 0 as 0. And finally, we can write then the fourier series.

So, the coefficient here cn il over this pi and then minus 1 over n the remaining part of the cn and we have e power in pi x over l and n varies from minus infinity to plus infinity, except this n equal to 0 because c0 is 0. So, that term will not be there in the series. So, that is the fourier series expansion written in complex form.

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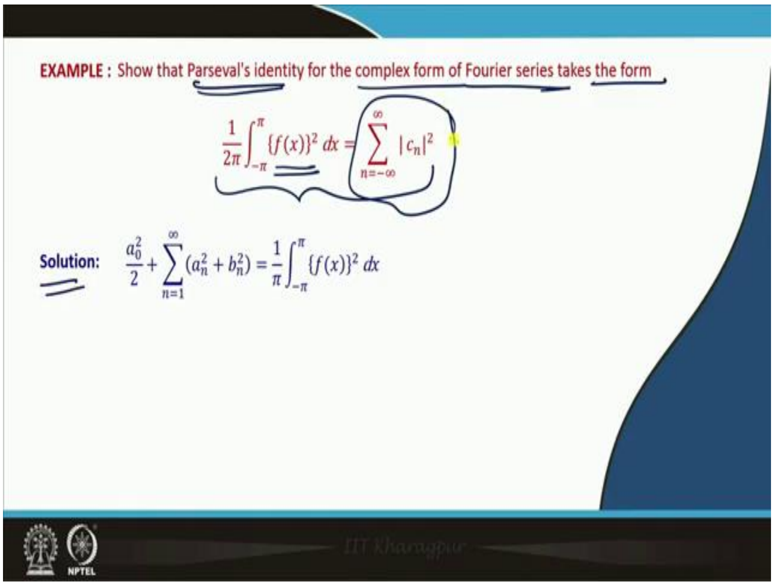

