# Fourier Series and Integral Transforms

**Professor Jithendra Kumar**
**Department of Mathematics**
**Indian Institute of Technology Kharagpur**

**Lecture 36**

## Fourier Series for Even and Odd Functions

So welcome back to lectures on Engineering Mathematics II. And this is lecture number 36 on Fourier Series for even and odd functions. So in this lecture we will first discuss what are the even and odd functions, again. And then the evaluation of the Fourier coefficients and hence the Fourier Series for even and odd function.

We have already discussed in one of the earlier lectures that, if the function, the given function is even or odd then the corresponding Fourier Series becomes simpler. So it will have either only sine terms or cosine terms.

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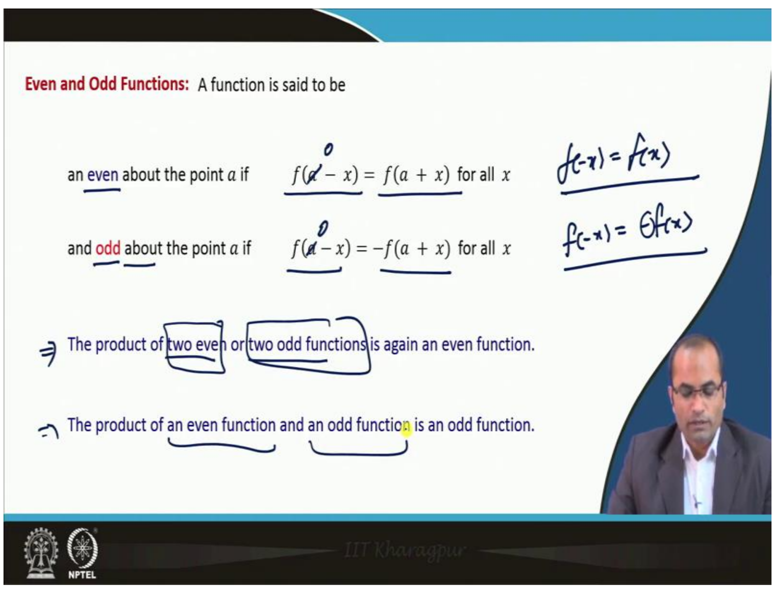


So now coming to the even and odd functions. So a function is said to be an even, an even function about the point a if we have this condition that if a minus x is equal to f a plus x for all x. Then we call that the function is an even function. On the other hand we will call the function an odd function about the point a, if fa minus x is equal to minus fa plus x and that is for all x. So that is the odd function.

So usually when we, in many examples we will see that this a is just 0. So what we have that f minus x is equal to f x for even function and f minus x is equal to minus f x for odd functions, when a is 0. So when we are talking about x equal to 0, so the y axis. So there is a property of this even and odd function so the product of two even or two odd functions is again an even function, because of that minus sign. So minus if multiplied by minus is again plus.

So this property holds for if we take 2 even functions, their product will be also even. Or if we take 2 odd functions their product will be also an odd function. So that property exactly will be used for evaluating Fourier coefficients. And moreover the product of an even function and an odd function, so if we have in the product one function is even and the other one is odd, in that case the product will be again an odd function. So having these properties of even and odd functions we will now evaluate or simplify the Fourier coefficients for such functions.

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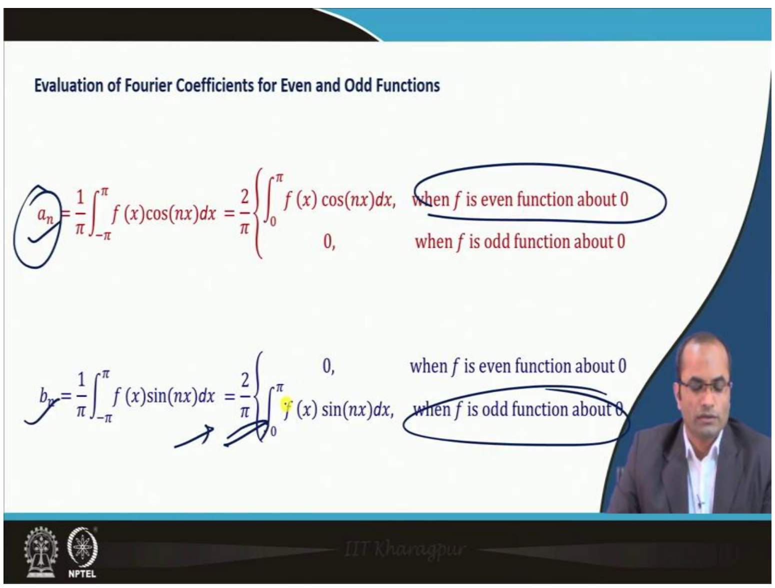


