Engineering Mathematics - II
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Lecture no. 04
Line Integrals

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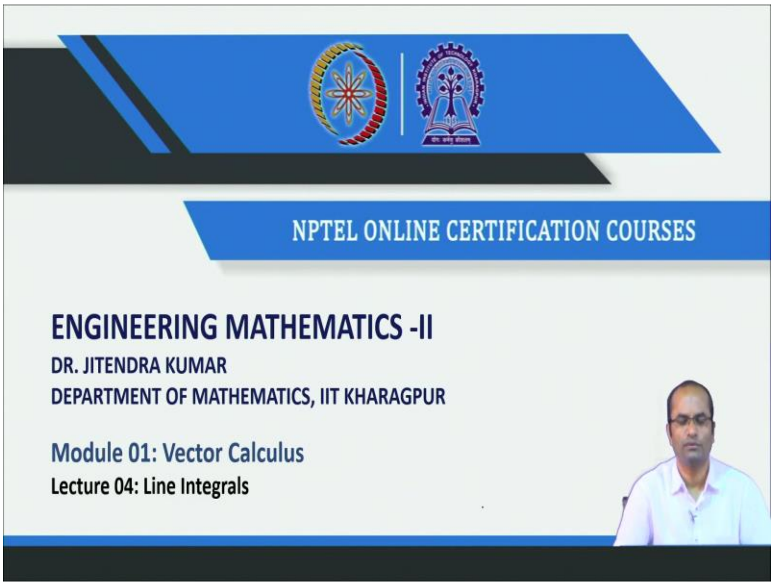

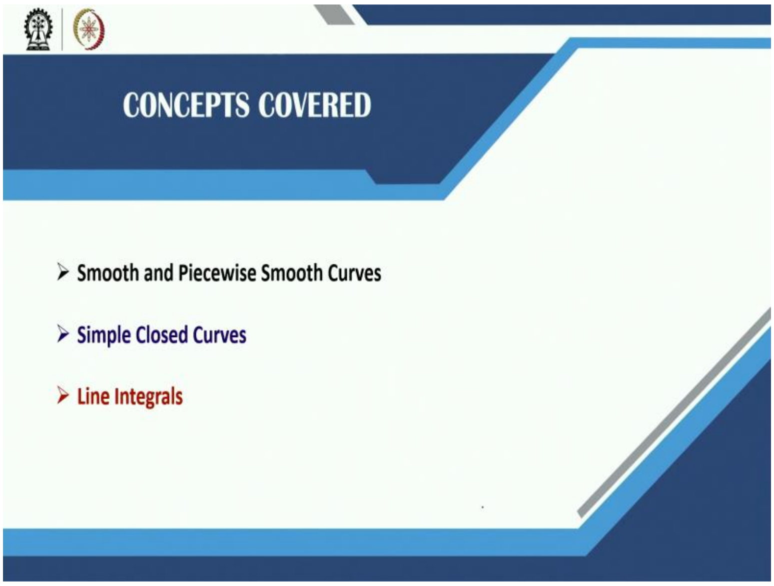

So, welcome to lectures on engineering mathematics II and this is lecture number 4 on line integral. So, today we will cover here what are the smooth and piecewises smooth curves. And also we will be talking about the simple closed curves and then finally, we will introduce this line integrals in vector setting.

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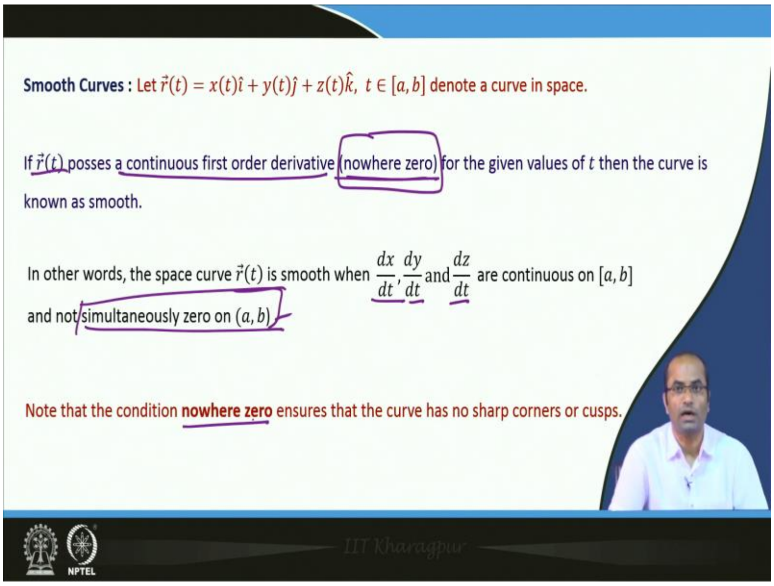

So, what are the smooth curve? So, let this rt, xt, yt, zt, which we denote for the curve. And this rt possess a continuous first order derivatives and nowhere 0. So, will emphasise here I mean later on you will explore that why this nowhere 0 is needed for the given values of T. So for all values of T, this vector should be non0 and then the curve is known as smooth. So, there are 2 conditions here the rt possess a continuous first order derivatives and they are now, nowhere 0 at the same point a given point T. Then the curve is known as smooth.

In other words, we can also say that the space curve R is smooth when these derivatives. So dx, dt, dy, dt and dz, dt are continuous on the given interval A, B and they are not simultaneously 0. So, again, there we have written nowhere 0 and now we have here that they are not simultaneously 0 that means for a given T, they all are not 0 all together in the whole interval AB. So, this condition nowhere 0 ensures that the curve has no sharp corner or curves this we will explore in the next example.

