Engineering Mathematics - II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur
Lecture 35 - Convergence of Fourier Series – II

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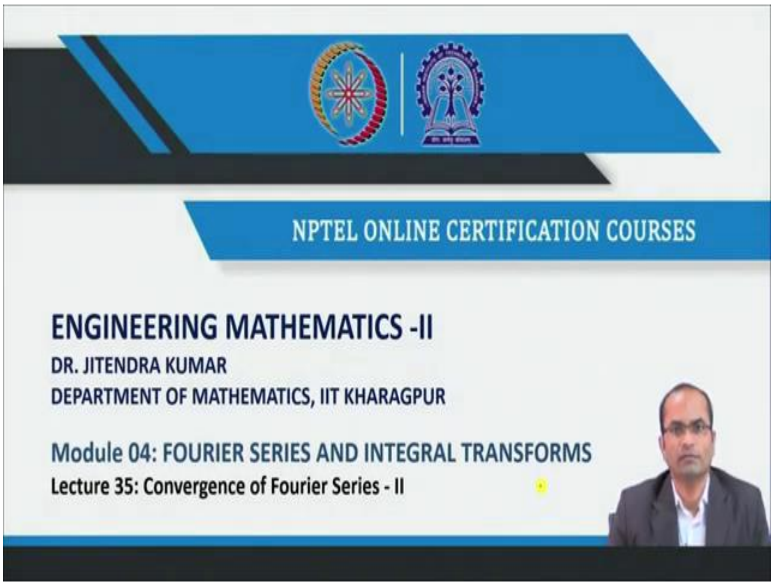

So, welcome back to lectures on Engineering Mathematics - II and this is lecture number 35 on Convergence of Fourier series. So, we have already discussed the convergence of Fourier series and now this is a continuation of that lecture, where we will consider different kind of convergences.

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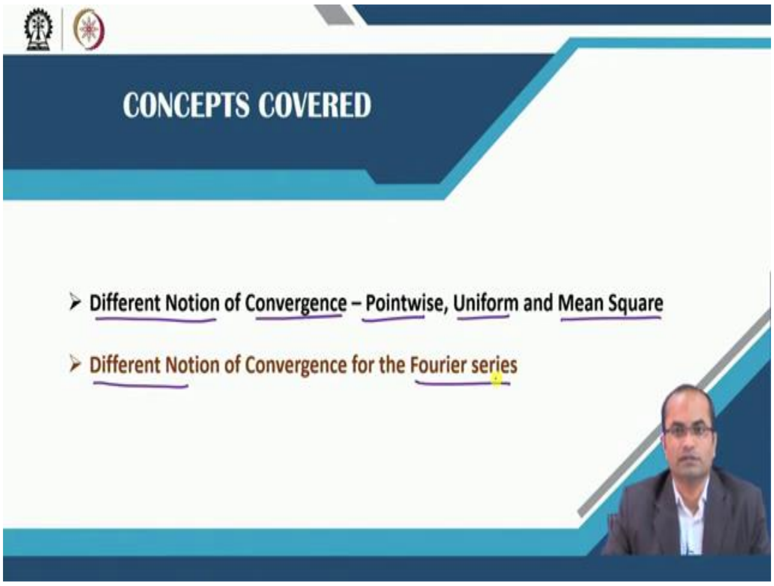

So, we will cover basically what are these different notions of conversions and mainly, we will be talking about the pointwise conversions, uniform conversions and conversions in the sense of mean square and the connection between all the 3 will be also demonstrated. So, and then we will come to the point that this different notion of convergence in the in connection of the Fourier series how this is related.

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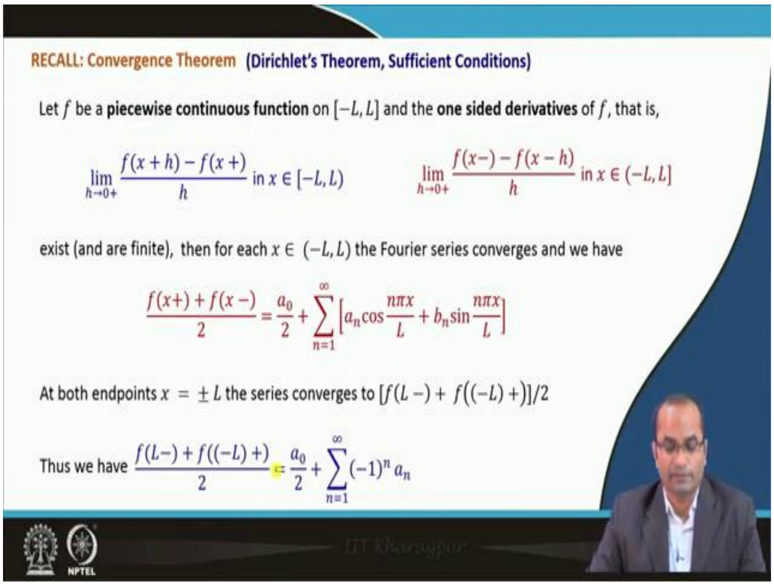

So, just to recall from the previous lecture, we have discussed convergence. That was the Dirichlet theorem and those were the sufficient conditions mainly the piecewise continuity and existence of one sided derivatives. So, just to recall again, if f is a piecewise continuous function and these one sided derivatives mainly this quotient, the limit of this quotient and the limit of this quotient exist. In that case, we call that for each x in this interval minus L to L, the Fourier series converges and we have this result that the value of this series here the sum of this series is going to be equal to the average value of the function at that point where this f x plus is the right limit at x and f x minus 1 is the left from the left side the limit of f at x.

So, at both the end points also we have discussed and again this series converges to this average value of the limits and thus we have at the end points, because when we substitute here x is equal to L this will be simplified. So, we get rather simple result at the end points.