So suppose we have this an given by this integral 1 over pi, minus pi to pi f x cos nx dx. Just for simplicity again we are considering this minus pi to pi. But we can also talk about minus L to L and then cos n pi x over L and the outside term will be 1 over L. So just for simplicity we are talking about the interval minus pi to pi.

So if this function fx is an even function, so what will happen the cos is an even function, so the cos here is even. And if this function fx is even function then the product is again even. And in that case the integrant here the whole integrant is an even function. So minus pi to pi this integral can be written as integral 0 to pi and then 2 times. So the 2 over pi will come outside. And then we have this integral for when f is even.

And if f is odd, so if f is odd and then we have even, so the product is again odd and in that case since the integral is minus pi to pi, then this will be 0. So if f is an odd function about zero this integral will become zero, meaning the ans will become 0. And if f is an even function, then those an we can simply compute with the help of this rather simplified integral. So now integral is varying from 0 to pi only instead of minus pi to pi.

Now similar situation happens for bn as well, so if this fx is even for instance and the corresponding sine is an odd here. So the integrant is odd and in that case this integral will become zero. So when f is even the integral for bn or the coefficients, the Fourier coefficients bns will become zero. And the other one will be simplified the 2 time and then 0 to pi fx sin nx dx.

So what we have here that if a function is odd or the function is even, then the corresponding Fourier Series either will have only the bns terms or it will have only the ans terms. So when f is even we will get only the ans, the coefficient ans, and when f is odd we will get the coefficients bn. And the correspondingly the Fourier Series will be simplified because in case of the even when we have only the ans there, so we have only the cos term in the Fourier Series and when the function is an odd function.

So we will have only bns there. And hence the Fourier Series contain only sine terms. So that is this simplification for even and odd functions, we are going to discuss now in context of the Fourier Series.

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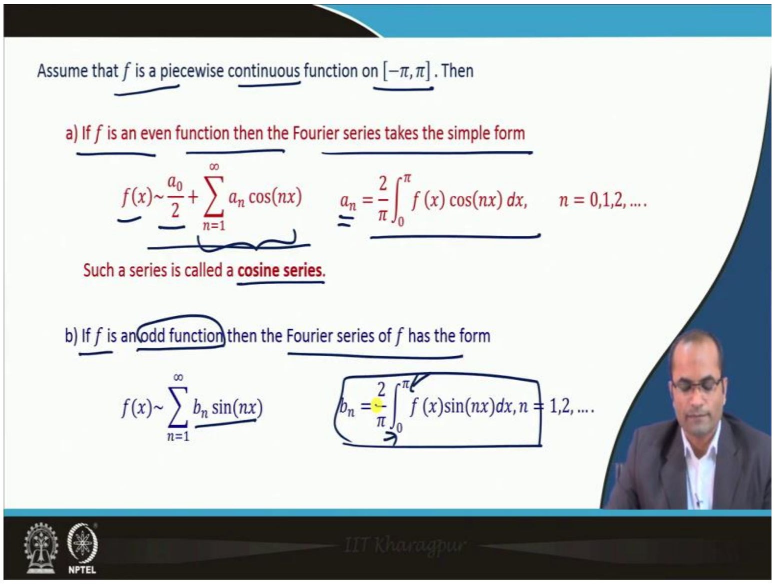


So we assume that f is a piecewise continuous function on this interval minus pi to pi. Again for simplicity we are taking minus pi to pi. All these results can be extended for a general integral minus L to L. So in that case if f is an even function then the Fourier Series takes the following simple form. So if it is an even function we have only the ans the Fourier coefficient ans will survive the bns will become zero.

And therefore in the Fourier Series we will see only the cos term. So the Fourier Series will be simplified to this a by 2 and having only the cosine terms, where this an can be computed with this integral 2 over pi, 0 to pi fx cos nx dx. And such a series is called cosine series because we have only the cosine terms in the series and therefore we call such a series as cosine series.

