Engineering Mathematics - II
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Lecture 39
Bessel's Inequality and Parseval's Identity

So, welcome back. This is lecture number 39 on Bessel's Inequality and Parseval's Identity.

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So, we will be talking about the Bessel's Inequality and the other result which turned out to be an identity, the Parseval's Identity and some of their applications will be discussed in this lecture.

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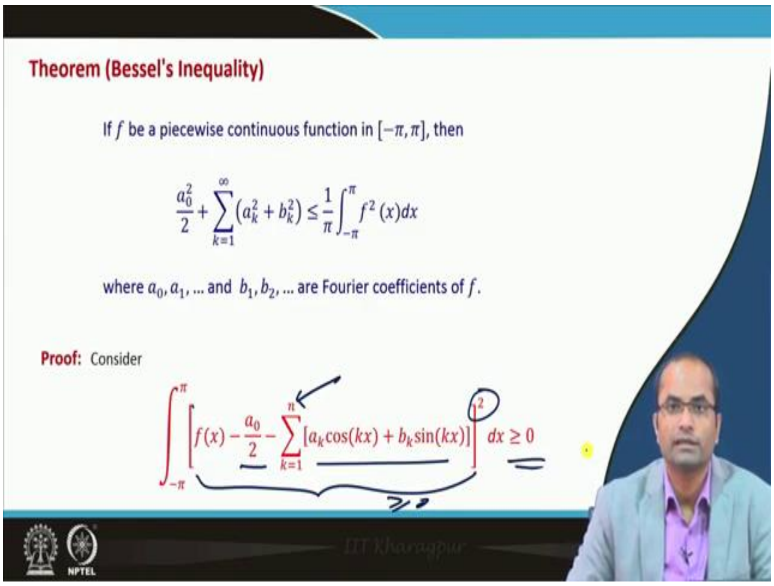

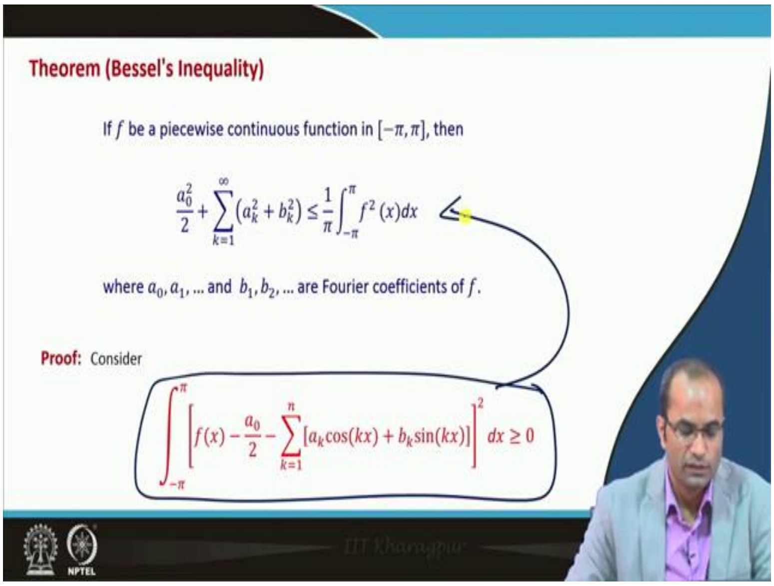

So, the first theorem, we have on Bessel's Inequality which says that if f is piecewise continuous function in this interval minus pi to pi, then we have the following inequality that a naught square by 2 and the summation here, k 1 to infinity ak square then bk square and thus sum here is less than or equal to 1 over pi. And then this integral minus pi to pi f square x dx, where these a's and these b's are the fourier coefficients of f, as the standard notations we have. So, we will look into the proof of this theorem, which is not very involved. So, if we consider this square and then integrate, so what is there in the square?

We have the f x and then the minus is actually the fourier series, the truncated fourier series, where the truncation is done after this n term. So, the summation here is moving from k 1 to this n and then we have the standard fourier coefficients or the fourier series. Here also the first term, the constant term is considered. So, we are taking now this difference and then the square. So, since we have this integrant non-negative. In that case, this is the integral will be greater than equal to 0 for sure. So, out of this inequality, we will see now that how to get exactly this, the so called the Bessel's Inequality.

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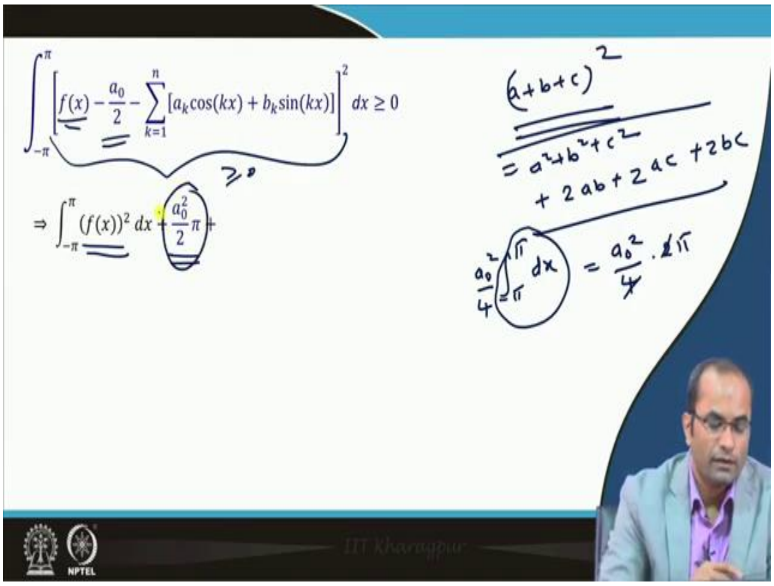

So, having this inequality which is an obvious result or the standard result because of the integrant here greater than equal to 0. So, we will just open the square term because this is the whole square, it is like there are 3 terms, so a plus b plus c and the square. So, we will have this a square then we will have b square, we will have also c square then 2 times ab will be there, then 2 times ac will be there and also 2 times bc term will be there. So, all these terms we have now see in our context. So, the first the f x whole square term that is there, then we will have a square by 4.

So, a square by 4 and then there will be the integral, so minus pi to pi, so integral minus pi to pi and then the a naught square by 4. And then we have dx and from here we will get this 2 pi. So, this is a naught square by 4 and then we have 2 pi and as a result, we are getting a square by 2 into pi.

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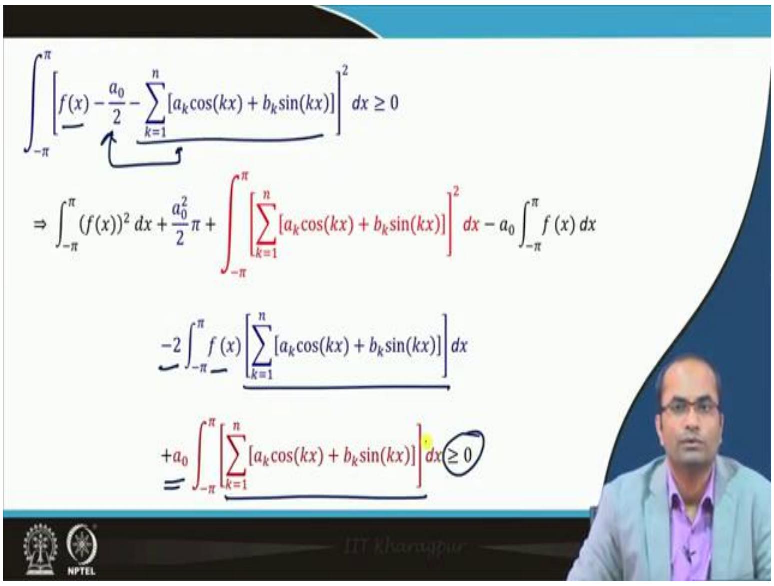

Well, then we have the whole square of the third one, so that is the whole square here of the third one and then the integral form minus pi to pi, then we have, so these are the 3 terms with the square of this, this and this one. Now, we have the products, so first we will consider the product of the first 2. So, f x is multiplied by this minus a naught by 2 and then it has to be doubled, so this 2 will get cancel and we have just a naught f x. So, a naught f x and then the integral, naturally the minus sign will also come because one of them is minus there. Well and then we have the product of f x with this summation.

So, with minus 2 of course, so minus 2 then f x and then this summation is there and then we have at last term the product of the 2, a naught by 2 and the sum there. So, we have a naught and by 2 will be cancelled with the 2 and we have the integral over this summation dx and this everything has to be greater than equal to 0.