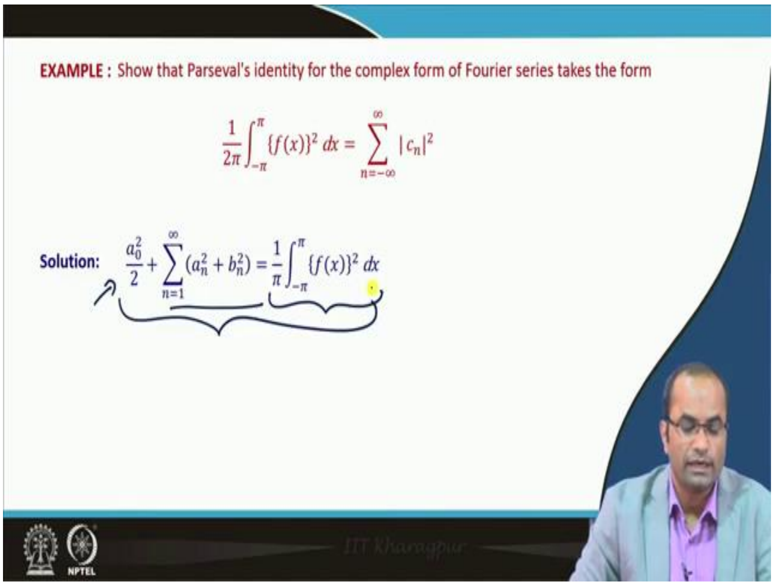

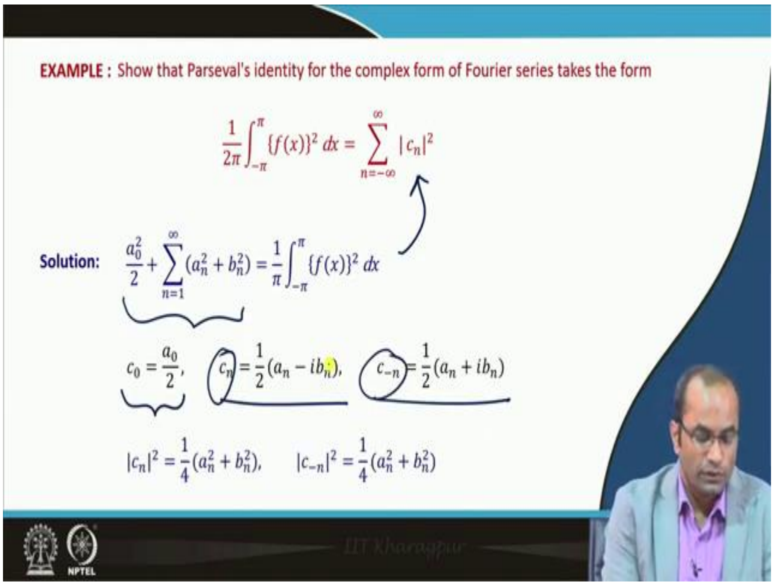

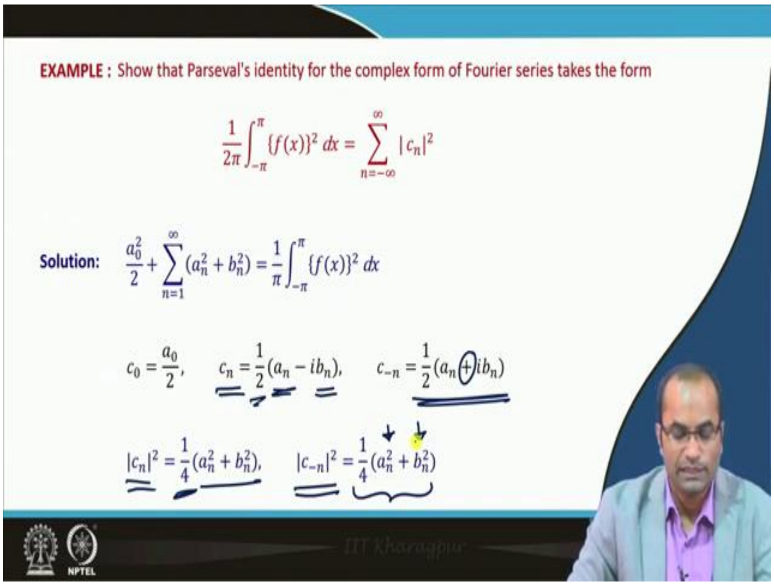

And now, we will get to the Parseval's identity which was already discussed in previous lecture. So, this Parseval's identities in the complex form of fourier series because now in the fourier series we have cn, earlier we had an's and bn's. So the, naturally the Parseval's identity will be also changed now in terms of cn not an's and bn's. So, Parseval's identity in this case takes the following form we have the similar term f x whole square dx minus pi to pi, but there is a simplified form here minus infinity to infinity cn square that 2 with the absolute value.

So, it is solution how to get this form, we will we can easily see now. Because the form of the Parseval's identity from the previous lecture, we know that it was a square by 2 and there was a summation over this an square and then the bn square and there was a term 1 over pi from minus pi to pi f x square.

So, this was the Parseval's identity which we have discussed for the normal form of the fourier series. And now, we know the relations that how cn, the new cn, the new fourier coefficient is related to the old ones. And with the help of that, we can convert this Parseval's identity to this form. So, recall that c0 was a naught by 2 and the cn was half of a n minus ibn and then cn, c minus n was half an plus ibn that was the relation when we have used these new notations cn and c minus n.

So, if we take the modulus of this cn because the complex number, so the modulus will be an square plus this bn square and 1 by 4. So, this is 1 by 4 an square plus bn square and the same thing will happen here it will have the same modulus because the only difference is that now we have the plus there, but in modulus that does not matter. So, again for cn modulus square will be 1 by 4 and an square and then we have this bn square.