If f is an odd function, so if f is an odd function then the Fourier Series will be simplified again and since this f is odd the bns will survive, all ans will become zero. That means in Fourier Series we will have only the sine terms, that means n goes 1 to infinity and bn sin nx will be the Fourier Series. And the bn we can compute with this help of other simplified formula, with where the integral goes from 0 to pi and there is a factor 2 there. So 2 over pi, 0 to pi and fx sin nxdx.

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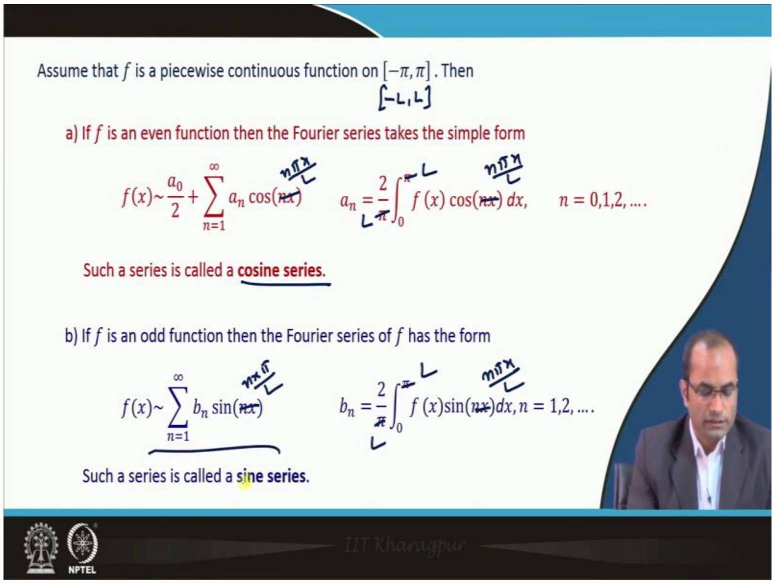


So these results can be also considered when we have minus L to L. For example the interval. So here this nx will be simply n pi x over L and this integral here instead of pi we will have just L and again here nx will be n pi x over L. So this will be more general formula so again the cosine series for the sine series it will take the form, so n pi x over L here and then the L and then we have L there. And again here for nx we have n pi x over L. So these changes will be made and then we have more general Fourier cosine or Fourier sine series for the functions which are odd and even respectively. So well such a series is called a sine series.

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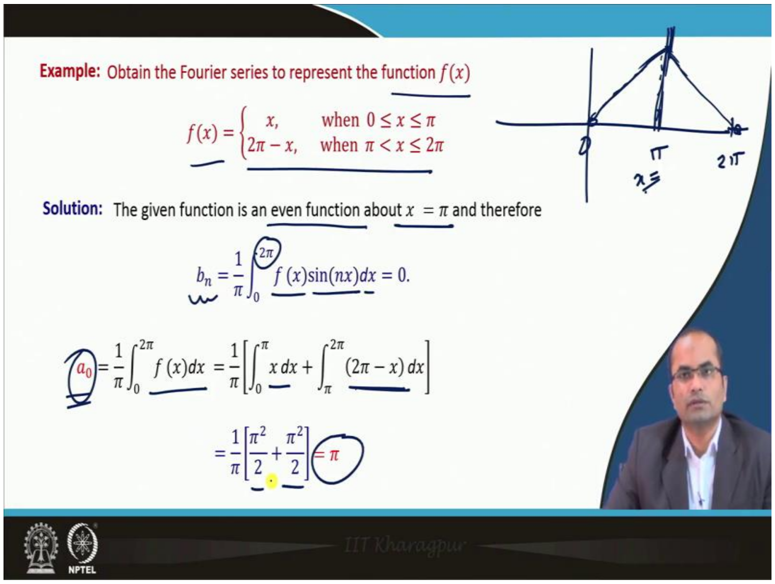


So now we can discuss some examples based on this simplification. So for instance we want to represent this function here fx which is given by x and 2pi minus x in the range pi to 2pi. And x is in the range 0 to pi. So if you look at this graph of this function 0 to pi, so for instance here we have pi and then we have 2pi there. So 0 to pi it increases, linearly.

So here we have again pi. And then from pi to 2pi, it is 2pi minus x. So it decreases to 0. So we have this function kind of head function, which is symmetrical around this x is equal to pi. So it is, it is an even function around this x is equal to pi. And then we can apply the result which we have discussed earlier.