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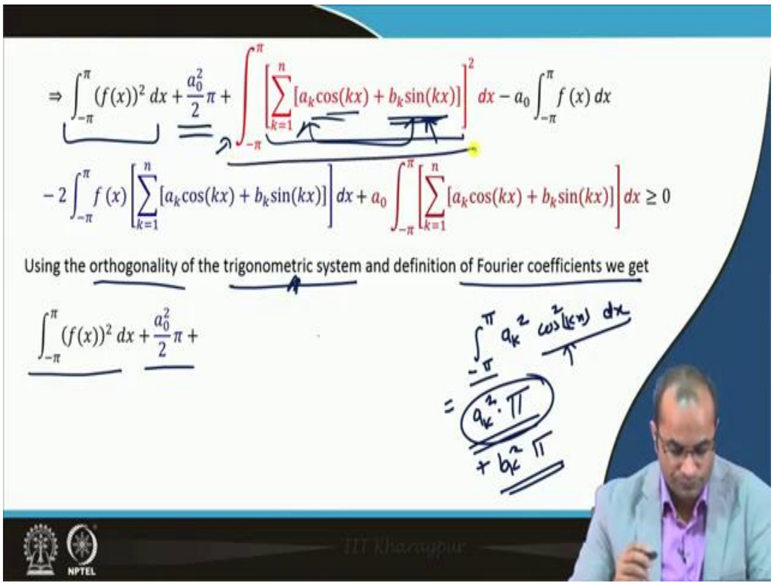

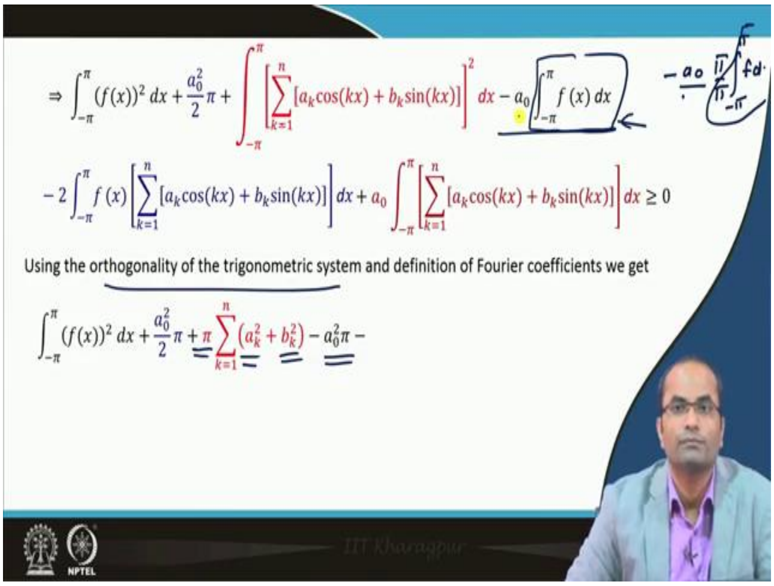

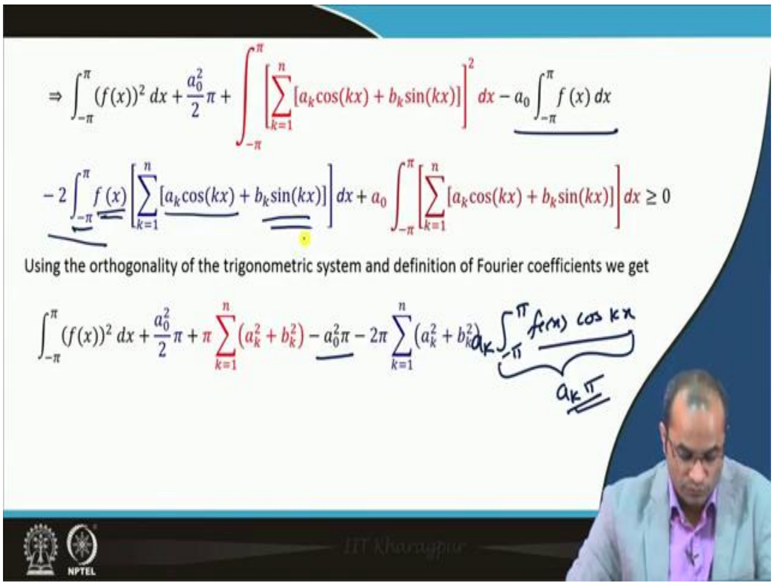

$$\Rightarrow \int_{-\pi}^{\pi} (f(x))^2\,dx + \frac{a_0^2}{2}\pi + \int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right]^2 dx - a_0\int_{-\pi}^{\pi} f(x)\,dx$$

$$-2\int_{-\pi}^{\pi} f(x)\left[\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right]dx + a_0\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right]dx \geq 0$$

Using the orthogonality of the trigonometric system and definition of Fourier coefficients we get

$$\int_{-\pi}^{\pi} (f(x))^2\,dx + \frac{a_0^2}{2}\pi + \pi\sum_{k=1}^{n}(a_k^2 + b_k^2) - a_0^2\pi - 2\pi\sum_{k=1}^{n}(a_k^2 + b_k^2)$$

$$\int_{-\pi}^{\pi} 1\cdot(\cos kx)\,dx = 0$$

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$$\Rightarrow \int_{-\pi}^{\pi} (f(x))^2\,dx + \frac{a_0^2}{2}\pi + \int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right]^2 dx - a_0\int_{-\pi}^{\pi} f(x)\,dx$$

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$$\int_{-\pi}^{\pi} (f(x))^2\,dx - \frac{a_0^2}{2}\pi - \pi\sum_{k=1}^{n}(a_k^2 + b_k^2) \geq 0 \Rightarrow \frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx$$

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$$\Rightarrow \int_{-\pi}^{\pi} (f(x))^2\,dx + \frac{a_0^2}{2}\pi + \int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right]^2 dx - a_0\int_{-\pi}^{\pi} f(x)\,dx$$

$$-2\int_{-\pi}^{\pi} f(x)\left[\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right]dx + a_0\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}[a_k\cos(kx) + b_k\sin(kx)]\right]dx \geq 0$$

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$$\int_{-\pi}^{\pi} (f(x))^2\,dx - \frac{a_0^2}{2}\pi - \pi\sum_{k=1}^{n}(a_k^2 + b_k^2) \geq 0 \Rightarrow \boxed{\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx}$$

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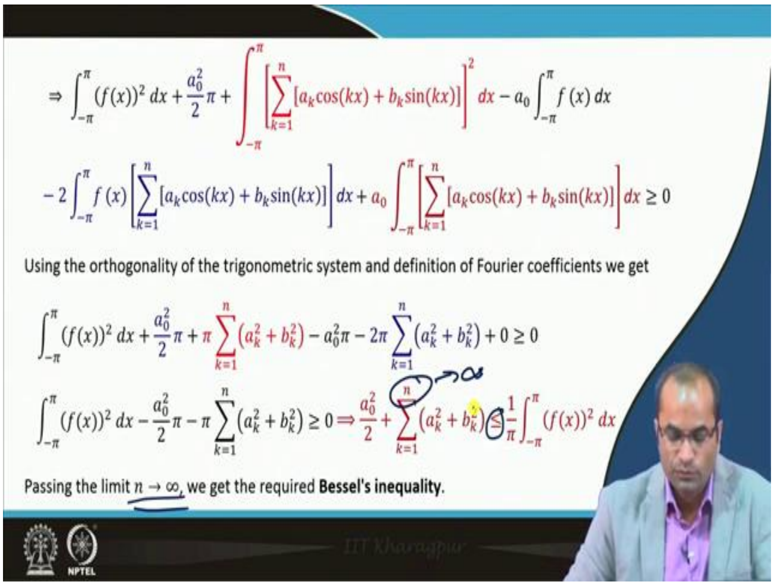

So, having this inequality now, what we will make use of the orthogonality of the, of the trigonometric system and also the definition of the fourier coefficients. So, with the help of these 2, we will be able to now look into the new inequality where many of the terms will disappear because of the trigonometric that orthogonality of the trigonometric system. So, the first term will survive as it is, so we have minus pi to pi f x whole square dx, the second term is already a simplified term, so we have a naught square and by 2 into pi. The third one, so we have again, the squares here of the sum which is sitting inside.

So, this square says that we have the square of each and 2 times the multiplication of the 2 functions. So, what is interesting now, the first we have ak square and cos kx square that is the first term for instance and we have the integral also together. So, the first term, it is like minus pi to pi, we have ak square and then we have cos kx multiplied by cos kx, so cos kx square and dx. And then there will be term similarly, bk square sine square kx and then the other terms will be having the multiplication of cos kx and sin kx. So, let us just first discuss this term.

So, we have ak square and then you remember this in trigonometric system when the candidate is from the same family, the same function is multiply 2 times then the value was pi.

So, we have here pi ak square and similarly from the second square when bk is sin square kx is coming, bk square sin square kx. There also you will get a similar term, that means the bk square and again this pi will be coming. When we have the product, so 2 times the product of ak, bk then cos kx sin kx, in all these terms, we have the 2 functions, one from the cost family other one from the sin family, and this orthogonality says that those term will be 0.

So, we have rather simplified form of this integral now, which only the few terms will survive. One is this pi ak square bk square, this is what we have just discussed. And that summation is of course running and then we have, all others will become 0 because of the orthogonality of the trigonometric system.