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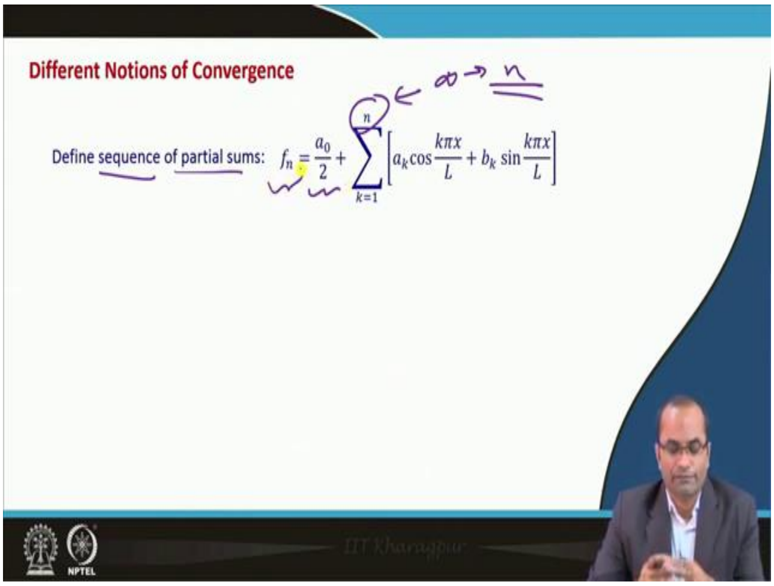

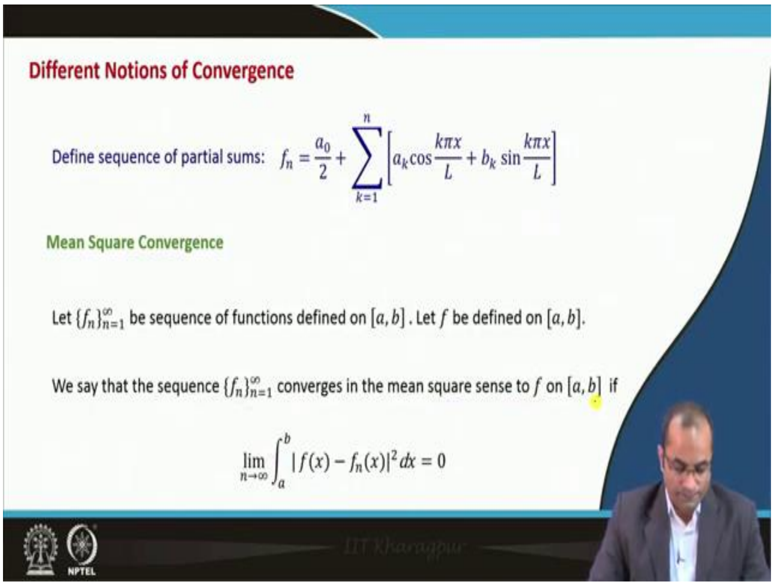

So, now, we will continue with the different notions of convergence the one which we have already seen in this previous lecture, where for each point x in the interval minus L to L, we were talking about the sum of the series converges to this average value of the limits of the function at x. So, now, we will define here because we need to define a sequence when we are talking about the convergence. So, we define here the sequence of the partial sums. So, instead of this infinity there we have just replaced that infinity by n to have the first these n terms of the series with together with the first term and this let us we call this sum as f n. So, having this f n for each value of n, we have f n that means f 1, f 2, f 3 and so on.

And naturally the question, now about the convergence is that what is happening when n approaches to infinity that means, we are exactly at the Fourier series when n approaches to infinity. And our aim is to now find out that what happens when n approaches to infinity to this f n. Well, so the first one, the first notion of the convergence, we are talking about this is mean square convergence and this mean square convergence is in the sense of the integral. So, this f nx at the sequence f n be the sequence of the function suppose in on the interval a, b and let f be also another function which is defined on this a, b. And then we say that this sequence here converges in the mean square sense, in the mean square sense to this function f on this given interval a, b if the following condition holds.

So, what is this condition here? We are integrating this a to b, the difference of the two. So, f x minus this f nx, f x what we have this limit actually of this f n and this f n, the difference and the square of the this difference when integrated over dx, if this is equal, when we take this n approaches to infinity then we say that this f n converges to f in the mean square sense, why in the mean square sense? Because this is a weaker version of the convergence, we will be talking about later, 2 more types of convergence because here we are letting only that this difference under this integral goes to 0.

So, as we know that this integral does not read for example, the value of the integrant at discrete at finitely many discrete points. So, in that case, though this f x and f n x, they may differ marginally at different points at finitely many points but still this integration may be equal to 0. And exactly this is what we are calling the, this is a weaker form of the convergence, still we are naming it as a convergence, but in the integral sense because this we are discussing here that this integral when n approaches to 0 goes to 0. So, again, as I already said that these two functions may differ at finitely many points, but still this integral value may goes to 0 and we call that this converges in the mean square sense to f. So, there are other stronger versions of the convergence, which we will just discuss.

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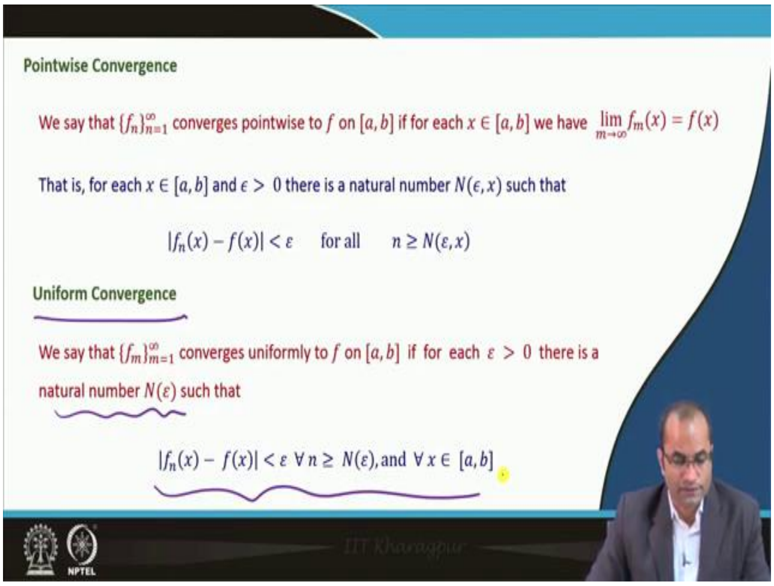

Now, the another one is the pointwise convergence and this is precisely the convergence which we have already discussed while discussing this Dirichlet theorem. So, we say that this f n converges pointwise to this f on this interval if for each x a, b, we have that this limit f or we can say n or m does not matter. So, f n x when n approaches to infinity is equal to f x, that means we are fixing x here and then this sequence of this f n x for this fixed value of x goes to f x again for that same value of x. So, here it is pointwise it is called pointwise convergence because we are fixing the point and then we are talking about that this f n x, the sequence of these real numbers for fixed x we have these sequence of real numbers f n x.

That this goes to f x as n goes to infinity and we call this a pointwise convergence and just to mention here that this is a stronger version than the earlier one which was under integral only. So, here we are at least talking about that for each x, once we fix x, f n x will converge to f x, definitely, if we have the pointwise convergence, it will imply the convergence in the mean square sense which we have just discussed before.

So, now the formal definition of this sequence or more mathematical definition we give, so for each x in this interval and for any epsilon however small there is a natural number n epsilon x. So, this n, the national number depends on what epsilon we choose and what x we fix such that this difference between f n x and f x can be set arbitrarily small because epsilon what we have chosen for all n greater than this n epsilon x.