So the given function is an even function about this x equal to pi. Therefore what we can have the bns will be zero. So bn we have this 0 to 2 pi and 1 over this L, 1 over L and 0 to 2L and then fx sin this because the period is 2pi, so nx only dx. This will be zero without any, any calculation we can just set to 0, because this is an odd function around this Pi.

So the integral this 0 to 2 pi will become zero. And then if we compute this a naught and then an. So a naught is 1 over pi, 0 to 2 pi fx dx. And we know the fx is x and 2pi minus x in this range. So we can compute this integral. So here we will have x square by 2 meaning this pi square by 2 and then here we will have minus sign with 2pi minus x wholly square by 2. So again we will get pi square by 2. And as a result we are getting this pi. So a naught we have computed as pi.

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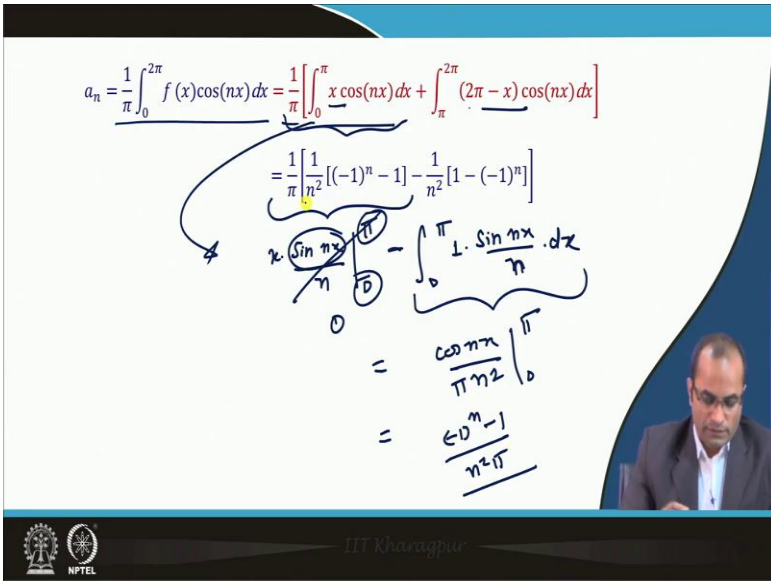


And then we can compute an as well, with the formula 1 over pi, and 0 to 2 pi and fx cos nx dx. So in this case again 0 to pi it is x and pi to 2pi it is 2pi minus x. So we can again simplify these results, which will give us, so if we can take a quick look here. So we have x and then we have sine nx over n and 0 to pi and then minus sign 0 to pi. So x the differentiation will be 1. And then we have again the sine nx over n and then dx.

So because of this sine and then we are substituting pi or 0. So this will become 0. And we have the result from here. So again with minus sine, sine nx in the further integration will become cos nx and divided by this n and again 0 to pi. So this will be and 1 over pi was sitting there already. So here n is square, 1 over n was there and again 1 over n will come and there is a pi outside this, outside this integral. So when we put this pi we have minus 1 power n and minus 1 over n square this pi.

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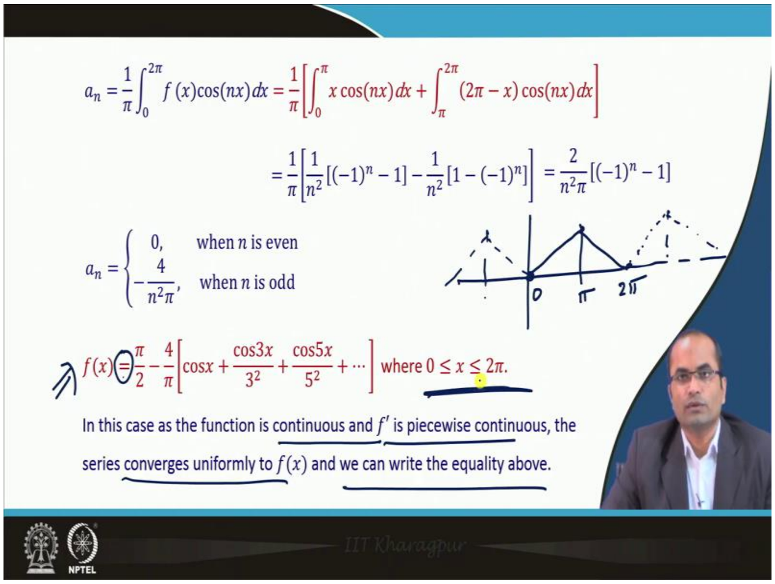


So which is exactly written here. And then similarly we can integrate the second integral which is 2pi minus x cos nx dx. And then we can further simplify this by taking this one over n square common and this minus will be adjusted here inside. So there would be just 2 times of the term. So 2 over n square pi and minus 1 power n minus 1.