And now we are at this term here minus this a naught and if we look at this one now, minus pi to pi f x dx. So, that is the fourier coefficient, a naught we can relate now here and what you will get, a naught square pi, because if you multiply by pi and divide by pi, and then we have this minus pi to pi, and then f dx. So, here, this 1 over pi and the integral that will become a naught, so we have minus a naught square and then 1 pi will be there.

So, that is fine corresponding to this, we are getting a naught square and pi. Now coming to this term here, we have minus 2 and then we have with f x and then this sum and then we have ak cos kx. So, what we have, we have the integral, we have the integral there minus pi to pi, we have f x and we have cos kx. Similarly, we have f x with sin kx. And again, we can relate this to the fourier coefficient because that is exactly corresponding to ak and then pi. So, here also, we will get pi and ak square. Similarly, here we will get pi bk square.

So, these terms, we can have now minus this 2 times pi and ak square and bk square. Then to this last term, we have a naught and then ak this cos kx bk sin kx. So, what will happen to this one and we are integrating this minus pi to pi these term, so, we have for instance this cos kx will be there for the integral this dx. So, this is again trigonometric from the trigonometric system and using orthogonality that be the one and this is a member of the cos family this will be 0. Again, one with the sin x and we integrate that will be also 0.

So, the last term is completely 0 because of the orthogonality and then we have this equality, inequality greater than equal to 0. So, this term and this term they are the same. So, we have minus pi there and here also we have minus a square term, so that can go to the other side.

So, first the simplification, so we have this f x square dx, we have this minus a naught, because this was minus a naught square, so minus a naught square by 2 and then we have minus pi from here ak square plus bk square and the inequality greater than equal to 0, which turns out to be this Bessel’s Inequality now, so we have taken these 2 to the other side and then we have this nice inequality which is relating the function, the integral of the function to these fourier coefficients.

And later on, you will see that indeed this inequality in the limiting case will become equality. So, when we take this limit now, it is exactly the Bessel's Inequality taking the limit and tending to infinity, we will have infinity there and then this is a precisely the Bessel's Inequality. And as I said later on, we will observe in this Parseval's Identity, that this inequality is actually the identities, it is equality not just the less than equal to but this is actually equal to that was proved later on.

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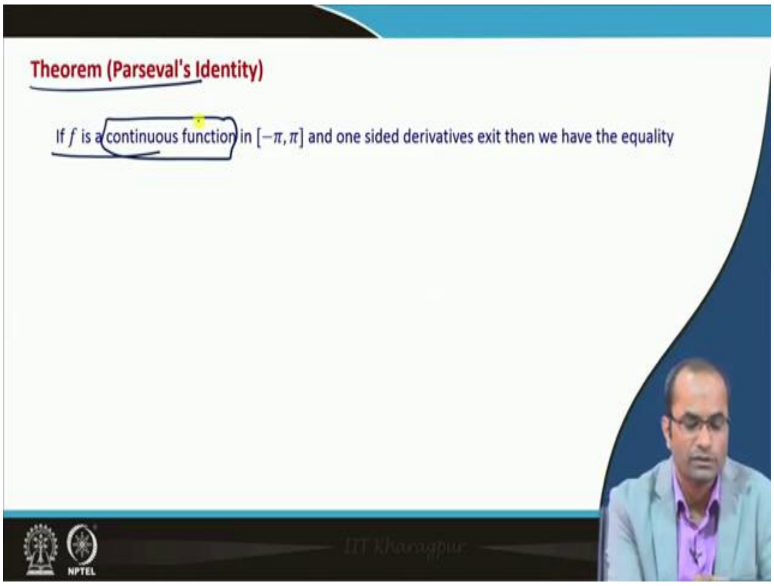

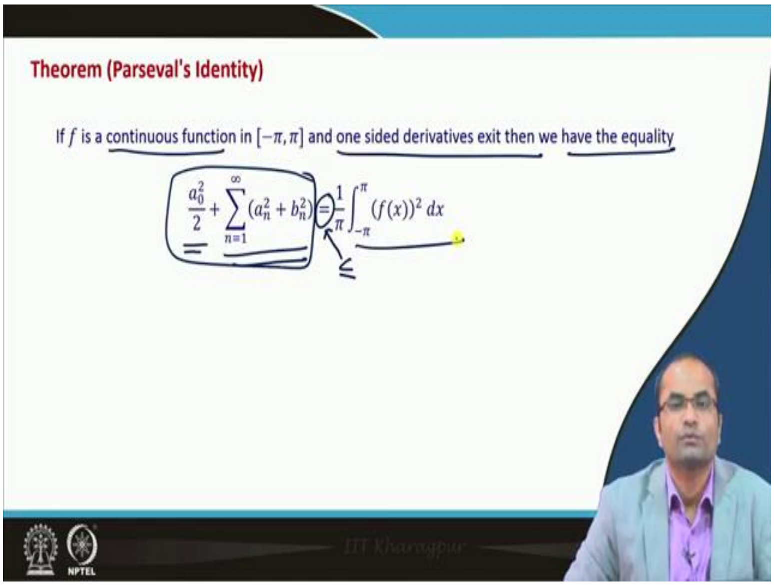

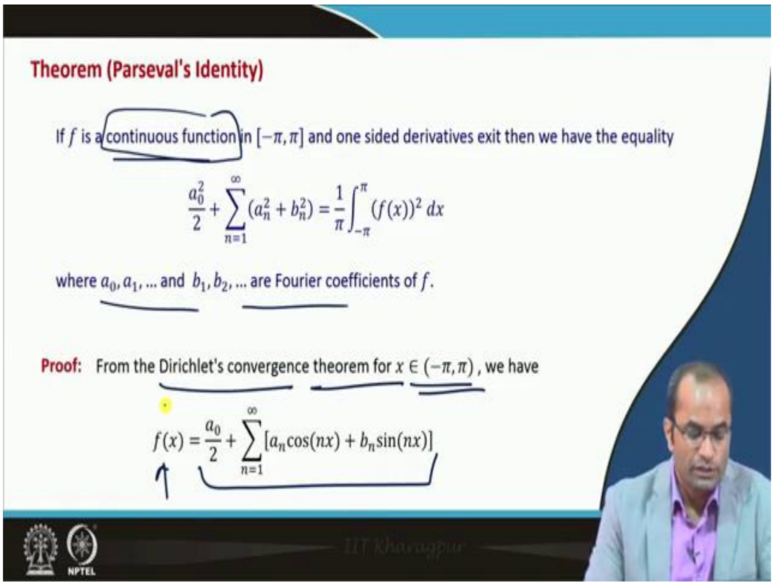

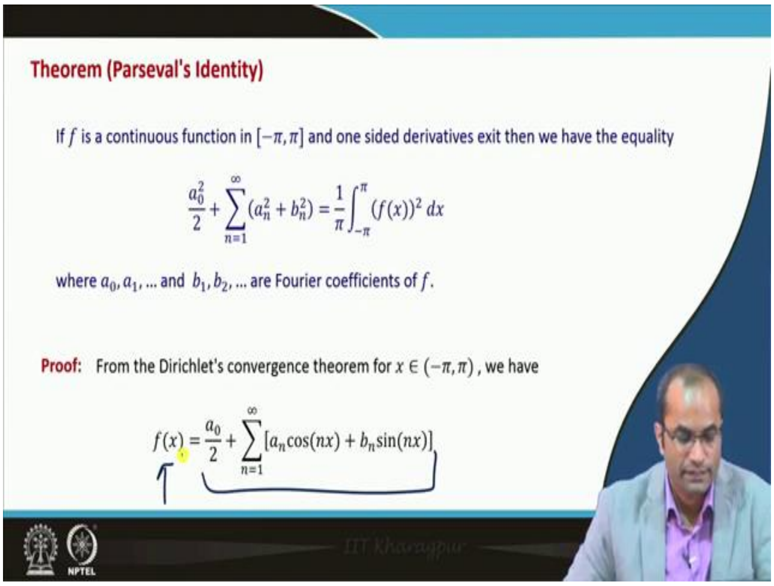

As this result is Parseval’s theorem or Parseval’s Identity says now that, though here we are taking a little more restriction for the sake of proving this result, but we do not have to take additional assumptions here, all the assumptions which we have taken before, like a piecewise continuity that is enough to have this identity which we are going to discuss now.

So, for the simplicity of the proof, we have taken this continuous function now. And one sided derivative exists, then what do we have, we have a naught square by 2, the same term what we have in Bessel’s Inequality, the only difference that inequality is replaced now, with equality, 1 over pi minus pi to pi f x whole square dx.

So, this is the Parseval's Identity which exactly is valid under the same assumptions at this Bessel's Inequality. But here, we have just restricted bit more this function for the simplicity of the proof. And these are exactly the fourier coefficients of the function f. So, from the Dirichlet's convergence theorem and that is the, we have taken these additional restrictions because we can apply the Dirichlet's convergence theorem and in this region from minus pi to pi because the function is continuous one sided derivative exists.

So, we have exactly the equality that means this fourier series converges to this function f x. If we do not take the continuity assumption there then we have to have here the average value and again the results can be proved, though it may be a little bit lengthy. So, here we have this convergence result that the fourier series converges to this f x.