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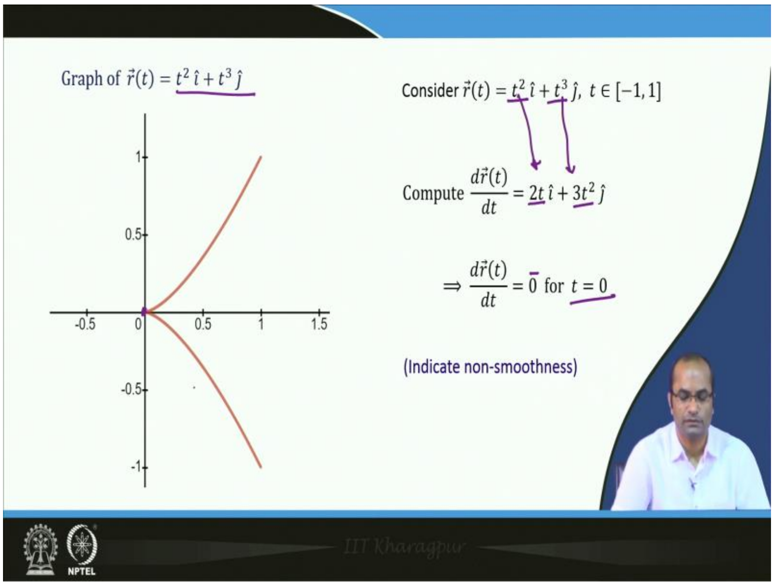

So, for instance, if you consider this rt as t square plus t square curve in 2 dimensions and t varies from minus one to one. And if we compute this tr over dt that means, the derivative of t square is 2 t and then tq is here 3 t square. So, we have this tangent vector 2 ti and 3 t squared j.

And if we compute this as t equal to 0. So, we see that this tangent vector is 0. So, the tangent vector should not be 0 that is, that the meaning is that at every point there should be a tangent vector which is non0 vector. So, this it self when we are getting this 0 this indicates that the curve is non-smooth. And if you plot this curve t square and I and t cube j. Then we see that, there is a corner here at the origin or when the t is equal to 0. So, there is a sharp over turn in the curve and therefore, this curve is not a smooth curve.

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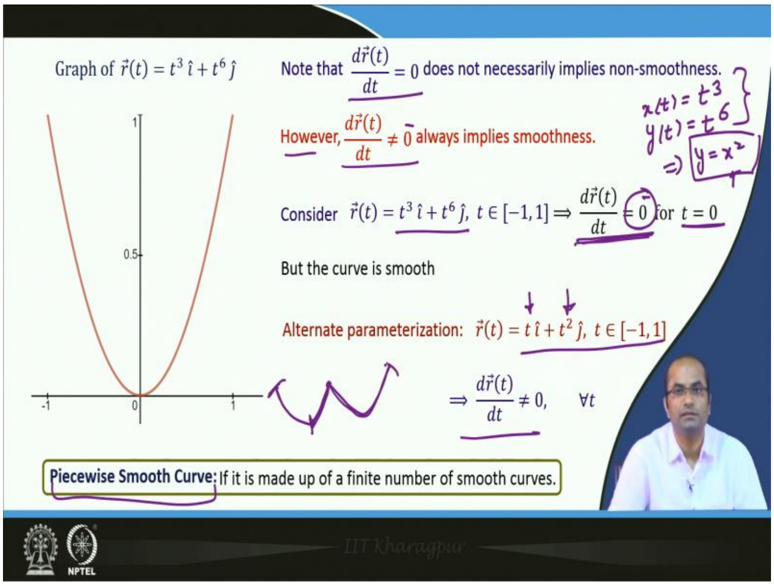

What one should also note that this condition that dr over dt equal to 0 does not necessarily implies non-smoothness. So, what does that mean? However, that whenever we have not equal to 0 that means there is a tangent vector which is not a 0 vector that always implies smoothness but not the other way around that dr over dt is 0 necessarily implies that there is a non-smoothness.

Like in earlier example, dr, dt was 0 and we have seen that the curve was non-smooth. But for instance, if we consider this curve here tq plus t square j and t varies from minus one to one. And we compute dr over dt and evaluate this at t equal to 0 then we will observe that the value of this tangent vector is 0.

And in this particular case, we will see that the curve is smooth. So, that we can observe clearly from the figure of this curve, if we draw a graph of this t square I plus t 6 j. Then we see that the curve is smooth at all points including this t equals 0. So, what is happening here?

So, we can have a different parameterization and this was one parameterization. So, if we take a close look here that the xt was tq and this yt was t 6. So, these 2 implies that we have here that y equal to x square. So, this is a parabola nothing else. The given curve written in this parameter form tqy plus t 6 j is a parabola. So we can have several parameterization. For instance, the other parameterization we can have like xt we will take t and then yt as t square.

So, in that case, this particular parameterization, if we just compute dr over dt. So the first component itself since it is t the derivative will be one. So, we have one plus, I mean one I plus 2 tj at its derivative or the tangent vector, which is always non zero whatever value of t we substitute there. So, what we learned here that there could be a form I parameterization of a given column, which may tell us that the derivative of this R with respect to t is 0. The tangent vector is 0 at some point of T, but this, does not necessarily implies that the curve is non-smooth. The curve can be smooth like the example we have seen here.

So, there is a term which we will be also using the piecewise is smooth curve. So, if it is made up of a finite number of smooth curves. So, we can have a finite number of smooth curves and there are sharp corners. But, these are made of from here to here. The curve is a smooth again in this part the curve is smooth and again in this part the curve is smooth. So, here we will call that this curve is piecewise is smooth, this is not a smooth curve as a whole, but in pieces, this curve is smooth.