So in this case what we have the an when n is an even number here. So for instance 2, 4 et cetera. So this will become zero and when it is odd so then these two will be added minus 1 minus 1, so we will get minus 2 and then 2 there. So minus 4 over n square pi. So having these ans, now we can just substitute in the Fourier Series, this was a naught by 2. And then the rest is coming 4 by pi and then we have this n square term. So 1 over 1 square, 3 square, 5 square and so on.

And in this region 0 to 2pi. So this, the result is having only cosine term because the function was an even function around this middle point pi of the interval 0 to 2pi. So in this case one should also note that here I have used this equality sign, why equality sign is correct here. In this case the function is continuous. So if you take a look at the function so this is, this is a continuous function in the whole range this 0 to 2pi. This is a continuous function.

And there is f derivative, we can compute the derivative and we will notice there is also a piecewise continuous function. So we have learnt already the convergence of the Fourier Series and we know that the Fourier Series converges uniformly indeed. The uniform convergence was the strongest convergence, which we have discussed in the lecture.

So we can write this equality there. That means for any value of x in this range the series will converge to the function fx, only the end point. But again if we look at the extension of this function as periodic extension, so the function becomes continuous also at the end point. So there is no problem to use this equality for all values of x including the end points.

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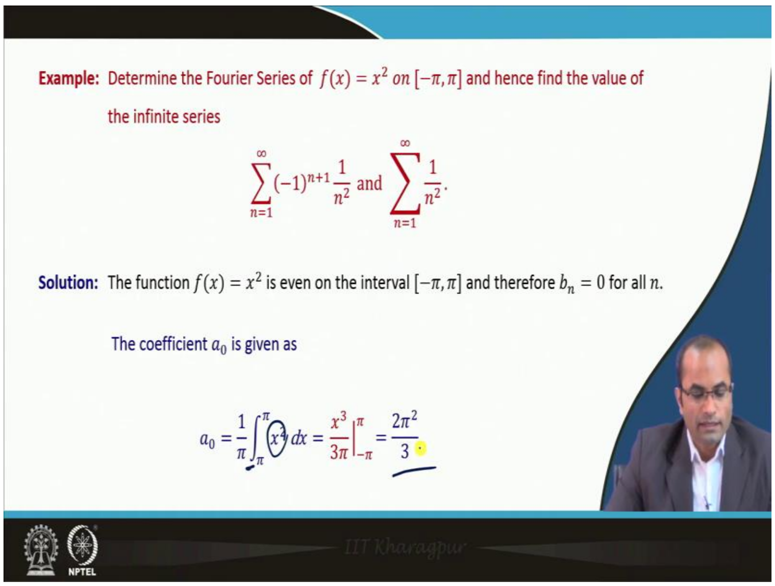


And now we will consider another example where we are want to determine the Fourier Series of the function fx is equal to x square in the interval minus pi to pi and then we can use the convergence result to show the this infinite sums, the value of this infinite sums. So here what we have? We have the x square. So the function x square in the range minus pi to pi. So we have minus pi to pi and we know that this x square function is, is an even function in this range minus pi to Pi. So again, we will get the cosine series and this therefore the bns will be against zero because of this even function.

So well this is an even function in this interval. Therefore this bns will be zero for all n. And then we can compute ans. So first the a naught, we are computing here, a naught is 1 over pi and then minus pi, so this is minus pi here, minus pi to pi x square dx. And then we integrate here, we have x cube by 3 and then minus pi to pi. So we can substitute it will be just added. So 2 pi square by 3 we will get the result of this a naught.