So, if for each f, x and epsilon, there is a natural number n such that this relation hold for each n greater than n, then we call that it is a pointwise convergence. And then we will be talking about the uniform convergence and there is a slight difference between the two because here for each x and epsilon, there is just a natural number n which may depend on epsilon and it may depend on x or it may depend basically on both such that this difference is less than epsilon for all n greater than n epsilon x. Now, in the uniform convergence this n, we will if there exist n free from this x, so n must depend only on epsilon. So, we say that this sequence here converges uniformly to f if for each epsilon there is a natural number n which depends on epsilon.

So, here n was depending on epsilon and x, here now n is depending on only on epsilon and such that that this difference can be made arbitrarily small for all n greater than epsilon and for all x belonging to this a, b. So, here this difference can be made arbitrarily small for all n greater than an epsilon and this n does not depend on x. So, for all x this should hold whereas, in the case of pointwise conversions for each x and epsilon we say that if there exist a n such that for all n greater than n this difference can be set less than epsilon. But here in the second case, when we are talking about the uniform convergence, the only difference is that if there exists a natural number n which depends only on epsilon not on x, then we call that this is a uniform convergence.

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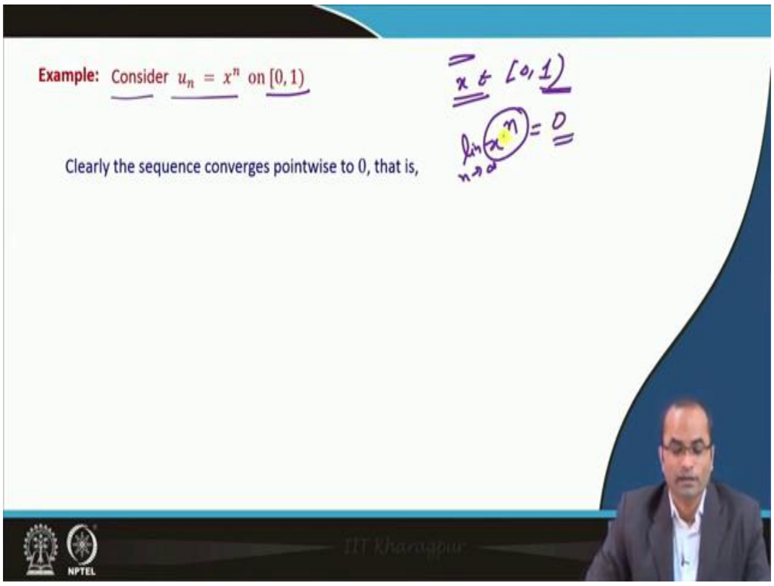

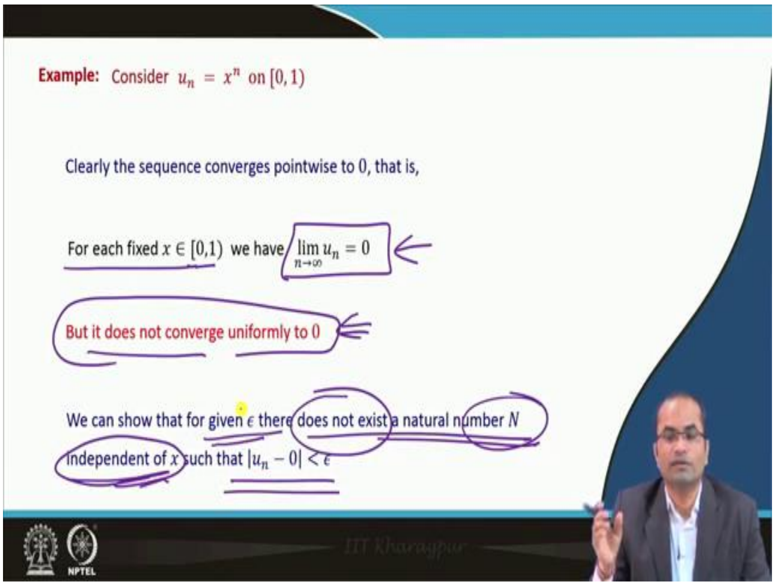

Well, just to differentiate the 2 types of convergence, we will take some simple examples to just clarify that how these 2 notions are different. So, consider this u n, a sequence here x power n and x is in between 0 and 1. So, what is interesting here if you fix x from the 0 and 1, so we are not going to 1, so it is open 1 there. So, any number if we take, we fix for x and then we talk about x n and take the limit as n approaches to infinity, then this is definitely going to 0 because x lies or x is smaller than 1. So that is what we call the pointwise convergence.

So, for each x in 0, 1, we have this condition that the u n is going to 0 as n approaches to infinity, but it does not converge uniformly to 0 this is what we will see now, that here we have only pointwise convergence and we do not have uniform convergence. Naturally the uniform convergence from the definition itself that is clear, the uniform convergence implies pointwise convergence and pointwise convergence implies the convergence in the mean square sense. So, here we will show that this does not converge uniformly, it converge only pointwise. So, for that, we need to show that for given epsilon there does not exist a natural number n, independent of x here this is more important. If this natural number n depends on x, we have only pointwise convergence.

So, if there exists a natural number n independent of x such that this difference between the u n x and the limiting value 0 is less than epsilon, then we have the uniform convergence. But here we will show that such n does not exist, n has to depend always on x and epsilon both in this particular case.

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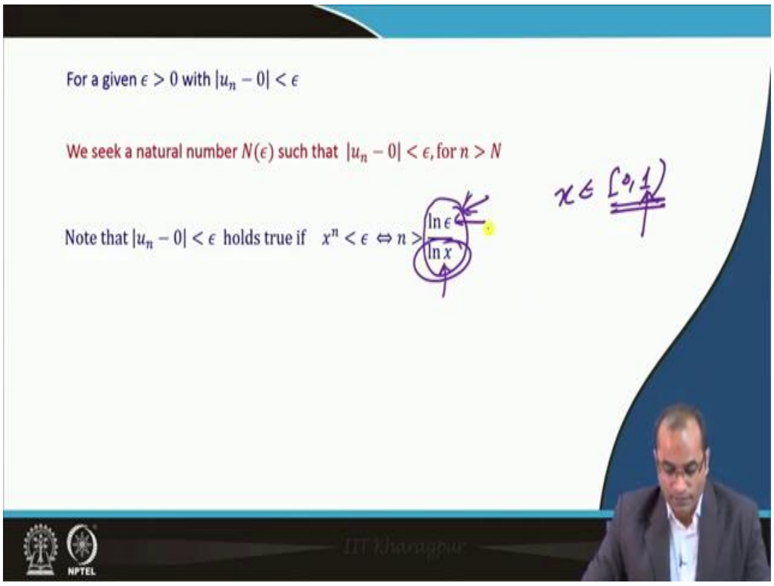