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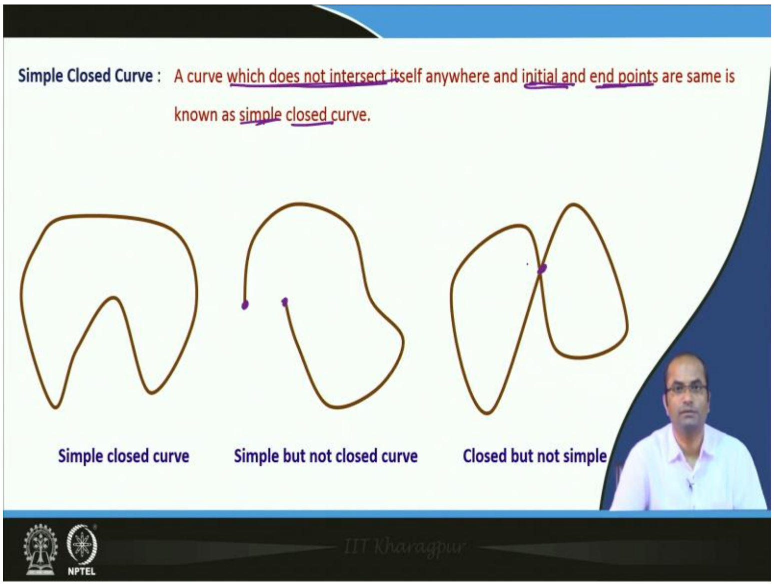

So, coming to the simple closed curve, another terminology which frequently using vector calculus. So, a curve which does not intersect here. So, the important point here is that which does not intersect itself anywhere and the initial and the end points are the same, this is known as a simple close curve. So, the closeness coming from the initial and the end points because they are same and the curve does not intersect itself that is what we use this terminology simple.

So, for instance this curve does not intersect itself and initial point and the end points are the same. So, the curve is closed. So, this is a simple closed curve, an example of a simple closed curve. In this case it is not a closed curve because these initial point and the end points they are different. So, this curve is simple but it is not a closed curve, it is simple because it is not intersecting itself. For instance here this curve is closed curve, but it intersect itself at this point and hence this is not a simple, so it is closed, but not simple.

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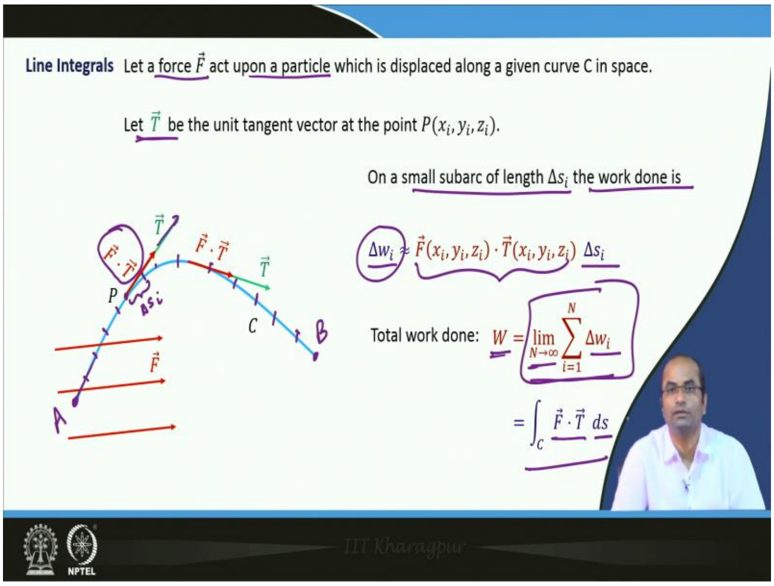

Okay, coming back to the main topic of todays lecture, that is the line integral. So we will introduce here that what is, what we call the line integrals in this vector setting. So, we have here, let us suppose. So, this is a particular example, but one can generalise this for many other or any other vector field. So, let a force field F act on a particle which is displayed along a given curve C.

So, we have a curve here the C and a particle is displaced along this curve here the given curve, it is being displayed from this point A to let us say, this point P. And we also assume that this vector t be the unit tangent vector at given point xi, yi, zi. So suppose, this is the point xi, yi, zi and then we will compute the total work done against this force field F to displace this particle from A to for instance B here.

So, what we do? We will break this curve into small, small sectors as we do always in the integral calculus also. The definition of the regular the integral which we had discussed already. So, here on a small subarc of length Delta SI. So, we say that this is our small subarc here of length delta SI, the work done will be the force into the displacement. So, the force because suppose this is the force field acting in this direction. So, we will find that what component of this is acting along the tangent of this curve.

So, for instance here, this is the tangent of the curve and the unit tangent factor we have defined by this t vector. So, we will compute the component by this. So, this is a projection of this F on this tangent which is given by, which is given by this F dot product with this tangent vector t. So, this is exactly the component of the force acting along the tangent at any point and then if we want to get this work done. So, this is the force and then the displacement is this dsI along the arc.

So, this is for one small arc and if we add from this A to B all these small arcs and then let the length of this arc tends to 0 in that case. So, here I have just shown here that at any point we have the similar situation. So, again this F.t will be acting along this tangent T. So, if we add all these work here. So, the total work done we can compute by this summation here I to N and then this delta WI.