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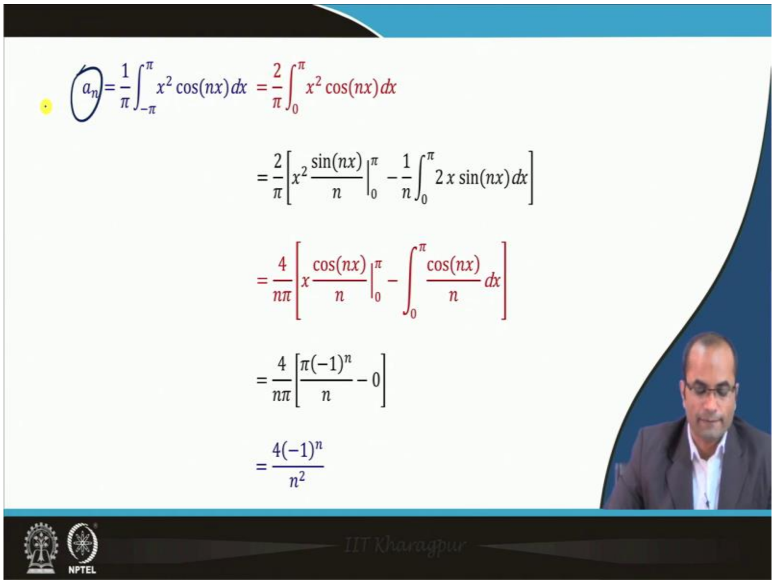


And then we can get a1 as an as well in general. So 1 over pi, minus pi to pi the function fx x square and then cos nx. So this also we can, because this cos is an even function, x square is even function. So we can use this 2 over pi and can integrate in the region 0 to pi. So this integration, again, we come do with the partial frac, with the idea of this product rule.

So here we have x square, we have x square and then this cos nx will be integrated to sin nx over n. Then we have minus, this one over n again because this cos nx will be sin nx over n. So 1 over n. And then 0 to pi and we have the function value, the integral, differentiation of this x square, which is 2x here. And then further because this will become zero when we put pi or we put zero. So this is going to be zero in either case.

And then we have the second integral, which has to be integrated again. So here the x and the sine will be cos. And then, again, we have the cos there x will become 1. And then, further we can integrate the sine here, which will become anyway 0 at 0 and pi. So only because of this term we will get the value. So when x is pi, so pi and cos pi, cos n pi will be minus 1 power n and divided by n. And there is a factor 4 over and pi sitting outside.

So what we have, we have 4 this pi gets cancel. So 4 minus 1n square and then minus 1 power n. And then we have this n square there. So we have 4 times minus 1 power n, n square. The value of the Fourier coefficients an and all bns are 0.

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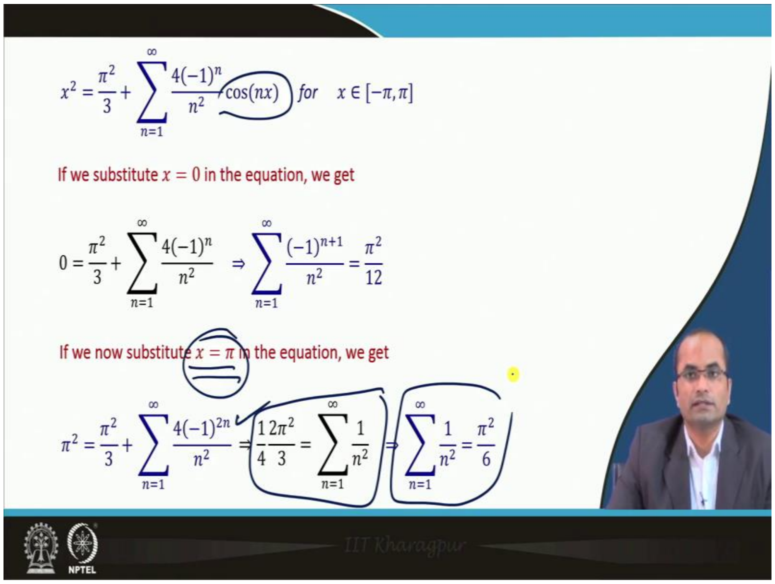


So then we can substitute in the Fourier Series the a naught and ans the bns are 0. So we will get this cosine series because the series will not contain the sine term. And again, we can put this equality, because this series will also converse in the infactible converge absolutely and uniformly because we have again this x square function defined here for minus pi to pi and even its extension also. So everything will become, everything will become continuous.

So even including the endpoints the function in its, in its periodic extension everywhere it is going to be continuous. So at any point in this minus pi to pi or over the whole axis, when we have this periodic extension. Not just the x square will be extended to, over the whole real axis. But in this minus pi to pi and then we have to extend this, we should think this as a function and its period, in its Fourier Series is given here that Fourier Series will converge for any point x in R.