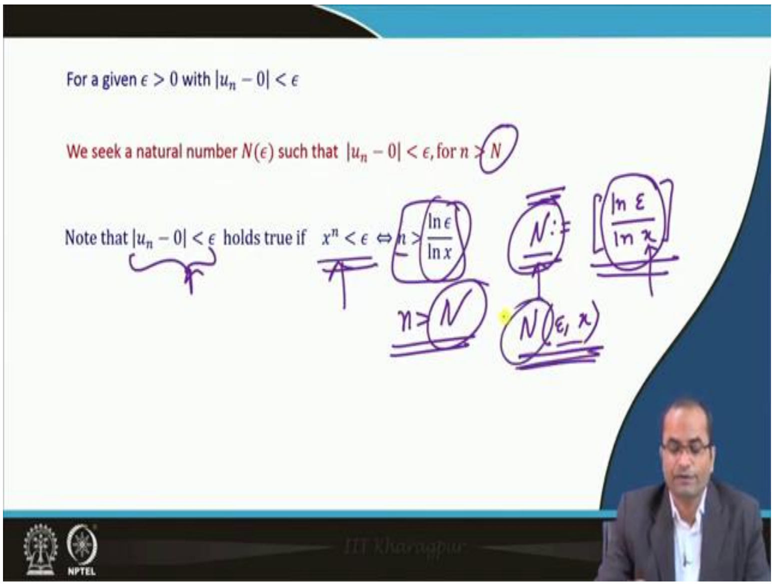

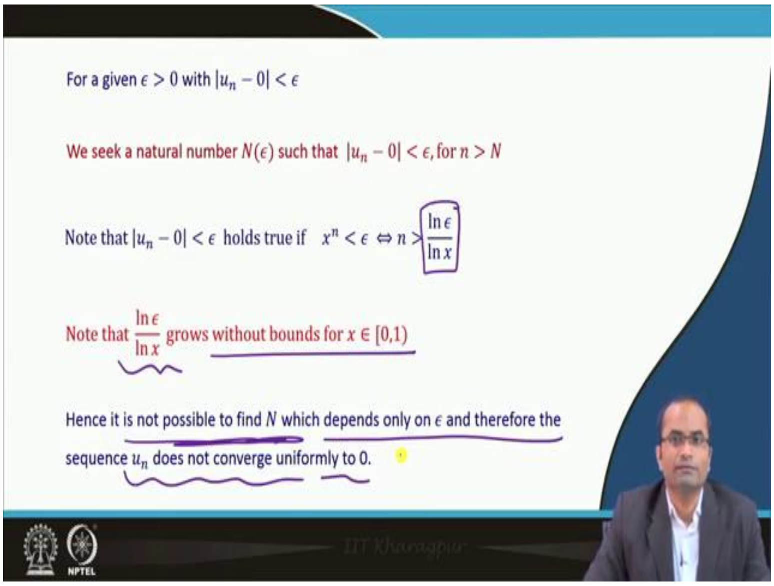

So, for given epsilon what we, with this u n minus 0 epsilon less than epsilon, we seek a natural number n which depends only on epsilon such that this difference can be set to less than epsilon, epsilon is any given arbitrary number and this should hold for all n greater than this big N. So, note that this holds true when we have basically u n minus 0 is, u n is x power n. So, x n minus x power n less than epsilon this is what we want here. And now, we are looking for whether this is possible for some n greater than, so that n greater than that capital N this is true.

So, here from this relation itself we conclude that this n has to be greater than this number ln epsilon divided by ln x then only this relation holds true. So, now the question is whether we can how a big N out of this ln epsilon such that the for all small n greater than that capital N, we have this relation? So, the question here is out of this expression here, can we find a big N which does not depend on x?

Clearly, it is visible from here that Ln x and x lies in the interval this 0 and 1. So, as this x goes to close to 1 this Ln x goes to 0 and this is arbitrarily large, I mean this unbounded number we are approaching now for Ln epsilon for any given epsilon, but this Ln x, so x is in the interval 0 to 1, we cannot bound this quantity Ln epsilon divided by this Ln x.

So, basically we cannot find such n, so that this relation holds. However, if we set this N, big N to just Ln epsilon divided by Ln x for instance, where this bracket functions means the rounding to integer towards infinity. So, we have this natural number n just coming from this expression and then having this rounded towards infinity to get this natural number. So, if we choose such n but the problem is that this n depends on epsilon and it depends on x. Once we fix the x, we take any number x from the 0 to 1, then we can get such n and then this relation will hold because for all n greater than this, this holds, so for n greater than this big N. So, big N is even bigger than this given Ln epsilon Ln x.

So, if we take any n which is greater than this N epsilon x, then definitely, we have this relation there that u n minus 0 is less than epsilon. But in that case, this n is depending on epsilon and x both and that is again it confirms from the mathematical definition that it is a pointwise convergence, because we cannot set this epsilon, this n free from x, this will always depend on x because of this quantity here Ln epsilon over x. So, this quantity grows and bounded for x in 0 and 1. So, hence it is not possible to find this n which depends only on epsilon and therefore the sequence does not converge uniformly to 0.

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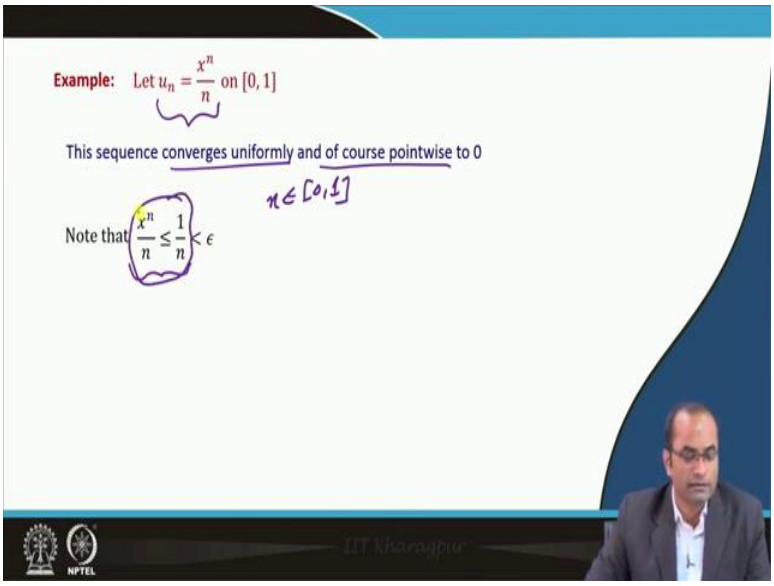