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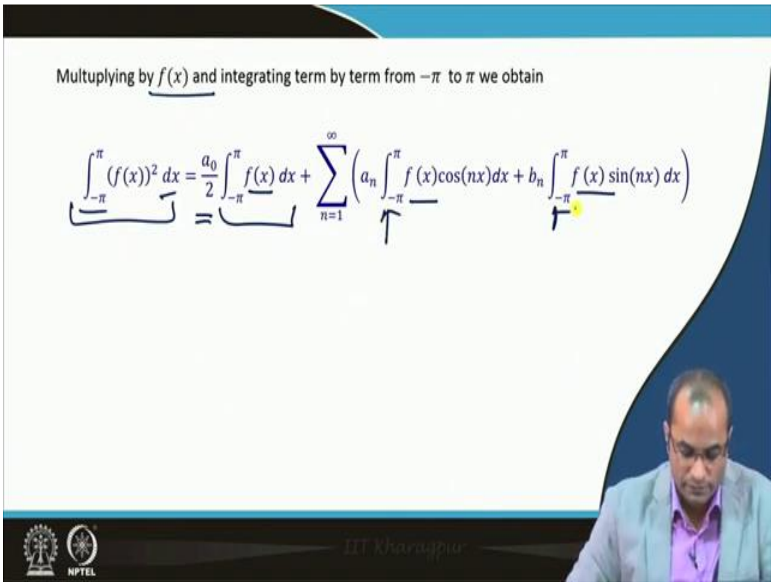

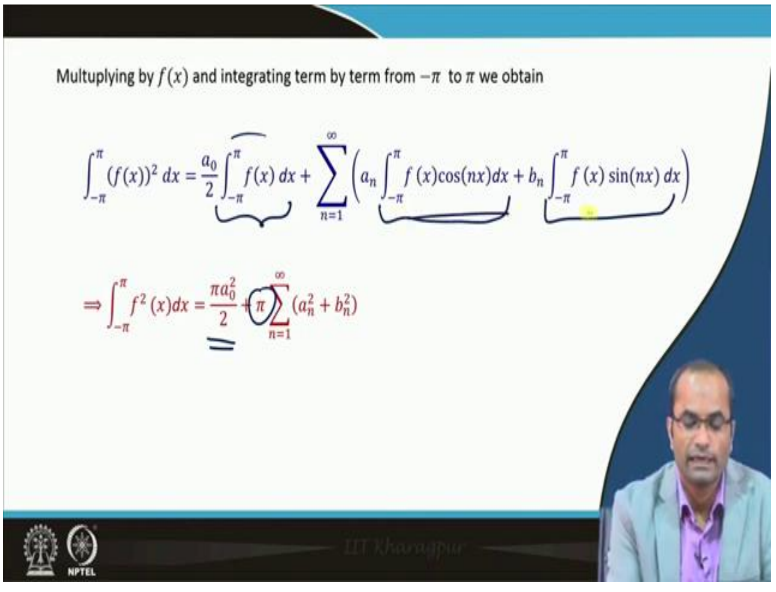

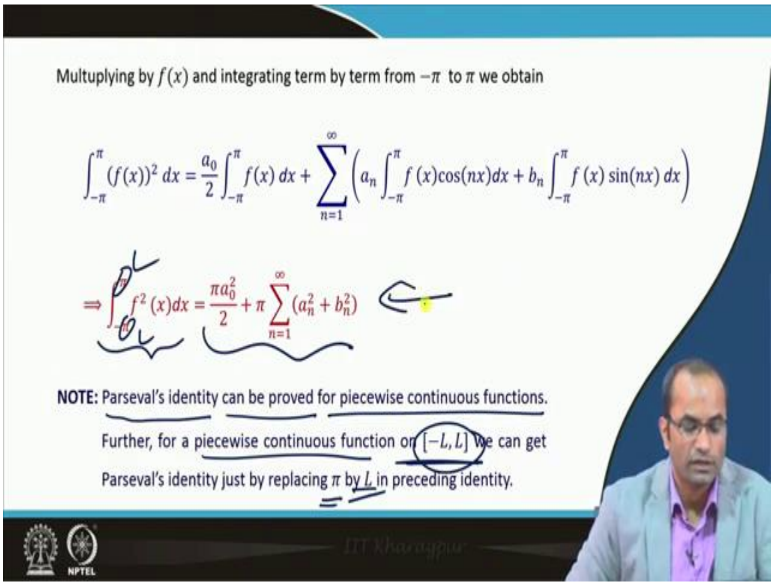

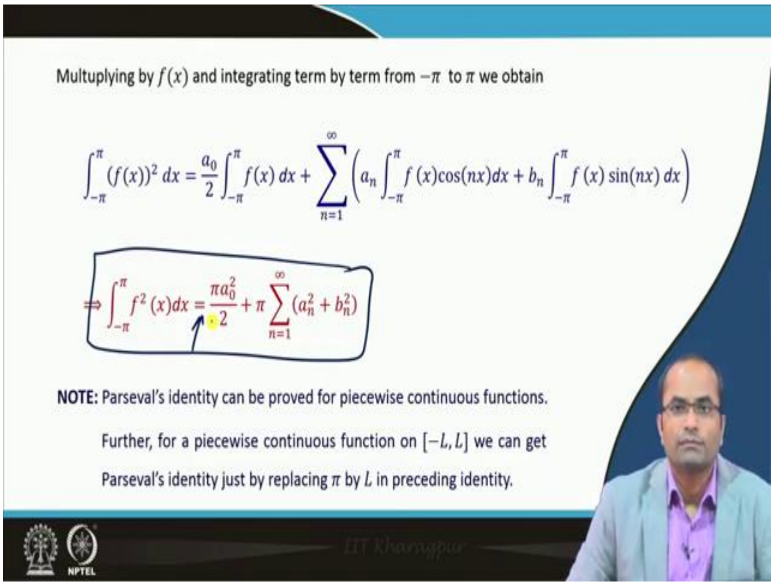

And now, if we multiply this about equality by f x and then integrate them by term from minus pi to pi, so what will happen, the left hand side we will get this f x square and then when we integrate, so we have this expression here. And the right hand side, we have a naught by 2, we have multiplied by f x and then we have integrated, similarly here also we have multiplied by f x and then we have integrated.

So, again we can use the fourier coefficients, the definition of the fourier coefficients here for instance, which 1 over pi, so we have pi and then this will become again a naught. So, a naught square by 2. And then here we have again use the fourier coefficient with this 1 over pi, so the pi will come outside then, this is again an and this has also bn.

So, we have this equality which is just, if we divide by pi, we are getting this the so called Parseval’s Identity. Just a note here, that this again which I have mentioned already that this

Parseval's Identity can be proved for piecewise continuous function, so we do not need any additional restrictions what we have taken here as continuity. And further piecewise continuous function on this interval. So, most of the results, we are considering minus pi to pi again for the simplicity of the calculations, but we can always work with a more general interval from minus L to L.

And we can again, for instance here can get the Parseval's Identity just by replacing this pi by L. So, here will be L and there will be also L and then everything will remain the same.

Well, having this Parseval's Identity and Parseval's or Bessel's Inequality, we can now look for some applications and indeed, this one is more useful, the Parseval's Identity because we have more precise result that we have the equality of these coefficients. So, this can be used in many cases to prove the sum of the series again as we will see in the examples below.

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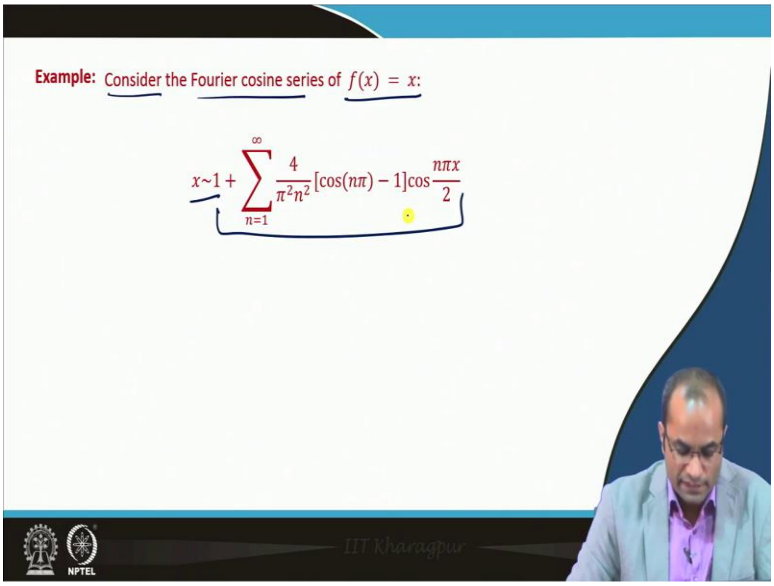