So here we have x from minus pi to pi. This will converge to the given function x square in that region, including the boundary points. So now if we substitute x equal to zero for instance in this series, so cos 0 will become one and we have the desired series or the sum of the desired series.

So we have 0 there and pi square by 3 and 4 minus 1 power n, which can be written in this form. So we have minus 1 power n plus 1, n square the value is pi square by 12.

So if we substitute now x is equal to pi, that we have to see that what is desired values our desired series and then for what values of x we can get that series. So here we will put x equal to pi. So here we will have, then minus 1 power this 2n. So all the positive terms will come. So this is the series, which we want to evaluate here one over n square and the value is coming pi square by 6.

So this is a nice application that we are using this convergence theorem, we can get the value of the series of various series, we will also observe in few, in next lectures again that many complicated series we can get with the help of this convergence result.

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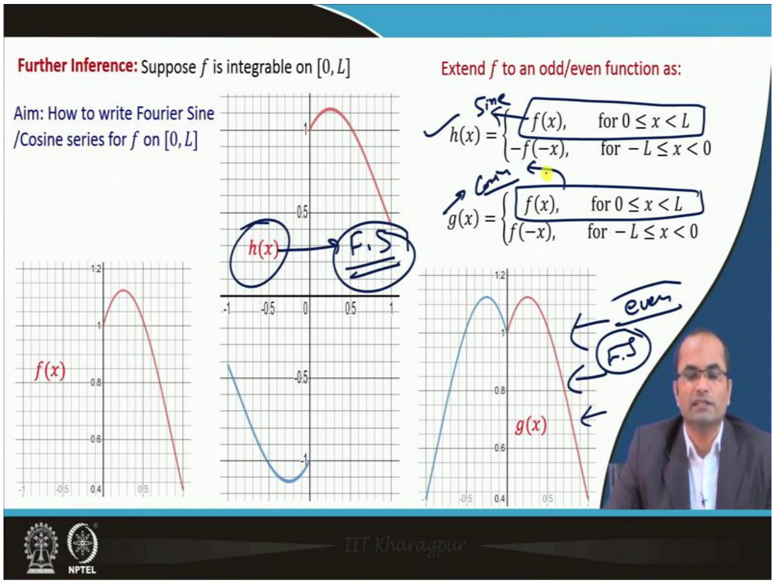


The further inference here if we can reduce, suppose this f is integrable and then what is our aim is to write, for instance, the Fourier sine or cosine series. So here the scenario is different now. Suppose the f is given, it's a piecewise continuous or it is integrable both ways we can write down the Fourier Series. So suppose this f is integrable in this domain 0 to L. And then our aim is for instance to write Fourier sine or cosine series.

So the aim here is different now, we are not writing just the Fourier Series, we want to write the Fourier sine or Fourier cosine series. And in this case what we should do then, we can extend this f to an even and odd function. So the details of this will be explained in the next lecture. So just to give some overview and the relation to what we have just done odd and even function.

So we can extend f to even and odd function such as, we define a new function hx, where the given fx is already there in the interval 0 to L. And in the minus L to L, we have made this odd extension of the function. Or what we can do, we can introduce this new function gx as fx again, the same function the given function in 0 to L. The function was defined in 0 to L. And then in the minus L to L we have made even extension, an even extension. So that this gx became an even function.

So this is an even function and this is an odd function, this is an odd function. So we can do such extensions of the function the validity of overall given function is there in 0 to L. Here also we have in 0 to L the given function. So we can thus think as follows. Suppose there is a function here fx which is given in 0 to some range, 0 to 1 it is defined by this one.

And then there are two possibilities, there could be many possibilities, but here we are extending so that the function become, this hx becomes the odd function. So that when we write, it is a Fourier Series, so Fourier Series of hx will have only the sine term, because this is an odd function. So the Fourier Series of hx will have only the sine terms. It will have only a sine series.

And when we stand here for gx which is an even function and we write the Fourier Series of this function then that Fourier Series will contain only the cosine terms. So we have the expansion of hx in terms of sine, we have the expansion of gx in terms of cosine. But what is interesting that in the, in the interval 0 to L, where the function is given, we have the same function fx which in this case will be represented by the sine series, in this case it will be represented by the cosine series. And that is the interesting the so minus called half range sine or half range cosine series we will learn in the next lecture.