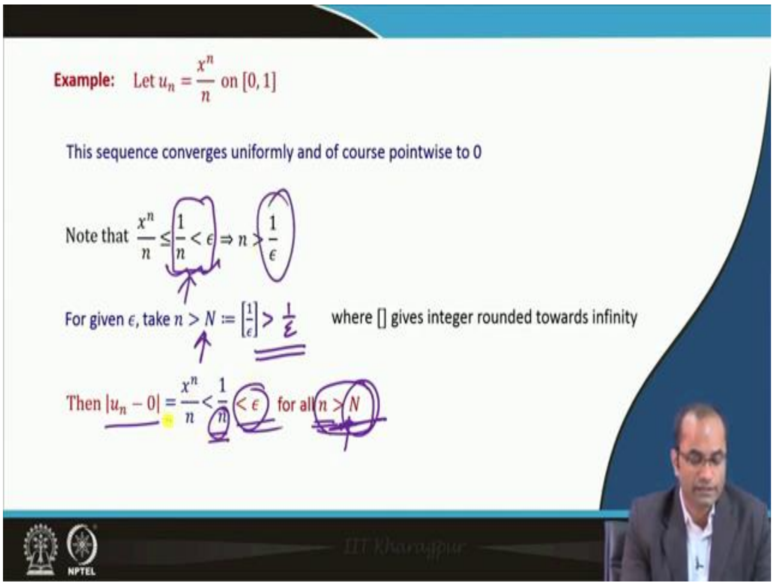

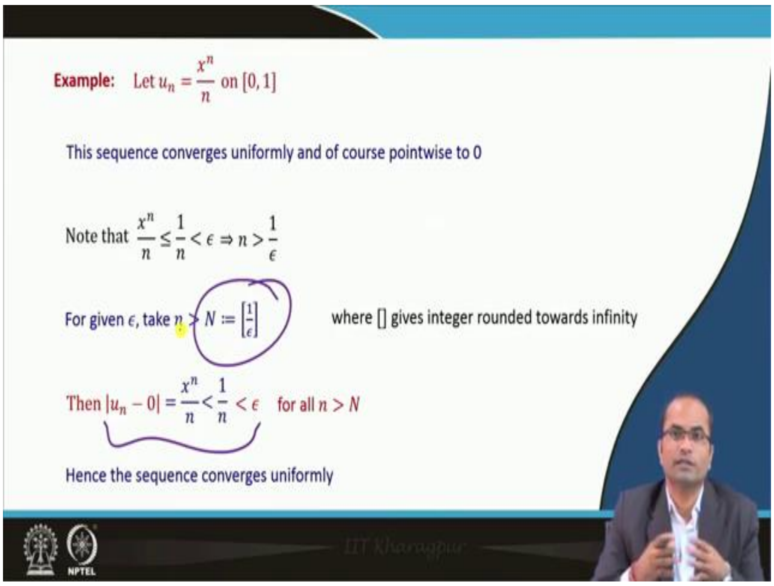

So, we will quickly go through one more example, where we will see that if we consider for example, u n as x n over n. And in that case, this sequence converges uniformly to 0, this is what we will show now and hence of course, pointwise because once we have uniform convergence, the pointwise will implies automatically. So, here we should again consider that this x n over n this quantity is always less than or equal to 1 over n and that is the key point where we can get a big N free from x and epsilon. So, because this x lies between 0 and 1, so we can have this bound here that x n over n is less than equal to 1 over n and then we can set this to less than epsilon here.

So, from this we can now construct n so that because here the relation is that this n should be greater than 1 over epsilon, then basically we have x n over n less than epsilon. So, in this case, what we have that for given epsilon, if we take this big N, which is just 1 over epsilon rounded towards this plus infinity, then we are done with that the difference is less than epsilon for all n greater than this big N. So, this we can easily see again. So, u n minus 0 and this is just x n over n and x n over n is less than 1 over n.

And this 1 over n from this relation itself is less than epsilon, so for all n because this N, this big N was even bigger than was bigger than this 1 over epsilon. So, if we choose here, if we choose this n greater than this big N, then definitely this 1 over n is going to be less than epsilon because this relation here 1 over n less than epsilon and here this n if we replace by something bigger than this 1 over epsilon, definitely that relation will hold. So, we have the uniform convergence in this case, because we are able to find this n which is free from x, it depends only on epsilon. So, this in this case, the sequence converges uniformly.

So, the uniform converges in the 3 notions what we have discussed, the uniform convergence is the stronger version of the convergence. So, uniform convergence implies this pointwise convergence and this pointwise convergence implies convergence in the mean square sense or in the integral sense.

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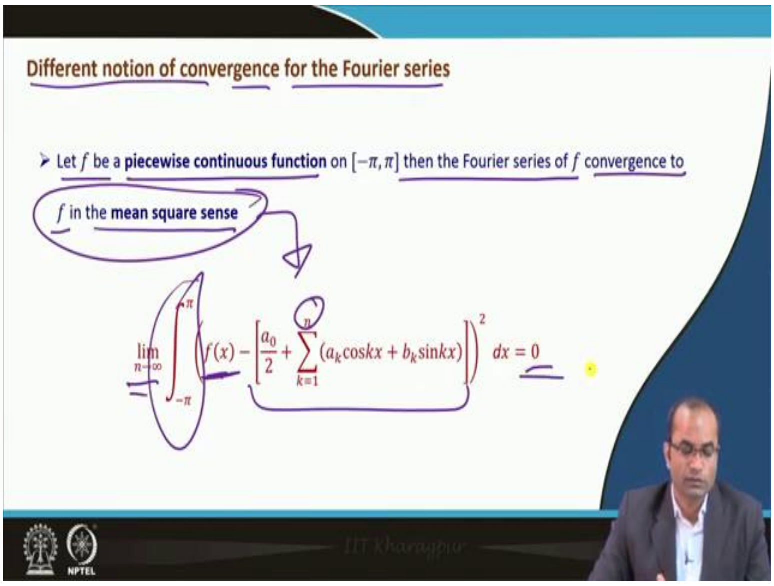