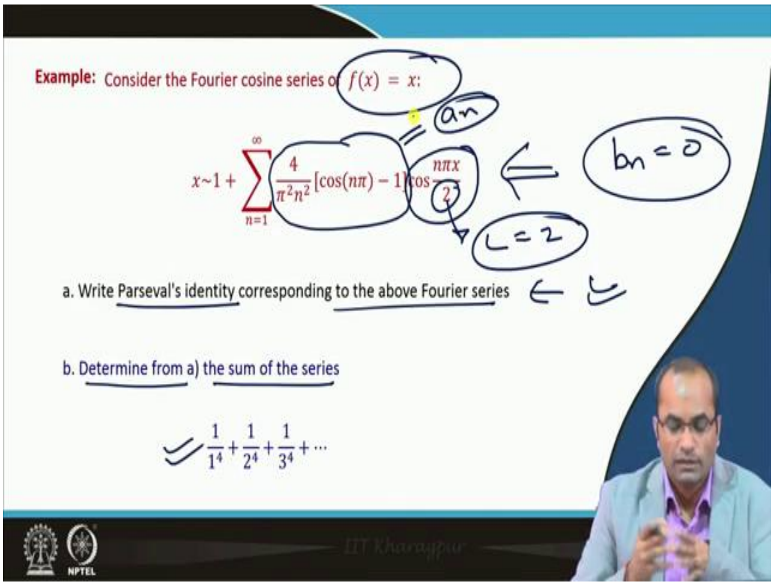

So consider, for instance the fourier series of this function f x equal to x. So, the fourier series is already given for f x is equal to x and we have not mentioned anything else the interval et cetera that we have to extract from the given series. And then, so the first question is that we need to write the Parseval's Identity for this fourier series and the second will be that determine once we have written the Parseval's Identity. We want to determine the sum of this series here, 1 over 1, 4 plus 1 over 2, 4 plus 1 over 3, 4 et cetera.

So, there are 2 tasks. The first one we will write the Parseval's Identity for this given relation, for the given fourier series, where we have to identify that what is the period means L and what are the coefficients. Because for writing the Parseval's Identity, we need the function which is anyway given there, we need the fourier coefficient, which is also clear from here the bn's are 0 because there is no sin term and we have cos and pi x over L. So, again the straightforward the L is 2, L is 2, bn is 0, and then here we have an.

So, without explicitly given in the problem, we can extract from the given fourier series, all these coefficients a ns, bn's and also this interval L.

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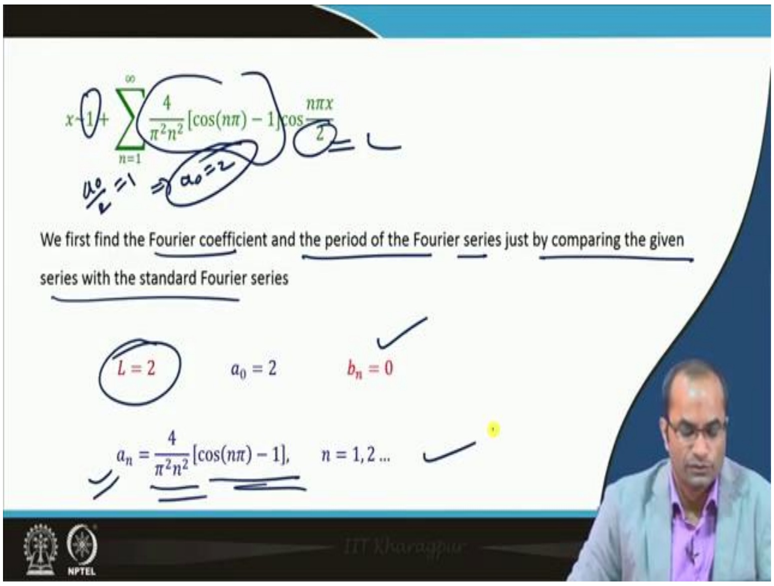

So, having this fourier series, we have to first find the fourier coefficients and the period of the fourier series just by comparing this with the standard fourier series. So, as I discussed already L is 2 then a naught, that will be coming from here because the first term is a naught by 2 and that is given as 1. So, a naught will be 2, the bn's because there is no sin terms. So, all bn's will become 0 and an is directly sitting here as 4 by n square pi square cos nx minus 1 and then in is going from 1 to n, so on.

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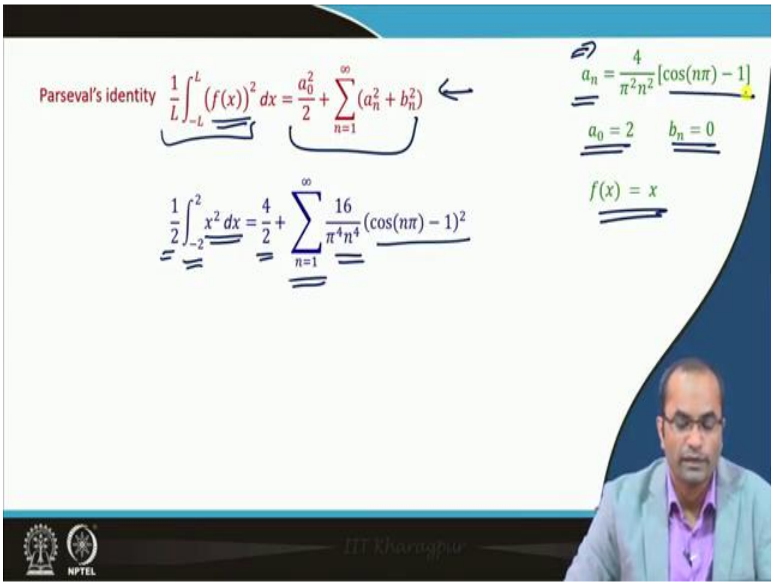

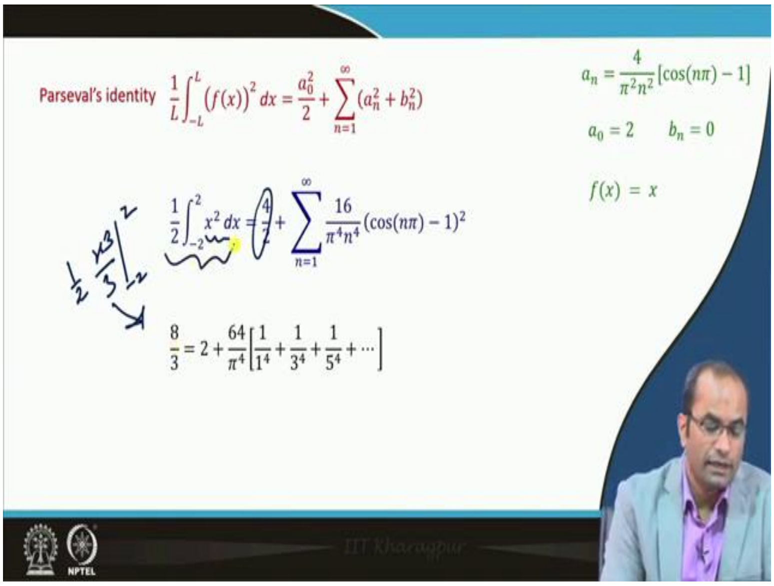
Parseval's identity $\frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$
$a_n = \frac{4}{\pi^2 n^2}[\cos(n\pi) - 1]$
$a_0 = 2 \quad b_n = 0$
$f(x) = x$
$\frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{4}{2} + \sum_{n=1}^{\infty}\frac{16}{\pi^4 n^4}(\cos(n\pi) - 1)^2$
$\frac{1}{2}\frac{x^3}{3}\Big|_{-2}^{2}$
$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$
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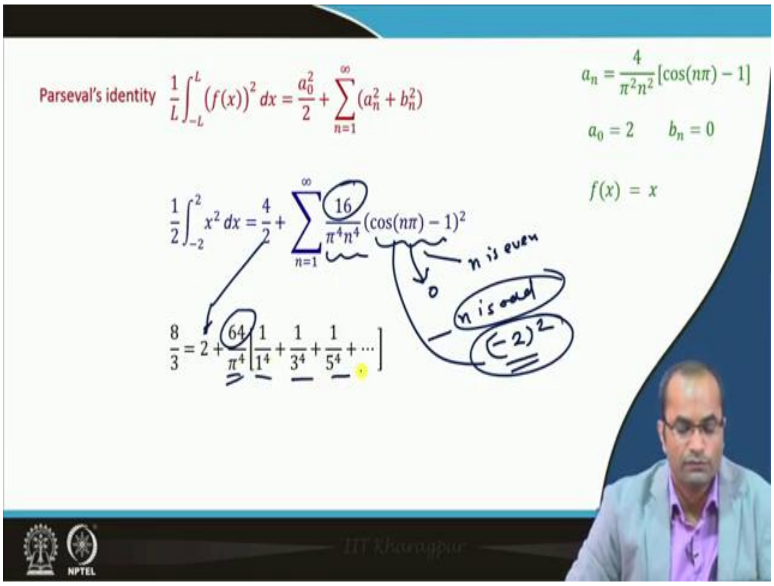
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n is even
0
n is odd
$(-2)^2$
$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$
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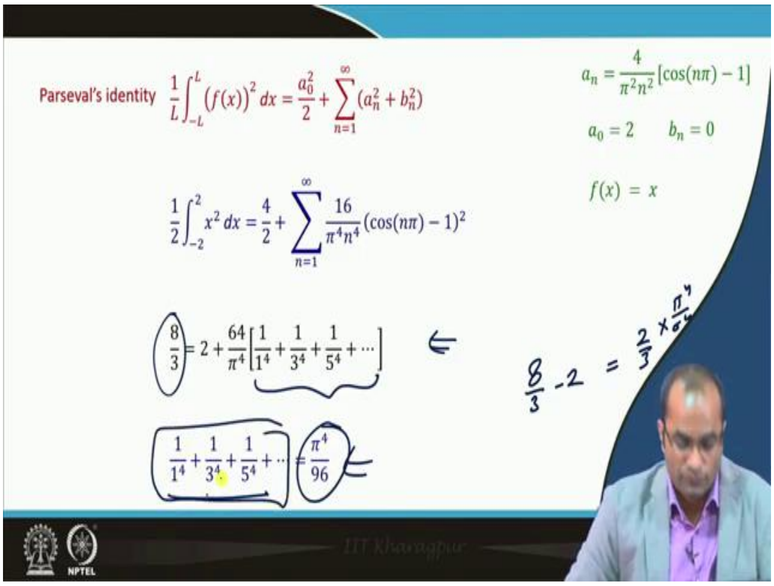
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$\frac{8}{3} - 2 = \frac{2}{3} \times \frac{\pi^4}{64}$
$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$
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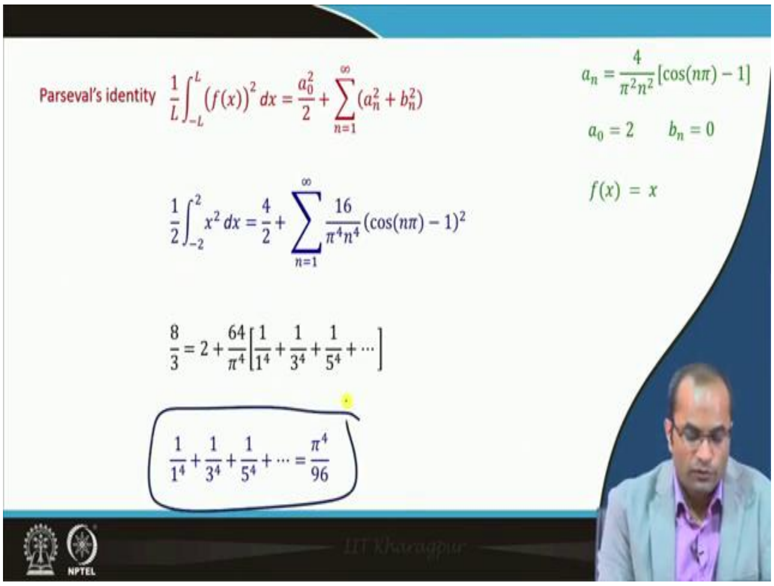