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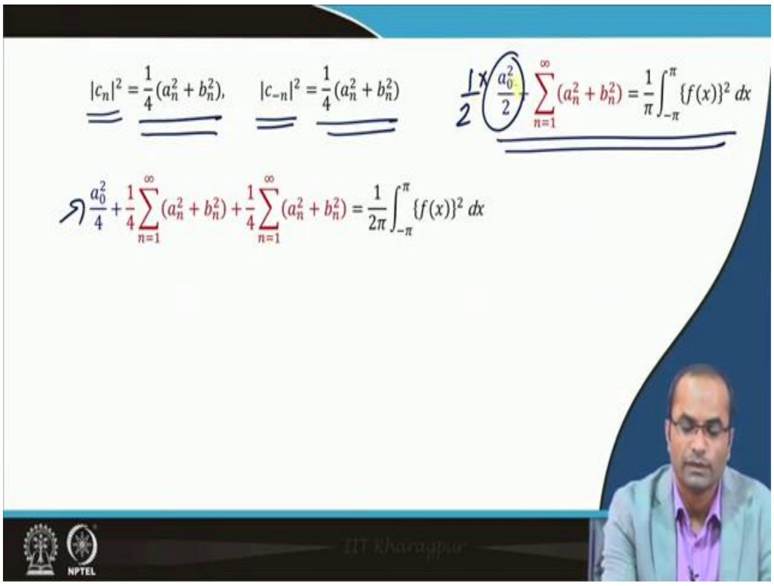

$$|c_n|^2 = \frac{1}{4}(a_n^2 + b_n^2), \qquad |c_{-n}|^2 = \frac{1}{4}(a_n^2 + b_n^2)$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx$$

$$\frac{a_0^2}{4} + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx$$

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$$\frac{a_0^2}{4} + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx$$

$$c_0^2 + \sum_{n=1}^{\infty}|c_n|^2 + \sum_{n=1}^{\infty}|c_{-n}|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx$$

$$\boxed{\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx}$$

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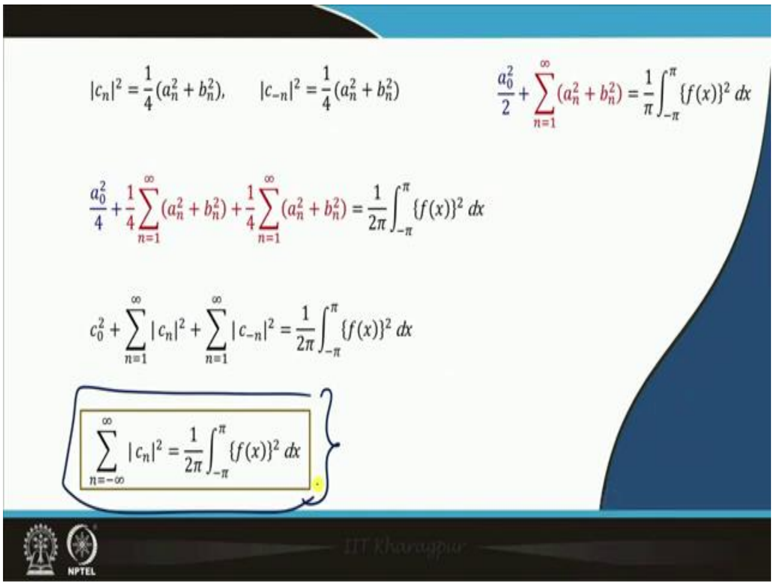
$$\boxed{\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx}$$



Well, so this cn square is half an square plus bn square and c minus n square is also the same and we have this traditional form of Parseval's identity in hand. So, now we can convert into the forms of cn's. First, the first term a naught by 2, as it is a naught square by 4, so we had divided here by 1 by 2 both the sides or multiplied by 1 by 2 to the whole series.

So, here the 2 will come, then here 1 by 2, and then here also 2 pi, so this is matching exactly 1 over 2 pi minus pi to f x whole square, the first term is a naught square by 4. And now this half of this an square plus bn square is written as 1 by 2, and so 1 by 4 plus this 1 by 4, so this is 1 by 2 and then we have the series. So, 1 by 2 is written as 1 by 4 plus 1 by 4 and then the rest is summation an square plus bn square. So, what do we have the half of this an square, bn square and from 1 to infinity and then again we have here one fourth and 1 to infinity an square plus bn square.

So, this is exactly c naught square because the c naught was a naught by 2, so here we have c naught square and this we know already that this is an square plus bn square with 1 by 4, we have cn square. And then for the second one, because we have already splitted into 2 to incorporate, plus and minus, so here we have the c minus n. So, we have considered this c minus n square within 1 to infinity and then we have the right hand side as it is 1 over 2 pi minus pi to pi f x whole square and dx.

So, now we can combine all the terms left hand side, we have c naught square, we have the positive powers of that is c1 square, c2 square, et cetera. We have also all negative power c minus 1 square, c minus 2 square, et cetera.