And then we take this limit that the number of segments of this total curve is N and if let N tend to infinity. So, then we will get exactly the work done along the arc. And this limit here we right as the integral over the curve C and this force F the dot product t and the ds. ds is this element on the arc.

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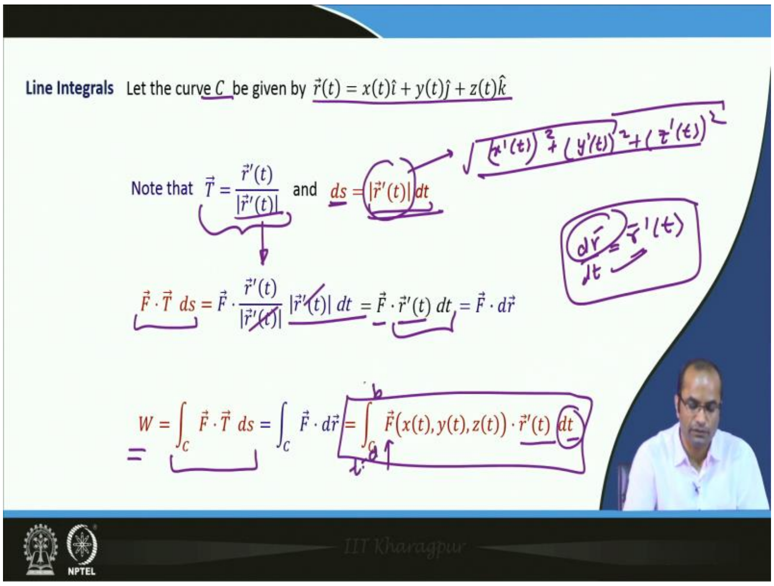

So, now, we will explore that how to evaluate this integral and what are the possibilities, different ways of evaluation. So, suppose this curve C is given by this vector function xt, yt, zt and you note that we have already learned this that the unit tangent vector, we can express in terms of the derivative of this R. So, the derivative of R divided by its magnitude will give the unit tangent vector and the ds this arc length we have also learned in the very first lecture that the absolute value of this or the magnitude of this R prime t will give and multiplied by dt that is the element here this ds we can express. This is if you remember this was just the square root of the xt whole square and plus y derivative t whole square and plus z derivative t whole square. So, this was already discussed there. So, we can compute this ds in terms of this derivative of R.

So, this F.Tds that was the integrant therein, in this curve integral. So F. and the t we have substituted from there R derivative divided by the magnitude of this R derivative, and then this ds is R derivative magnitude into dt. So, this will get cancel and we will get here the F and the R derivative dt, which can be also written F.dr, because dr, dt what we have written this R prime T.

So here we can also write dr as R prime dt. Well, so the total work done we can express in this form of this curve integral or we call line integral F.T. This is the unit tangent vector ds. And now we have seen that we can write as this F.dr, or we can write as F.R derivative and dt. So, this is the simplest form of the evaluation what we will do? We will substitute in F this parametric form xt, yt, zt. And this R prime t will be evaluated the dot product will be integrated over dt the single integral which we have learned and the t will vary for instance, whatever t goes from A to B depending on the problem.

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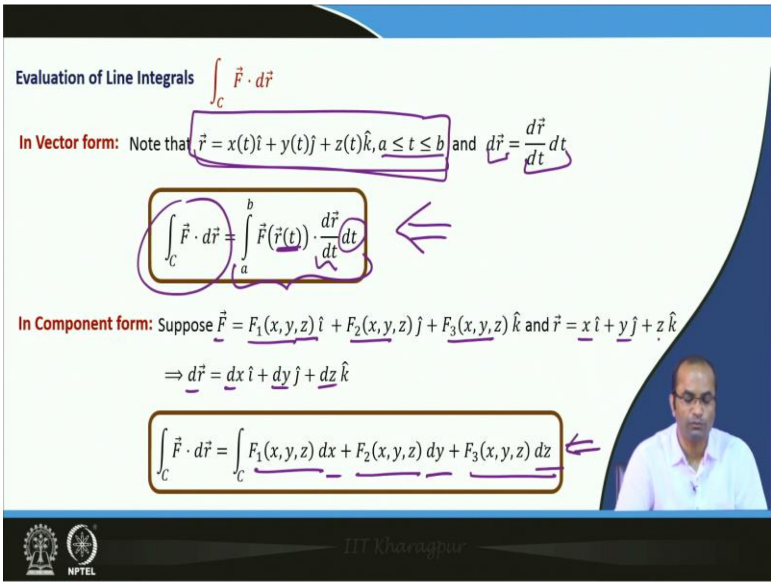

So, let us continue now, the evaluation part that suppose, this is the integral. So, there are 2 ways one can evaluate this the one which we have just seen that if this is the equation of the curve and then we can evaluate this dr as dr, dt, dt, this we know already. And this curve integral can be evaluated with the help of this single integral that in F, in the vector field we will substitute those parameters form xt, yt, zt and then dr, dt will be evaluated this dot product will be integrated over t and this t varies from A to B.

In component form. Suppose this F has these 3 component F1, F2, F3 and naturally x is, R is this x, y, z. So, what we can do? The dr will be dx, dy, dz and in that case F.dr can be evaluated so just this product will be F1 dx, F2 dy and F3 dz. So, what we will observe in the examples, in some cases when the parametric equation is easily available or given in the problem, then we can apply this form here, in the vector form. And when we have the cartesian coordinate or in the component form it is given then we can exactly do this integral dx, dy, dz and we will observe in particular examples that how to evaluate when we use this component form or the vector form.