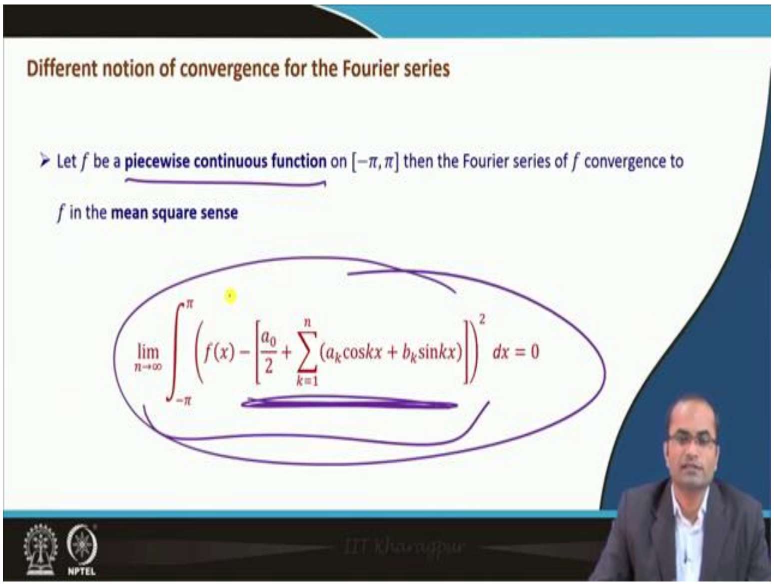

Well, so how these different notions of convergence is related to the Fourier series or their connection to the Fourier series, now we will discuss here. So, we have these nice results now that if f be only this piecewise continuous function, then the Fourier series of f converges to f in the mean square sense, so that is one result that we need only piecewise continuity. We do not need any extra condition for this weaker form of the convergence, which is mean square convergence. So, for mean square convergence, we do not need any other condition than just the piecewise continuity.

So, that is sufficient for writing the Fourier series and definitely, it will converge in the mean squares sense which is also without formal proof, it is a trivial from looking at the construction of the Fourier series which was coming by setting or by evaluating those Fourier coefficients by integrating the function f or by multiplying that f with cos nx or sine nx and then equating again, so the whole construction of the Fourier series was based on equating the integrals and as a result, that we are always getting this convergence in the mean square sense, which is only in the sense of the integral.

The second, so just again to explore a bit more here that what do we mean by this mean square sense that affects minus this quantity here for, this is a sequence, the sequence of this sum here and as n goes to infinity in this integral that this must go to 0 this is what we have the convergence in the mean square sense. So, this is always the case, we need only piecewise continuity that is also sufficient for writing the Fourier series and then in this sense we have always reserved without imposing any extra condition on f.

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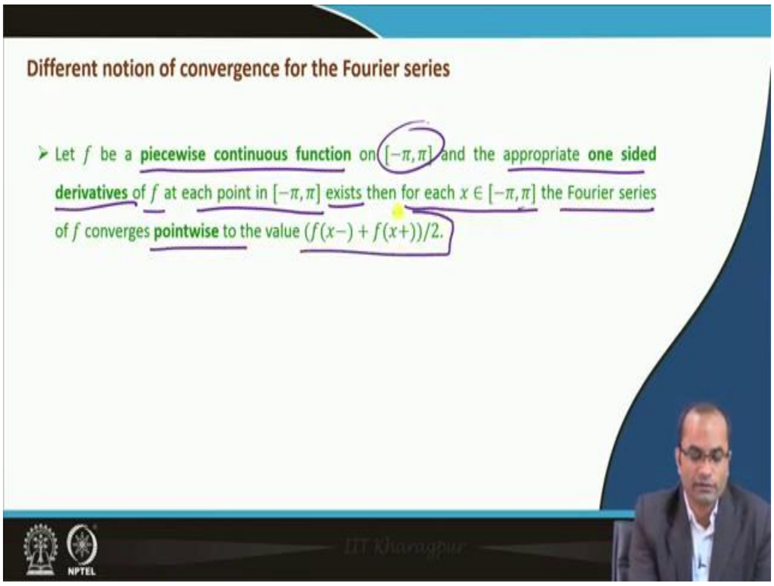

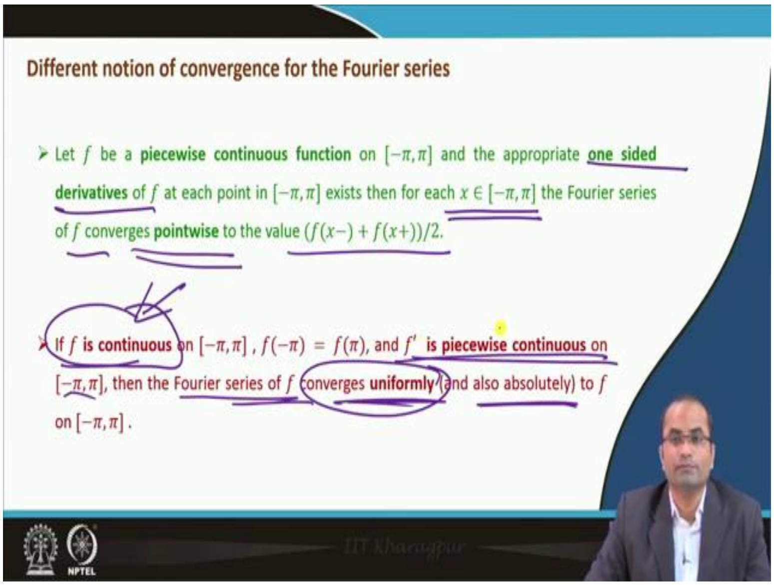

Then the second one, we have exactly the Dirichlet theorem which says that if the function is piecewise continuous, the appropriate one sided derivatives, appropriate means we have already discussed before those left and right derivatives, etcetera. So, if those derivatives, the one hand one sided derivatives of f at each point in this interval exist, so here just for simplicity, we are taking minus pi to pi but we can always take it rather general integral from minus L to L. So, appropriate one sided derivatives of f exists then for each x in this interval that Fourier, so it is a pointwise conversion. So, for each x in this interval the Fourier series converges pointwise to average value here of the limits.

So, this was exactly the Dirichlet theorem, which we have discussed. And there we have actually the pointwise convergence, because we are talking about that for each x for each fixed x, the Fourier series converges to this value. The third one, which is the stronger version of convergence is when f is continuous. So, to get the stronger version of convergence, naturally, we have to impose more conditions on f. So, for the weaker for the weakest among these 3 was the mean square sense and there we did not literally impose any condition only that piecewise continuity is enough for that weaker notion of convergence, but when we are talking about the pointwise convergence, then we have added here the existence of one sided derivative.

And now, we are talking about the uniform convergence and we have to add a little more, that is the f has to be continuous and f prime should be piecewise continuous on this interval, then the Fourier series of f converges uniformly and also absolutely. So, if we just take the absolute value of each of this term in the sequence, so that also converges. So, we have here the stronger notion of convergence once we have continuous f and f prime piecewise continuous. So, we have discussed the 3 notions of the convergence, the one is the weaker one in the mean square sense, where we do not need any condition other than piecewise continuity, which is sufficient for writing indeed, Fourier series.