So, all these can be coupled now, and we have the whole summation which is running from minus infinity to plus infinity over this modulus c n square and the right hand side is 1 over 2 pi and this integral minus pi to pi f x whole square dx. So, this is the Parseval's identity for the complex form of fourier series. So, we can apply this in one of the examples in the next slide.

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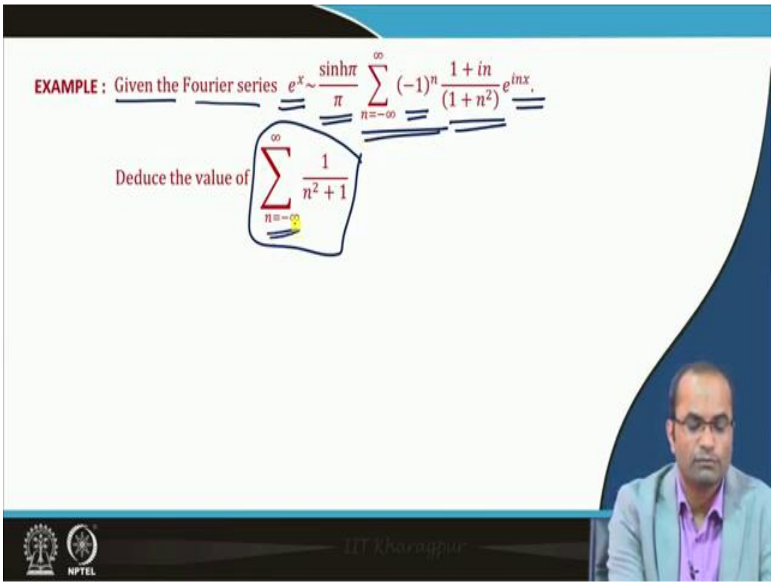