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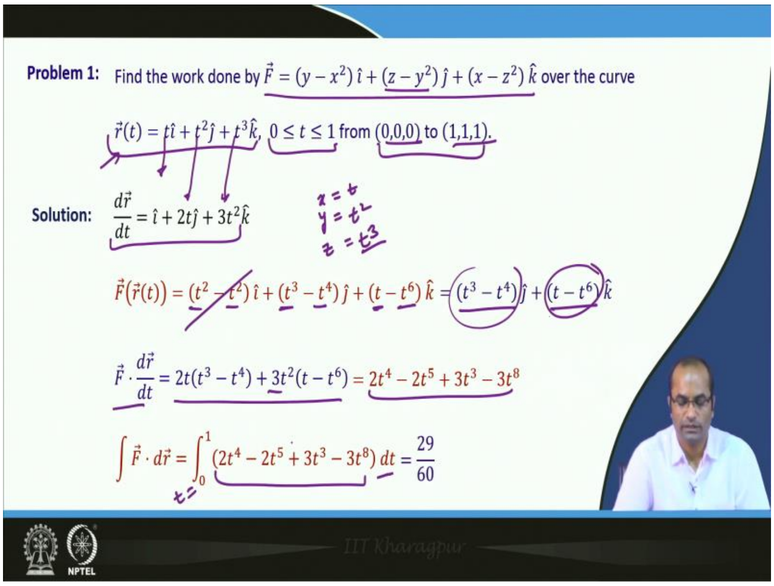

So, suppose we have this problem find the work done by this force. So this is the force field given, we have the 3 components over the curve, the curve equation is also given that is the parametric equation or the vector function of this one variable is given. P varies from 0 to 1. So, this will naturally come from this vector function that when t is 0 then we are at the 0, 0, 0 point. When t is one we are at 1, 1, 1 point. So, we are, our curve goes from this 0, 0, 0 to 1, 1, 1 along this given position vector okay.

So, the solution naturally when this, the parametric form is given you will use this vector form for evaluation of this work done. So, we need to evaluate dr over dt which is easy. So, the partial derivative t with respect to one. Here the t square will be 2 T. tq will be 3 t square.

And then we have the F, when we substitute here this parametric form that means this x is t and then yt will be t square and zt will be t cube.

So, we can substitute their y minus x square. So, t square and then minus x square that is t square again. Here we have z minus y square. So, the z is t cube and minus y square that is t 4. And third component we have x minus z square. So, the x is t and minus is the z square that is t power 6, the third component. So, here this cancel out and we have this 2 components t cube minus t square and t minus t 6 4 F. And when we take this dot product the F dr, dt we will get this 2 T.

So, 2 t the jF component, here will be multiplied by this one t square minus t 4 and 3 t square will be multiplied by t minus t 6 which can be simplified and we get this simple expression for in terms of t for F.T. Now, we can integrate. So, F.dr that is the limit for the T. t goes from 0 to one and this is F.dr, dt and over dt we can integrate this simple integration the value will be coming 29 by 60.

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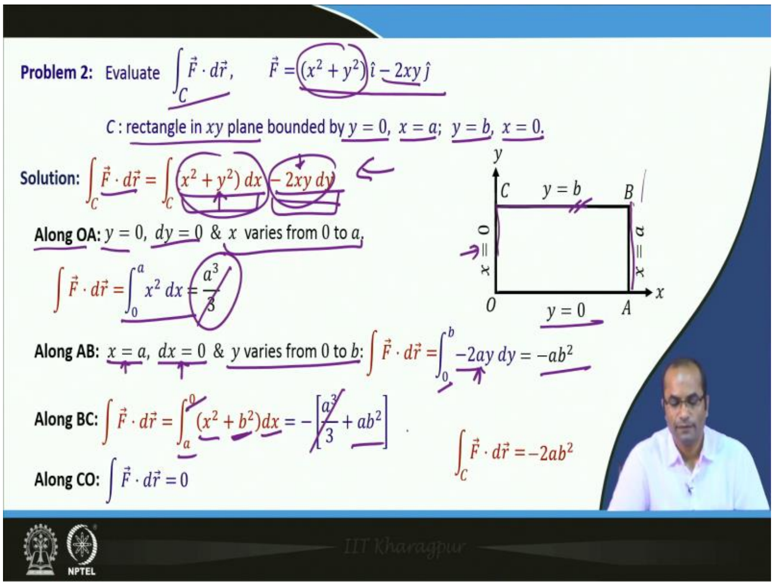

So, the second problem where we compute again this line integral F.dr, where F is given as this function x square plus y square. The Ith the first component, the second component is minus 2 xy. And in this case, the C the curve is the rectangle in the xy plane, which is bounded by y equals to 0. x is equal to A. xy is equal to B and x is equal to 0. So, we can draw this easy, y equals 0 and this is the line.

And then we have x is equal to A, this is the line here, and then we have y is equal to B line and then x is equal to 0 line. So, having this we will now evaluate this line integral along this boundary of this rectangle. So, F is given x square plus y square and then minus 2 xy. So note that in this particular case, since we have given these rectangle in this cartesian coordinate that here y is 0 on this AB line x is A and CV line y is B and so on, this will be much easier if we write in the component form directly xy form, not having the parametric form first and then convert it. That obviously, we can do but that will be much more difficult as compared to this because this is very trivial now.

So, F.dr that means the x square plus y square that is the first component will be dx and then the minus 2 xy will be with dy. So, this line integral we will evaluate. So, there are 4 parts of the boundary, the first we will consider the along OA. So, what is the special along OA, the y is 0 that is given there. So, y is 0 meanings the dy is 0 and x varies from 0 to A. So, this is the information available along this OA axis.