The second notion was the pointwise convergence and we have the Dirichlet theorem, which says that f is piecewise continuous and those one sided derivatives exist. So, in that case we have the pointwise convergence. And in the third case, we have imposed more conditions now, that f is continuous and this f prime is piecewise continuous and in that case, we have the uniform convergence the stronger convergence on f.

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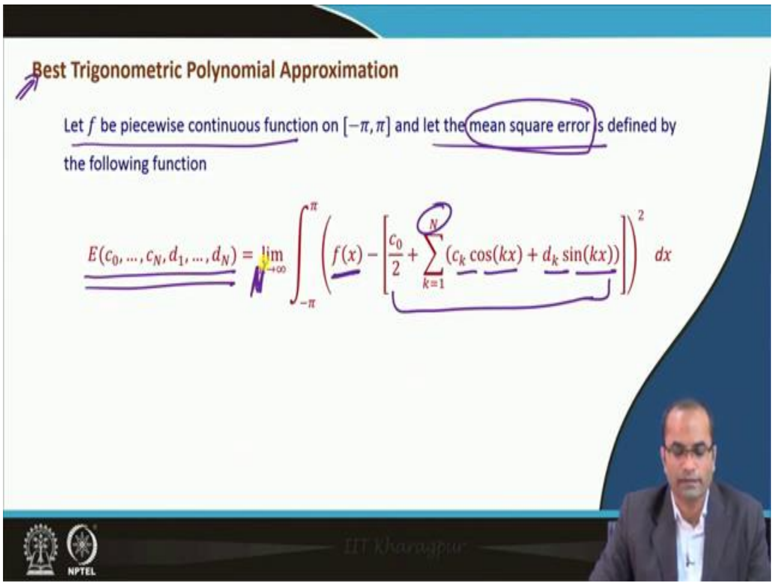

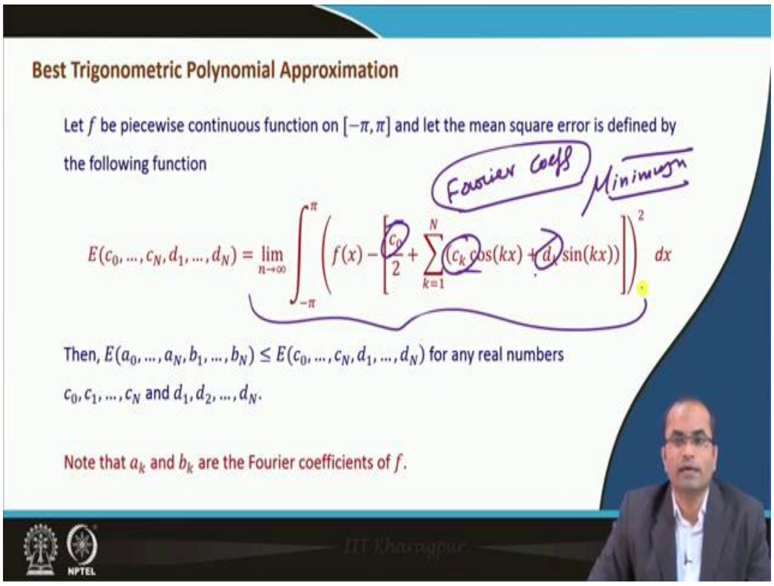

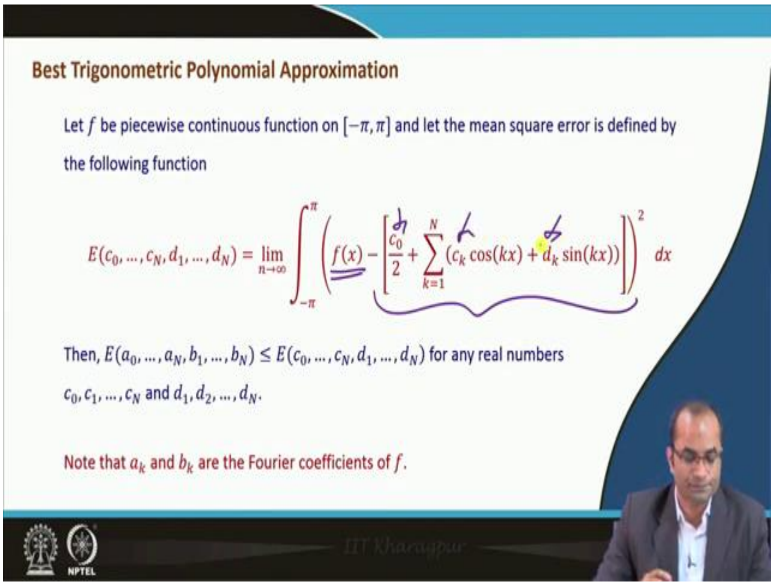

So, there is in this connection we have one more nice resource. So, I will just mention here that best trigonometric polynomial approximation, we are familiar with the trigonometric polynomial and trigonometric series already. So, here there is a very nice result that if f is piecewise continuous and let the mean square error, so we are defining some kind of error here mean square error with this following function E.

E is the mean square error between this f x and we have a trigonometric polynomial here, some c naught by 2 then c k and d k, some coefficients cos k x sine k x and k goes from 1 to n. And then this is indeed a bigger the capital N. So, this goes to, I mean this is what we have defined this error, the difference between this f x and this polynomial here and with this square and then integral.

So, what the result now we have, then this can be proved here, though we are not proving in this lecture. But it is one can prove this result that E when we have taken this a naught, a 1, a 2, a n and b 1, b 2, b n that means here if we replace a and b to this c and d, that means we will get exactly the polynomial coming from the Fourier series means if these a's and b's are Fourier coefficients Fourier coefficients, then this error is going to be less than or equal to any other real numbers if we choose for these coefficients.

Among all these errors, the one where we have exactly the Fourier coefficients is going to be the minimum one, for any real number here, this and this and we have that a k and b k are the Fourier coefficients.

So again, just to summarize what do we have this result? That such a mean square error is minimum when we have the c naught, c k and d k exactly the Fourier coefficients. So that polynomial or that truncated Fourier series will give the best approximation to the function f, we cannot have a better approximation by choosing some other numbers here, we have to choose the Fourier coefficients then only this mean square error is going to be a minimum.

So, we have the best trigonometric polynomial by choosing these coefficients of the polynomial as Fourier coefficients. So, if we choose these as Fourier coefficients, then such error is minimum such error is minimum, in that case we cannot have better error, the less error then this one by choosing some other coefficients.

Now, we have to choose the Fourier coefficients to get this error to minimize this error. So, this is a nice result in the sense of the approximation that we can approximate the function f x by such, so called the Fourier polynomial or the truncated Fourier series, and the error in this mean square sense will be minimum by choosing these coefficients as Fourier coefficients.