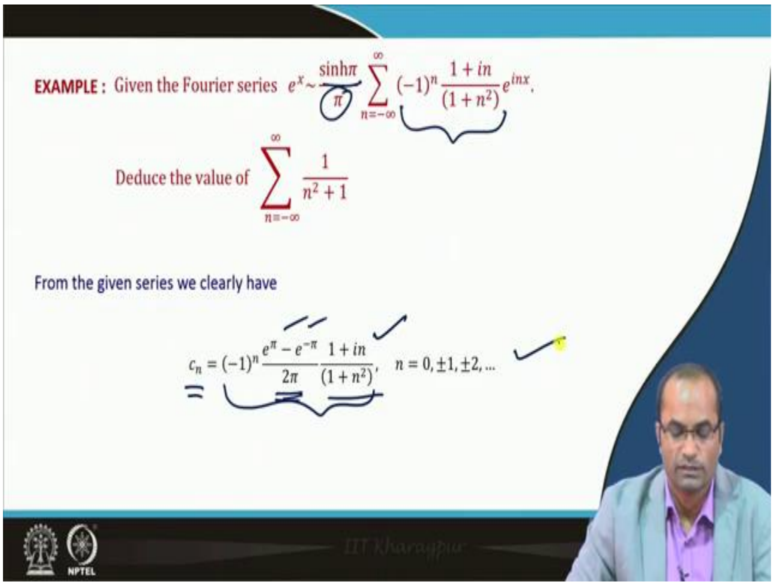

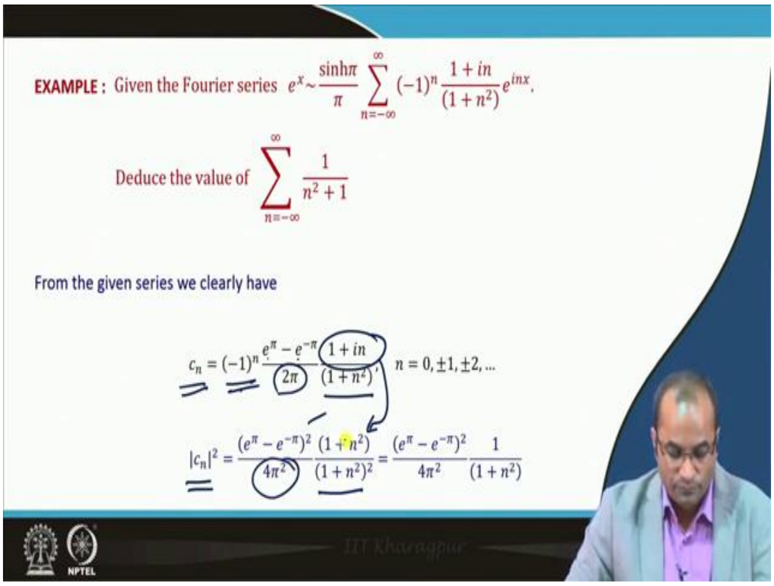

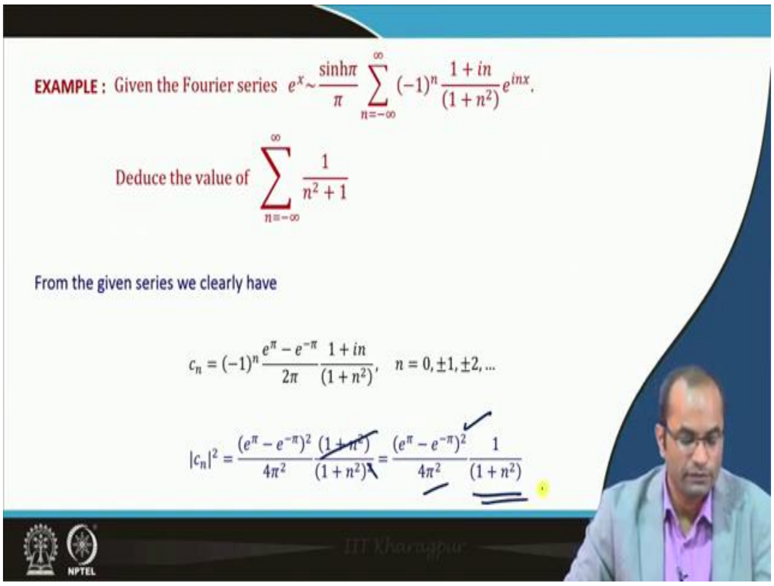

So, the example here, given the fourier series e power x as sin hyperbolic pi over pi, and then we have the summation here minus infinity to plus infinity minus 1 power n and 1 plus in over 1 plus n square e power inx. So, that is given fourier series of e power x, and we want to deduce this value here of the summation 1 over n square plus 1, where n is from minus infinity to plus infinity.

So, from the given series what we have, we can compare it now, so, cn's are directly given there, so these are the cn's, minus 1 power n and then we have sin hyperbolic pi that is e power pi, e power minus pi divided by this 2 and then there was a pi there so we have 2 pi, we have 1 plus in and we have 1 plus n square and the n varies from 0 to plus minus 1, plus minus 2, et cetera.

So, it is modulus now, because we need in the Parseval's identity, so we can get it and then, so this is the square. So, we have positive there e power pi minus e power minus pi whole square, there will be 4 pi square term there and then 1 plus n square whole square and for this we have the modulus 1 plus this n square, so the modulus of this because all other are just real number they will be squared, but this 1 plus in modulus will be 1 plus n square. And then, so here we have e power pi minus e power minus pi square 4 pi square and then this will cancel out, so we have only 1 plus n square.

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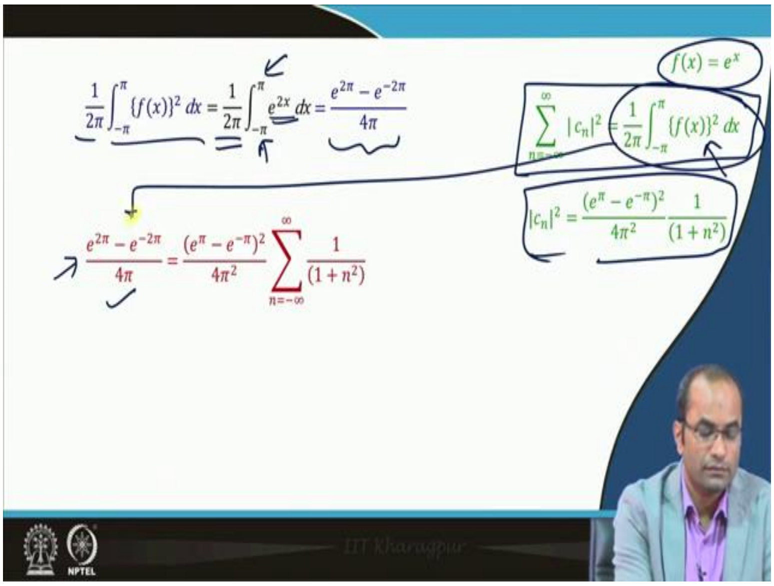