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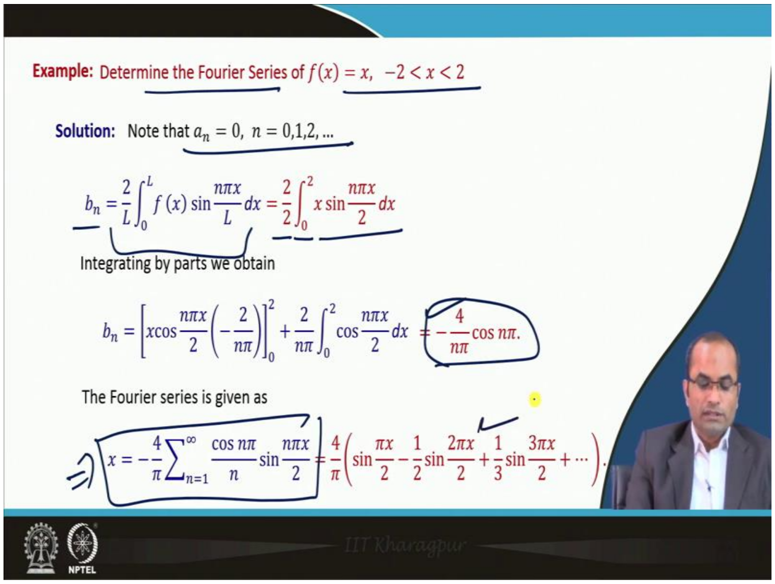


Just the last example again. So in this lecture I have taken this fx is equal to x, defined in the region minus 2 to 2. And the same example I will consider in the next problem, where the interval will be taken as 0 to 2. So when we take this interval 0 to 2 in the next lecture, our aim will be to write down its sine series or cosine series by extending the function from minus 2 to 0.

So if we make that extension from minus 2 to, minus 2 to 0 as an even extension, then we will have cosine series. And if we extend that function from minus 2 to 0 as an odd function then we will have only sine series. So that precisely we will be talking about in the next lecture. But what is now here the function is defined minus 2 to 2 and we want to write Fourier Series.

So here we are not aiming for a particular series like sine or cosine. Here we want to write sine series. But naturally if we look at this function from minus 2 to 2, this is an odd function. So this is an odd function, then naturally the Fourier Series will have only the sine terms. So without any further extension or anything if we write down its Fourier Series, it will have only the fine series.

But in the next lecture we will see that if the same function fx is equal to x is given only in the region 0 to 2. And then we want to have its expansion as a sine series or cosine series, then accordingly we have to expand it, we have to extend it in a odd function, as a odd function or as a even function. So that we will discuss anyway in the next lecture.

So here the aim is to determine the Fourier Series, for example, for this function. And as we have seen this is an odd function so all ans will become 0. For bn we have to use this formula 2 over 0 to L fx sin n pi x over L and L is 2 here. So we have to 2 by 2, 0 to 2 and this x sin n pi x over L integrating this by parts we can obtain, we can simplify this bn to this 4 over n pi and the cos n pi, which is minus 1 power n.

So the Fourier Series now, if we write down the Fourier Series of fx this will have only the sine term. And this is exactly we should be clear about it, that in this case we have written the Fourier Series of this fx is equal to x function. And the function was an odd function in the given interval.

The topic of the next lecture is the defining the half range sine and cosine series. So there we have the objective that the given function we want to express in terms of the sine function or in terms of the cosine functions. In this lecture we are just talking about the Fourier Series. However, if the function is an odd function, automatically we are getting the sine series, if the function is an even function automatically we are getting as cosine series.

But in the next lecture we will aim for either for cosine series or for cosine series by extending the function as even or odd function. So this is the sine series and we have these references here used for preparing this lecture.

So just to conclude again that if f is an even function, we have rather simplified version of the Fourier Series where an can be computed by this formula 2 over L, 0 to L fx cos n pi x over L. And if f is an even function we will get only the so, so this is odd function here. So even function will get cosine series and for odd function we will get the sine series where the bn can be evaluated by this function.

In the next lecture what we will see now again how to write Fourier sine or cosine series for f. So there we will aim for a particular series. We will not just write the Fourier Series of a given function that we have already done in the lecture, for instance. And what we have observed that the if the function is an even or an odd function the Fourier Series takes rather simple form. So that is all for this lecture and I thank you for your attention.