So, having the knowledge of these coefficients, we can apply the direct result of the Parseval's Identity. So, we have the an, we have a naught, we have bn's, and we have f x, this is all we need for writing the Parseval's Identity. And remember, this is the Parseval's Identity now, 1 over L minus L to L, f x square dx and the right hand side where these coefficients are summed up. So, then 1 over 2, minus 2 to 2 and then we have x square dx, right hand side we have a square.

So, the 4 by 2 and then the summation n, 1 to infinity and we have the 16 over, so an square that is 16 over pi square and n4 and then this cos n pi minus 1 square and the bn square, so 0.

So, this is the Parseval's Identity which we can simplify a bit more. But here now, x is square, so the integral is x cube by 3 and then we have minus 2 to 2 and then half is also sitting there. So, then we can have, we can simplify this and we will get this 8 by 3, the number, the left hand side.

And the right hand side, we have then 2 coming from here and then we have this, so when n is even, when n is even this will become 0, this will become 0. And when n is odd, when n is odd this will become minus 2 whole square that is meaning 4. So, only these odd terms will survive and that to with minus 4 factor so, 64 will be appearing there, pi square we can get it outside and then we have this n4 which as a result, we are getting 1 4, 3 4, 5 4, et cetera.

So, this is coming directly from the Parseval's Inequality that 1 over 1 4, 1 over 3 4, so these odd terms are adding up to, so here we have 8 over 3 then we have minus 2, so 8 over 3, then minus 2, so we have 2 by 3 and then this will be multiplied with this 64 pi 4 by 64. So, that will give us this pi 4 over 96, the sum of this series having these odd terms with the power 4.

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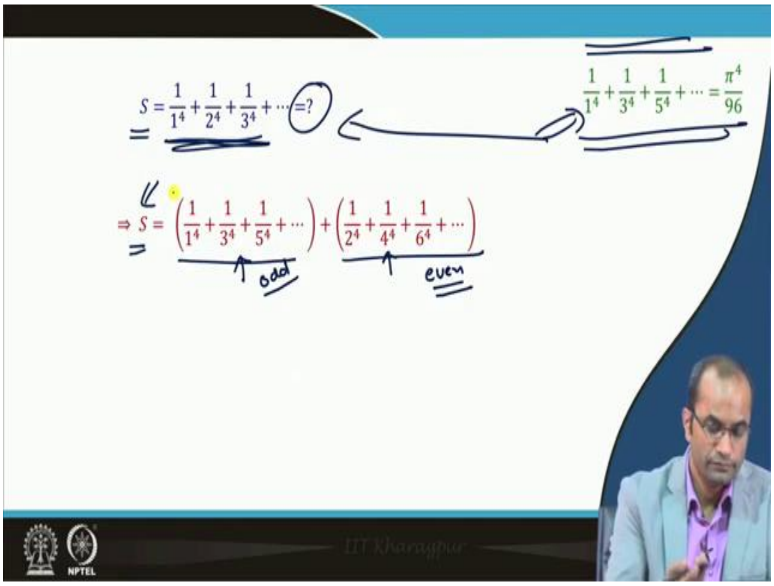

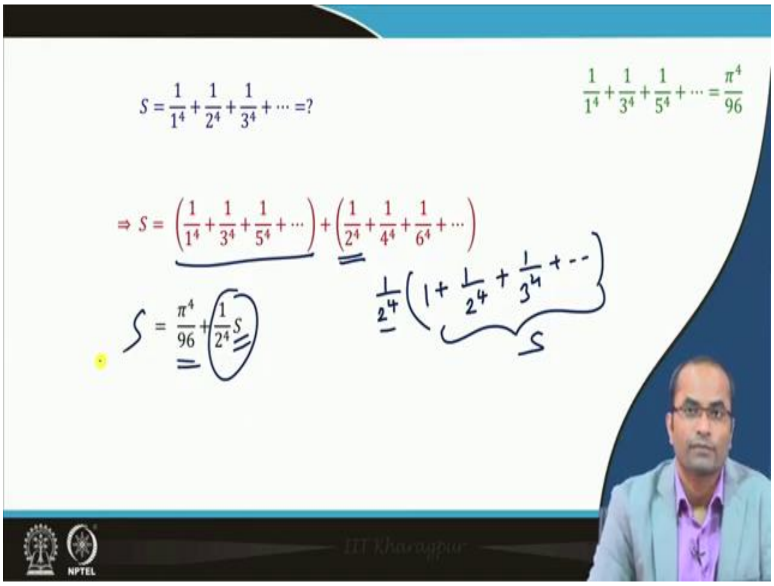

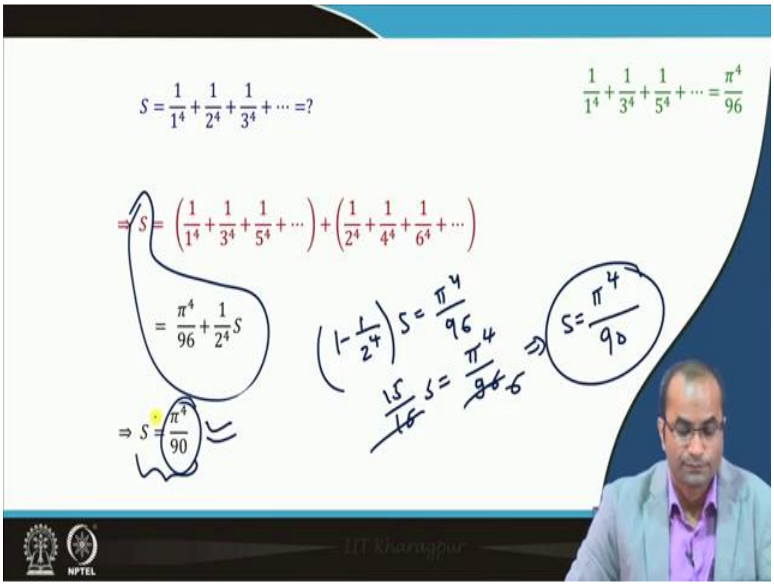

The aim was to get, so this is what we have now from the direct result from the Parseval's Identity. But in the question, this is asked that what is S here, 1 over 1 4, 2 4, 3 4, et cetera, the question is what is the sum of the series? So, with the help of this, we have to now get the desired series or sum of the desired series.