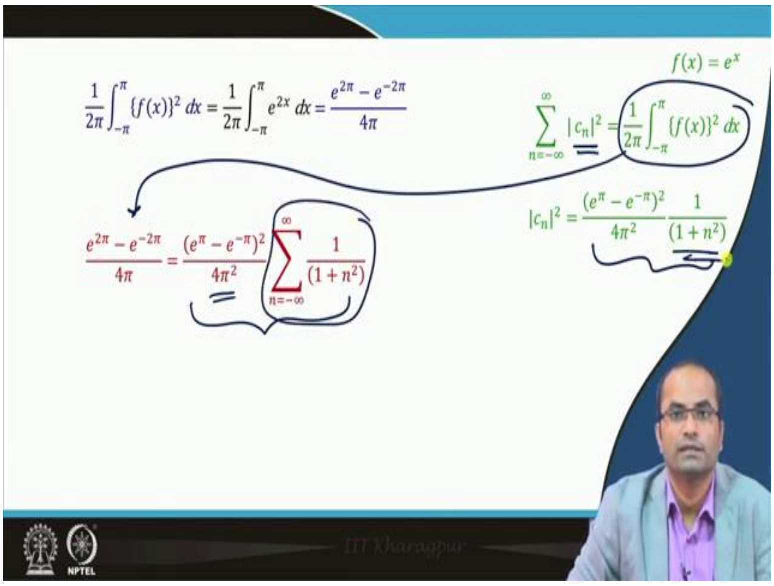

$f(x) = e^x$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(x)\}^2\, dx$$

$$|c_n|^2 = \frac{(e^{\pi} - e^{-\pi})^2}{4\pi^2}\frac{1}{(1+n^2)}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(x)\}^2\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2x}\, dx = \frac{e^{2\pi} - e^{-2\pi}}{4\pi}$$

$$= (e^{\pi} - e^{-\pi}) \cdot (e^{\pi} + e^{-\pi})$$

$$\frac{e^{2\pi} - e^{-2\pi}}{4\pi} = \frac{(e^{\pi} - e^{-\pi})^2}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(1+n^2)}$$

$$\sum_{n=-\infty}^{\infty}\frac{1}{(1+n^2)} = \frac{\pi(e^{\pi} + e^{-\pi})}{(e^{\pi} - e^{-\pi})}$$

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$$\frac{e^{2\pi} - e^{-2\pi}}{4\pi} = \frac{(e^{\pi} - e^{-\pi})^2}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(1+n^2)}$$

$$\sum_{n=-\infty}^{\infty}\frac{1}{(1+n^2)} = \frac{\pi(e^{\pi} + e^{-\pi})}{(e^{\pi} - e^{-\pi})} = \pi \coth \pi$$

So, we have the modular cn square ready which is given already here and then this is the Parseval's identity for the complex form of fourier series and our function is e power x. So, we will compute this other side of the fourier series 1 over 2 pi and the f x whole square term 1 over 2 pi, we have e power x and then the square, so e power 2x dx.

So, which is written here e power 2 pi minus e power minus 2 pi divided by this 4 pi because e power 2x and divided by 2 will be the integral and then we have substitute plus pi and then minus pi there. Well, so, we have e power 2 pi minus e power minus 2 pi divided by 4 pi that the other side of this Parseval's identity.

So, this integral side we have this value and then we have the cn square already with us. So, we have e power pi minus pi square, 4 pi square, and then the summation is exactly over this 1 over 1 plus n square that is the desired value, we want to compute this one.