So, along this OA we will evaluate this line integral F.dr that is here, so that dy is 0 so the second part will vanish only the first one will survive. Where we have x which is running from 0 to A and the y is 0. So in this first integral, what we will do the y we will set to 0, and this term will go to 0. So we will have just the x square, x square dx term, we have the x square, dx term, which can be evaluated and then we will get this A cube by 3.

Similarly, along this AB line, what we do? We have x is equal to A, we have this dx is equal to 0 because x is constant, so naturally the dx will be 0. The y varies from this 0 to B in this case. So again the same situation in the given integral we will substitute x is equal to A, the dx will be 0 and the y will vary from 0 to B. So our integral will be 4 y. So, this integral will become, the limit of y 0 to B and we have the dx 0. So, the first term will go to 0, the second we have minus 2 xy. So, 2 and x will be replaced by A and y, dy.

So, this integral, again we can evaluate that is minus AB square, the third one when along we are going, moving along BC. So, along this BC here y is equal to B that means dy is 0 there is no variation in y. So, the second term in our integral here will vanish and we have x square plus this y square dx. So x square and the y square, so since y is B there, so, that will be B square and then over dx, x varies from then A to 0. So, this we can also evaluate, the value will be minus A 3 by 3 plus AB square.

Similarly, along the CO, where x is 0, so the dx term will be 0 and the second term will be also 0, because in the second term, we have this x and since x is 0, the second term will also vanish and the first will vanish because we have dx there. So, in this case, so this integral is going to be 0, if we add all these 4 integrals, since here A cube by 3 is there, here minus A cube by 3 is there minus 2 AB square, here also minus 2 AB square. So, the value we will get minus 2 AB square.

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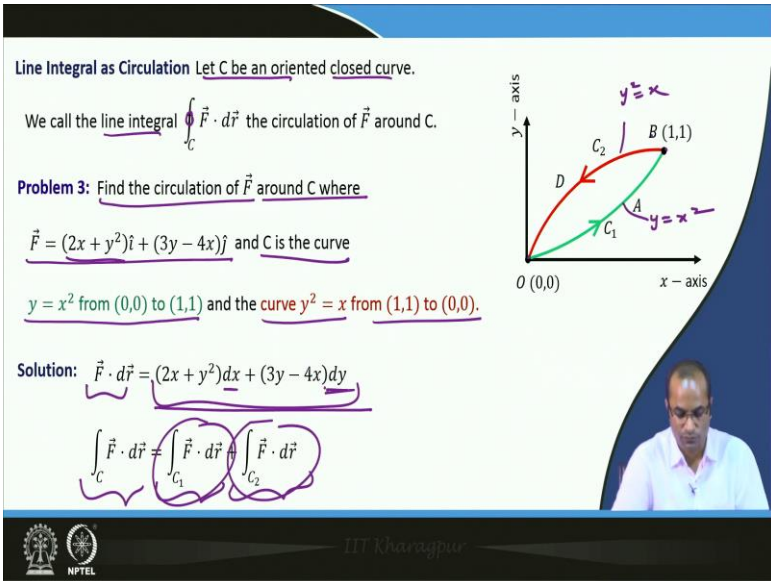

There is another terminology used here the line integral as circulation. So that let the C be an oriented or the simple closed curve. So we call this line integral, this is for the closed integral we usually use. So such a line integral the close line integral is called the circulation of F around C. So it has physical meaning of course, we are not going to the details of that. So, the problem 3 is find the circulation of F around C, where F is given by this and C is the curve. y is equal to x square the parabola from 0, 0 to 1, 1.

And then another curve y square is equal to x, another parabola from 1, 1 to 0, 0. So, we have this curve here given, which is made of these 2 parabolas. And the vector field F is given and you want to get the circulation of a F around C, that means the line integral of this F along this curve C, the closed curve C. So, this is the curve given there we have this parabola y is equal to x square. y is equal to x square and we have their yi square is equal to x another one.

So, this varies from 0, 0 to 1, 1 and then getting back to 0, 0 through the another parabola from 1, 1 to 0, 0. So, the solution so, we need to compute this F.dr. F is given here. So, dr we will write in this dx, dy form. So, this F.dr we can have in this component form as 2 x plus y square dx and so on. So, this line integral we can break into the 2 line integrals over the C one path that is F.dr over C one F.dr over C 2 and F.dr is given by this component.

So, in this case also you do not have to write the parametric equation because there are 2 separate curves there 2 different parabola. So, either way you have to, to convert both of them to different parametric curves and then we can again evaluate each of these integral that is fine or directly since the relation y is equal to x square is given, so, we can also compute without converting into the parametric form.

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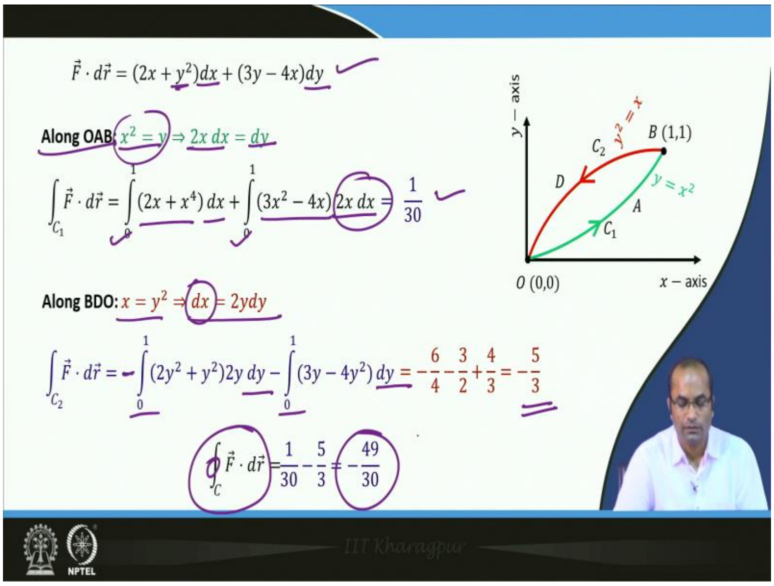