So, here the trick is that if this S, S is this complete sum, we can break into, we can split into 2 portion, one having only the odd terms. The other one is having even terms. So, these all are even terms, here we have the odd terms, 1, 3, 4, 5 et cetera, here we have all the odd terms, which is adding up to this complete S.

Not that this we know already that this is pi square by 96. And if we take common from here 1 over 2 4, so what we will get, 1, the first term, then again here 1, 2 power 4 and then 2, so, 1 over 3 power 4 and so on. So, this we will become again the S which is written here 1 over 2 4 into the sum S, which we can simplify now, to get this S from here. So, S will be coming as, so because the left hand side we have 1 minus 1 over 2 power 4. So, S is equal to pi 4 by 96 and here we have 16.

So, 15 by 16 S and then pi 4 divided by 96. See here we have 6 then. And that gives us S is equal to pi 4 divided by 90. So that is the result, we have here for the sum of the series given in the question, and the value is pi 4 divided by 90.

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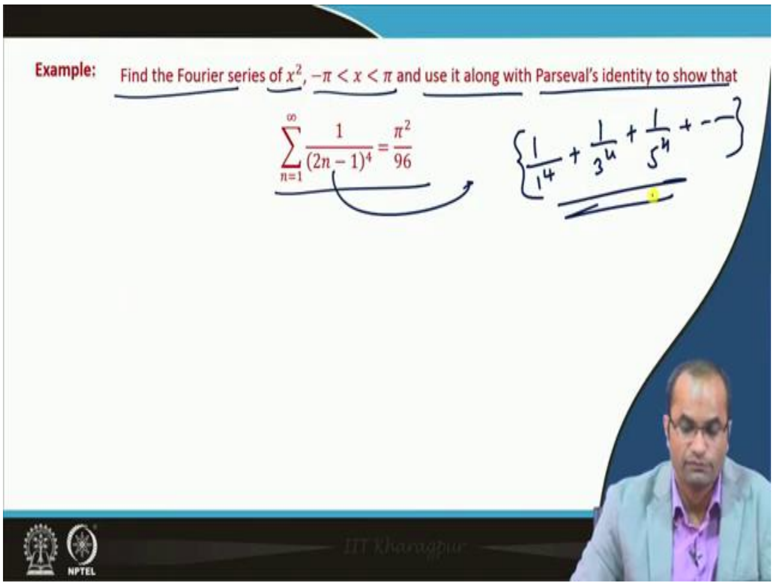

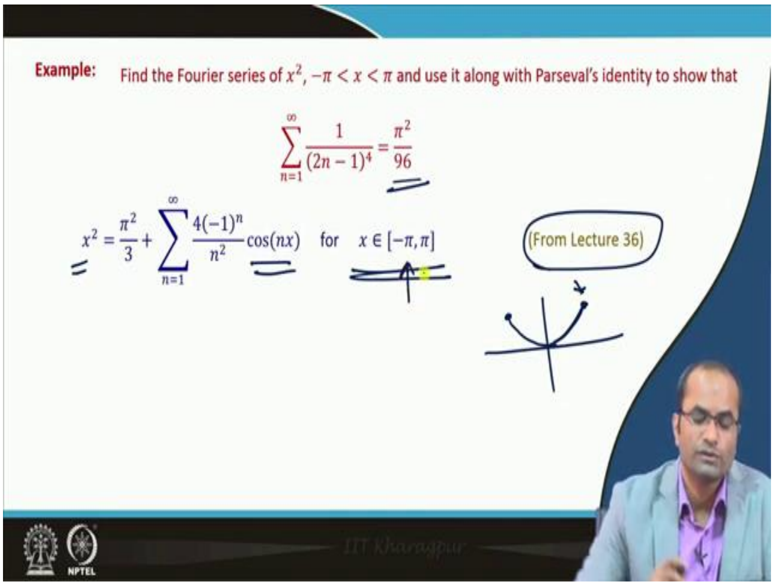
Example: Find the Fourier series of x^2, $-\pi < x < \pi$ and use it along with Parseval's identity to show that
$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^2}{96}$
$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$ for $x \in [-\pi, \pi]$
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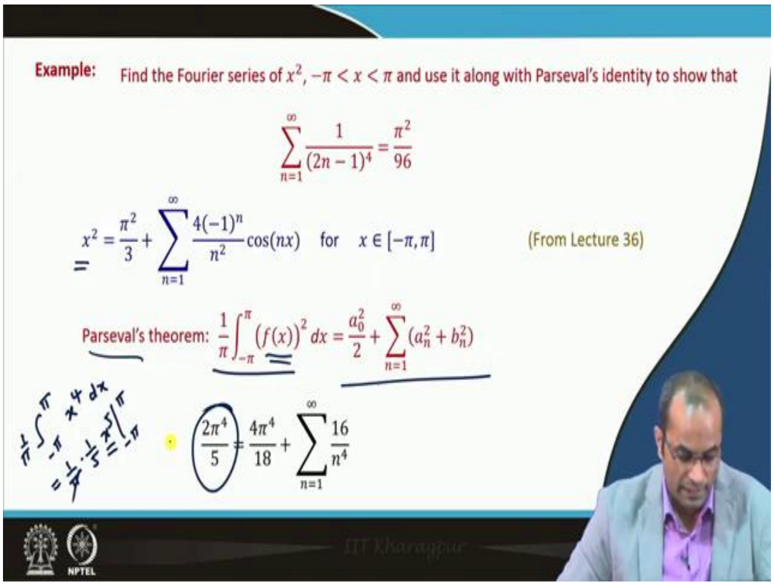
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Parseval's theorem: $\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$
$\frac{2\pi^4}{5} = \frac{4\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16}{n^4}$
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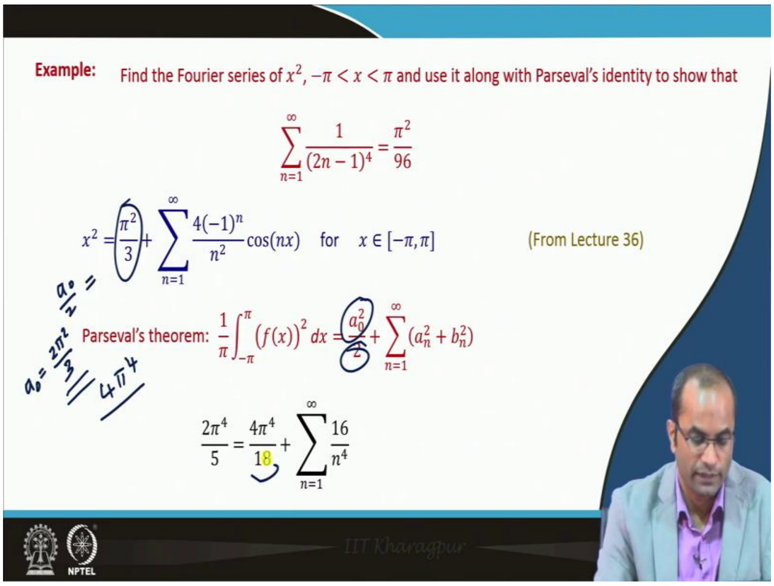
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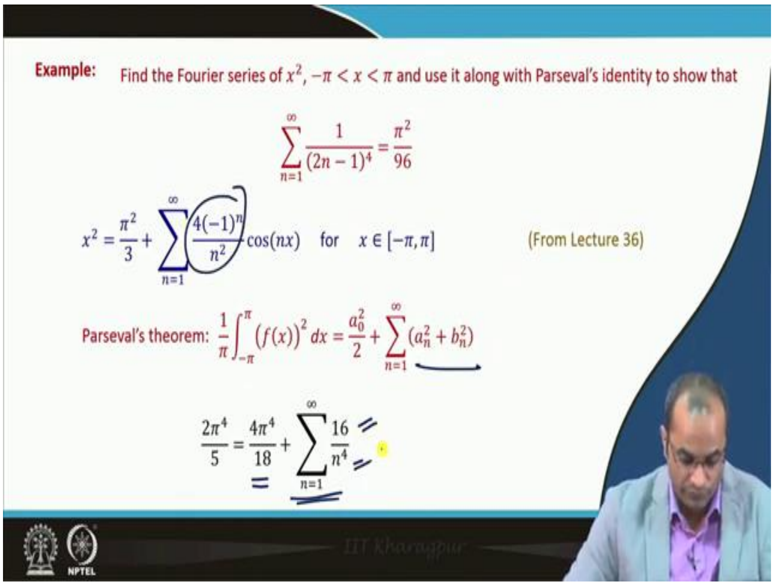