So, that can be now given as, so this e power pi, so this one here, the left hand side, we can write down as e power pi minus e power minus pi and the product with e power pi plus e power minus pi, say a square minus b square there. So, e power pi minus e power minus pi, the one term will get cancelled. So, in the division we have e power pi minus e power minus pi. In the numerator e power pi plus this e power minus pi and then there is a pi because of this cancellation.

So, we have exactly this is the sum of this given series 1 over 1 plus n square and from minus infinity to plus infinity, which can be further written in terms of pi and the coth hyperbolic pi, which is exactly this one here e power pi plus e power minus pi over e power pi minus e power minus pi.

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So, here these are the references, we have used for preparing this lecture.

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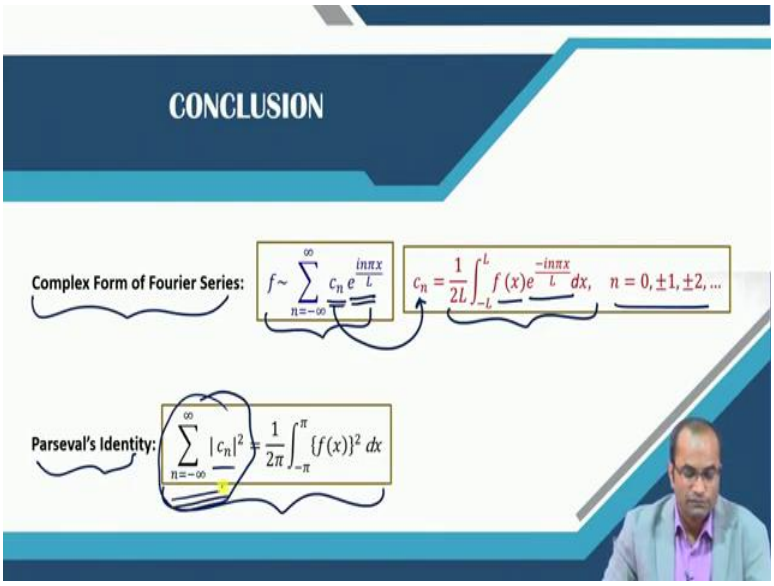

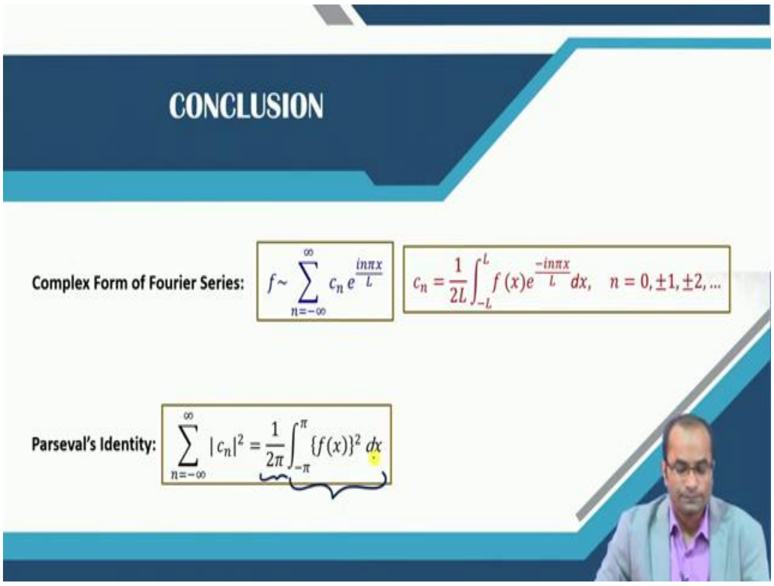

And now, just to conclude, so we have discussed in this lecture the complex form of the fourier series. This is the general form with this fourier coefficients and the exponential term e power in pi x over l, where the fourier coefficients are computed with the help of this integral 1 over 2L minus L to L f x and e power minus in pi x over L dx, where n can vary now the 0 plus minus 1 plus minus 2, et cetera. We have also discussed the Parseval's identity and indeed, use for computing the sum of a series.

So, the Parseval identity was we have just the sum of these squares of the absolute value or the modulus of cn, the sum was over minus infinity plus infinity and the right hand side, we have 1 over 2 pi and the integral minus pi to pi f x whole square dx. So, that is all for this lecture. And I thank you for your attention.