So, F dot.dr is given here and then along OAB. So, we will go along this curve C one, where y is given as x square. So, we can use this relation to compute the relation between dy and dx. So, y is equal to x square means 2 x, dx is dy. So, in one of the 2 integrals which will appear here because dx and dy is for instance the dy we will convert to 2 x, dx and y will be used as x square.

So, then everything will be converted to 1 variable. So, here the dx, the second also for 2, for dy we have use 2 x, dx and then x naturally goes from 0 to 1 in this case and, of course, the same integral so the 0 to 1 and the other one. So here we have 2 x plus x square. Because y square is x 4. So, here are 2 x plus x 4. The second term we have 3 y. So 3 x square minus 4 x and then dy is written as 2 x, dx. So, this simple again one dimensional integral we can compute the values coming as 1 over 30.

Along BDO, if we take the second path now, to get back to the initial point, that is y square is equal to x. So in this case, we have dx is 2y, dy. So in this situation, it is easy to replace this dx y, 2 y, dy and x by y square. So we will do so now, everything will be converted into y, the values or the range for y will be 0 to one again the minus sign because we are coming from y is equal to 1 to 0. So that is written here 0 to 1 with the minus sign. So this we can also evaluate the value will be coming minus 5 by 3. So, if we add the 2 the finally, we have this value of the complete the whole line integral the closed line integral f minus 49 by 30.

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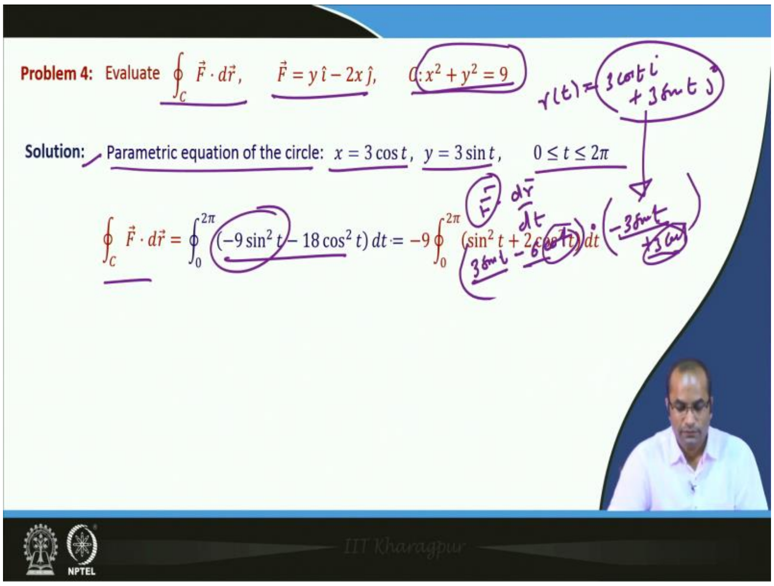

Well so, this is the another problem where we can evaluate this F.dr, where F is given and the curve is also given as circle. x square plus y square is equal to 9. So, the parametric equation we can write down for the circle because here the relation the x, y will be much more complicated to write down in the form of the component this line integrals better to do the parametric equation. So the, we know the parametric equation for the circle x is equal to 3 cos T. y is equal to 3 sine t and the t varies from 0 to 2 pi. So, we have the parameter equation of the curve and then we know that F.dr can be evaluated as F.dr, dt and then dt.

So, this we have to now compute this F.this dr, dt term. So, here this F was given yi minus 2 x. So, the y is like 3 sin t and minus this 2 x, 2 x is 6 cos t and then dr, dt we have to compute because x is given as R is given 3 cos ti plus 3 sin T. So, RP is xiyj. So, 3 cos ti plus 3 sin tj. So, from here we will get the derivative put there. So, the minus like 3 sin t will come and then in this place 3 cos t will come and then the dot products. So 3 and 3 will get this 9 sin square t and 6 3. So 18 we will get a cos square P from here.

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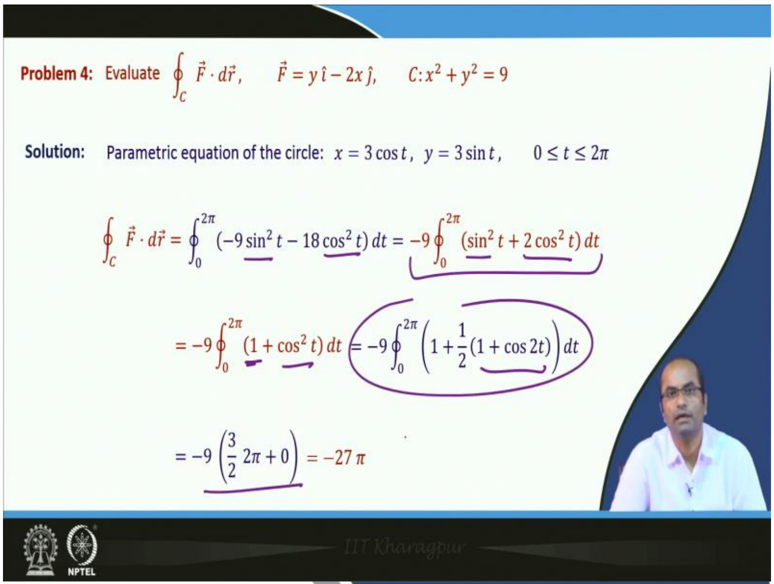