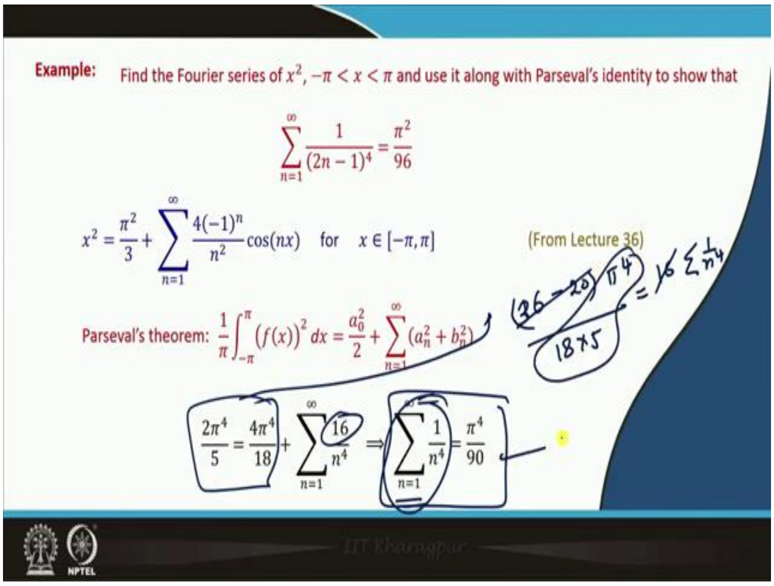

Well, so we will do one more example, that is the last example now. So, find the fourier series for this function x square given or defined in this interval minus pi to pi and use it with Parseval's Identity to show that the sum here is pi square by 96. So, what is the sum here again, so 1, the first term is 1 over, so 2 minus 1 so 1 4, then we have the second one, 1 over 3 4 and then 1 over 5 4 and et cetera. So, that is the sum which was already derived in previous case, but now in the different context, we will find again this sum as pi square by 96.

So, we have already done this fourier series in the previous, one of the lectures here lecture number 36 and that the fourier series of this x square which is even function, the fourier series will have only the cosine terms and for all x from minus pi to pi including the endpoints because this function x square was having this structure and the values at the endpoints are also equal. So, it will converge everywhere.

And then we have this x square equality for the fourier series and now we can write down the Parseval's Identity. So, the Parseval's theorem says that this will be equal to these relation of the coefficients. And then here, we have x square, so that will be the left hand side we have 1 over pi, and we have minus pi to pi x 4, x square and then square again, so x 4, so this is 1 over pi, and then we have 1 over 5 and x5 from minus pi to pi. So, which will be 2x 5 and this pi will 2 pi power 5 this pi will get cancels. So, we have 2 over 5 and x power 4.

The right hand side, we have a naught square by 2, so a naught by 2 is given already here. So, the a naught is 2 pi square by 3. So, if we square it, so we have 4 pi 4 over 9 and then 2 here, so 18, so that is also fine, the second term. And then we have the summation of an square and the bn square. So, the bn squares are 0 here, we have only an squares, so that is 16 over n 4.

So, that is the result, we have directly from the Parseval's theorem, which gives us now, the 1 over n4, the summation n 1 to infinity is equal to, we have to subtract here and then divide by 16. So, what is the result coming now just we can check again. So, it is a 16 minus 20 with pi 4 and divided by 18 into 5 and the right hand side we have 16 there and then the summation over n4. So, here we have 36 minus 20. So, 16 this is also 16 and the summation 1 over n4 is adding up to pi 4 over 90.

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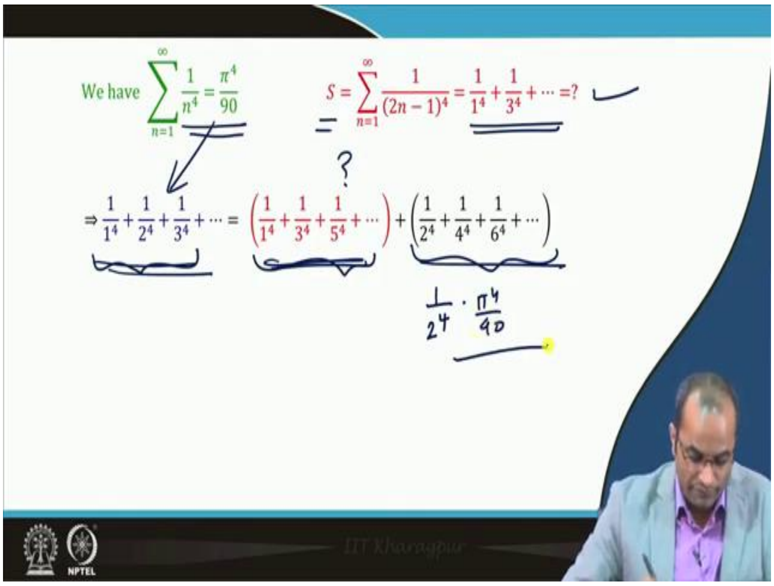

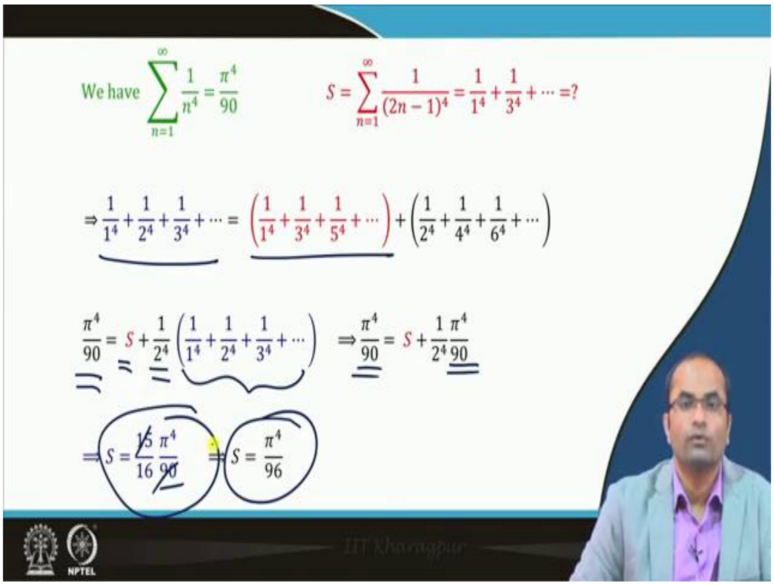

So, we have that 1 over n 4 is equal to pi 4 over 90, that is the direct result from the Parseval's Identity we got. And our interest here is to get this summation which is 1 over 1 4, 1 over 3 4 and so on. And, we will use the similar trick which was used earlier that the whole series can be splitted into 2 having these odd terms and then the even terms. In this scenario, we have all already this sum known, this we want to get and this we can again convert by taking this 1 over 2 4 into this given summation again which is pi 4 by 90.

So, having this now we can have this first of all pi 4 by 90, then this is the desired one and we take the common factor here 1 over to power 4 and then we have this series where again the sum is given as pi 4 over 90. So, here also pi 4 by 90 then we can subtract it and we will get directly this S, 15 by 16 pi 4 by 90 and again it can be cancelled out and to get this pi 4 by 96. So, that is the sum. Again, using this Parseval's Identity, we can do many more complicated sums.

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So, these are the references we have used now for preparing this lecture.

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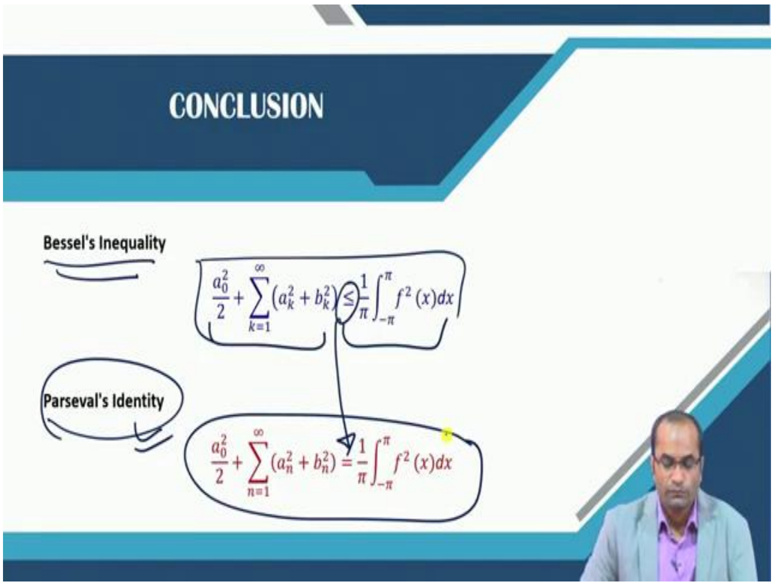

And just to conclude, we have discussed the Bessel's Inequality first, which was a trivial result followed by the integral, the squares which is always non negative. And then, with simple calculations, we obtain this inequality which is known as the Bessel's Inequality. The later on, what we realized that this equality under the same conditions can be set to equality and the name of this identity is the Parseval's Identity, which has several applications some of them we have seen for computing the sum of various series. So, that is all for this lecture, and I thank you for your attention.