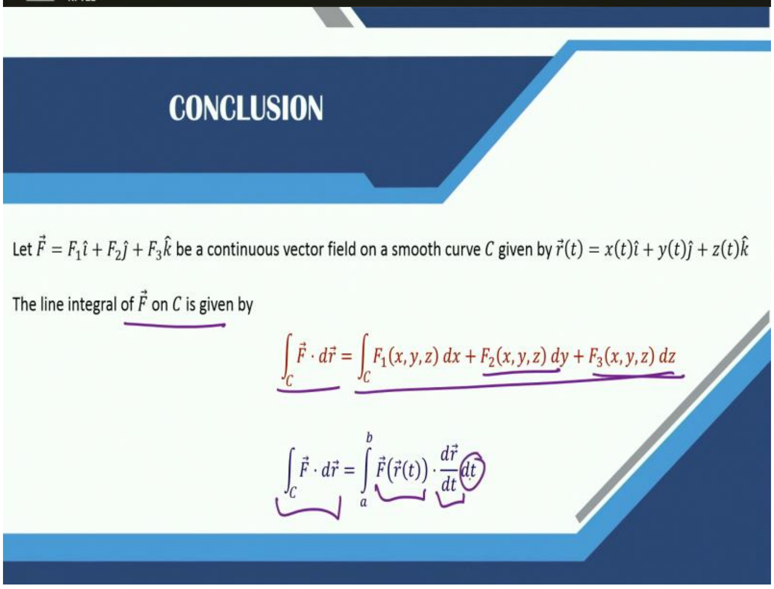

So, this integral we can evaluate as this minus 9 and the sine square t plus this 2 cos square T, which is sine square t plus cos square t we can take as one and then we have cos square t left there, which can be written in terms of this cos 2 t and then if we integrate this integral. So, we will get the value as minus 27 pie.

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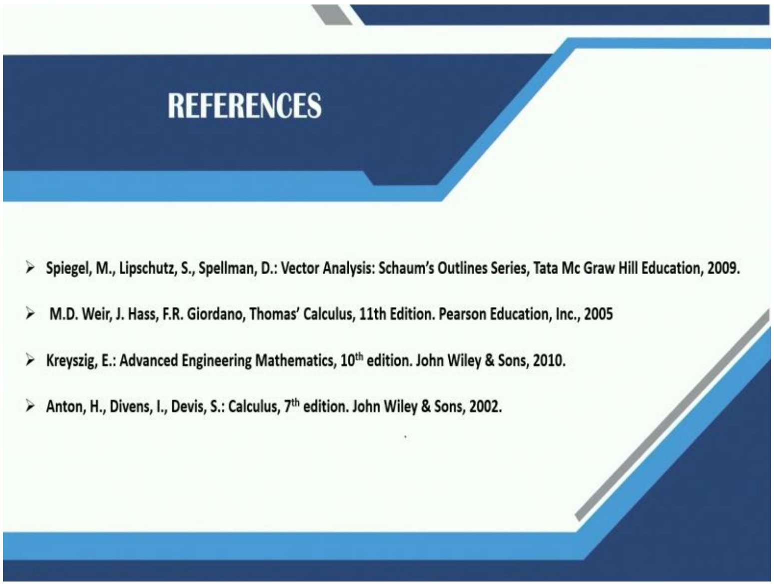

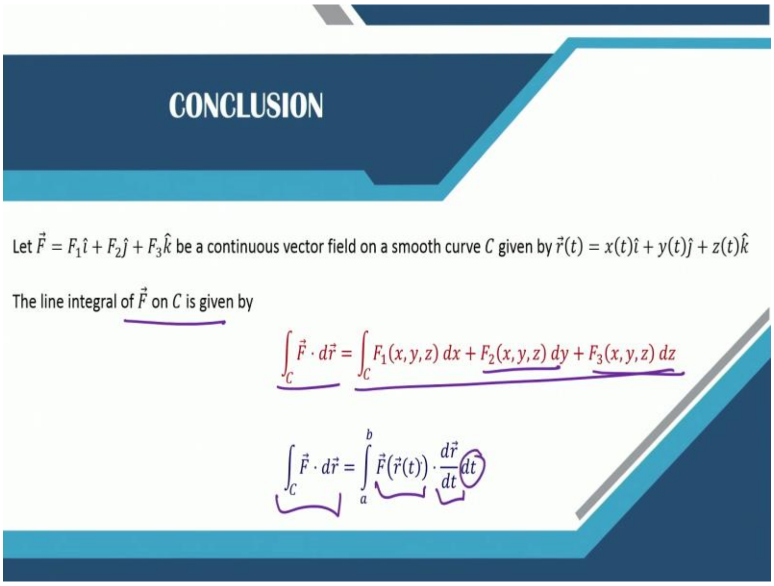

So, these are the references used for preparing this lecture and just to conclude. So, what we have learned? We have learned this line integral of F on a given curve C. So, that can be either evaluated in the component wise as f1 dx, f2 dy, f3 dz and we have seen several examples that how to do that.

The second evaluation if we know that the parameter equation is easy to get then we can simply do so. So the dot product of this F and dr over dt and then we can integrate or T. So, that is another way of evaluating the line integral. So, thank you for your attention.