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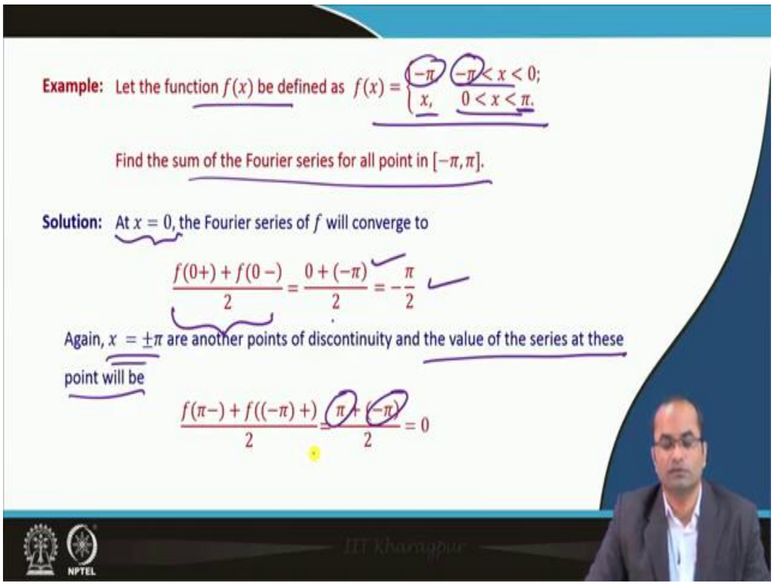

Well, just a quick example here now, so let this function f x be defined by this minus pi and x, then the sum of the Fourier series for all points we have to find. That means, we will just apply the Dirichlet theorem and we will see that what is the sum of the Fourier series for this function. So, we do not have to evaluate the Fourier series because we know already the convergence result. So, at x equal to 0 because that is also a point of discontinuity here, the Fourier series will converge to this average value f 0 plus and f 0 minus, the limiting value at 0 divided by 2.

So, what is f 0 here? f 0 plus 0 and f 0 minus which will come in from there minus pi, so we have 0 minus pi by 2 that means, we have minus pi by 2 the value of the Fourier series at x equal to 0. There is another point here at of discontinuity at plus pi and minus pi and these are the end points of the interval and the value of the series at this point will also converge to this average value f pi minus from this side and f minus pi plus from that side, so this is going to be pi and then minus pi divided by 2, so this will converge to 0. And at all other points, the series will convert to the functional value because the function is continuous there, so this average values there will be just exactly equal to the function value.

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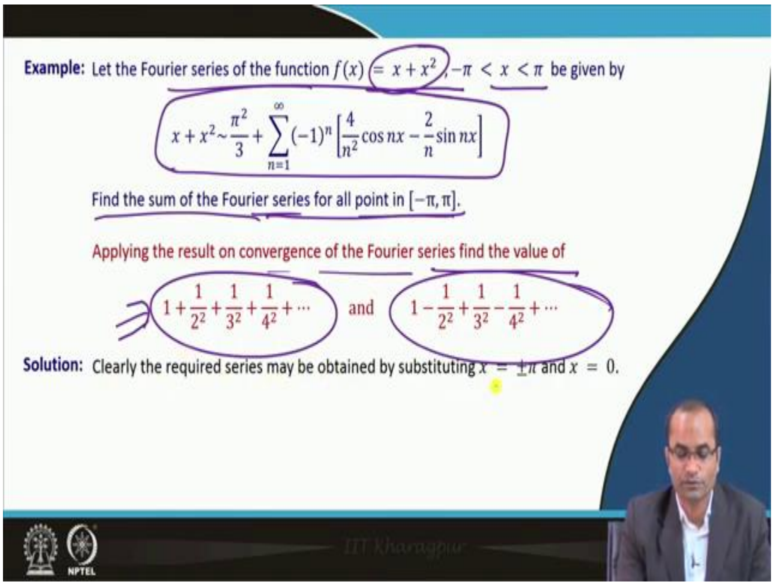

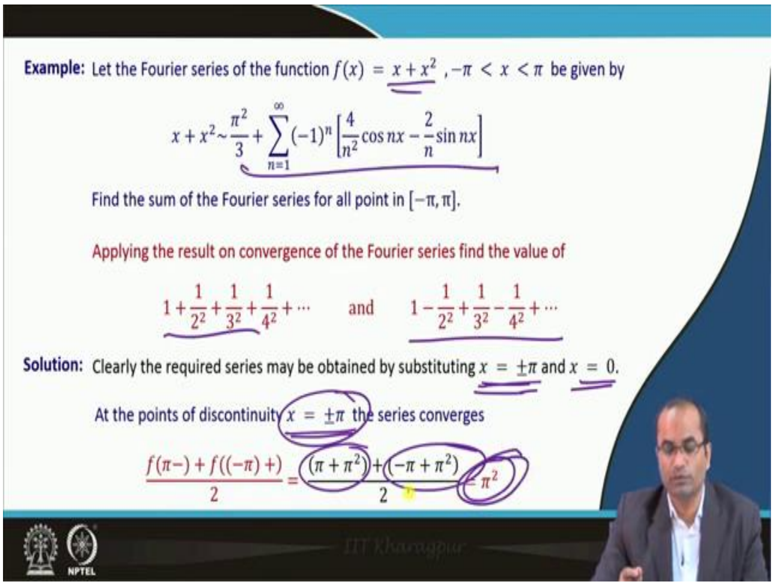

Well, so the last example where we will discuss that this Fourier series of this function x square plus x in this interval where the Fourier series is already given here. And, we want to find the sum of the Fourier series for all point in this interval and then applying the result on the convergence, we want to find the sum of this series here. So, that is interesting. So, concerning the Fourier the, the sum of the Fourier series at all points in the interval, we can apply Dirichlet theorem and we can easily find. So at the point of continuity, that this will be equal to exactly to the functional value and wherever we have discontinuity we have to take the average value of the two limits.

So now, about these Fourier about this series here, this we have to observe that if we substitute x is equal to plus pi and x equal to 0, so then we can get this series directly from the Fourier series.

So, if we put this x is equal to plus pi, first we have to see plus pi or minus pi that to what value the Fourier series converges. So, at this point of discontinuity, because here we have again the discontinuity, at one side we have pi plus pi square, other side we have minus pi plus pi square. So, there is a discontinuity, so this series will converge to this average value pi plus pi square and minus pi plus pi square divided by 2 that means, this pi square. So, we know that at x is equal to plus minus pi, the series converges to pi square and then the same point we will substitute in the series and that will be set to equal to pi square.

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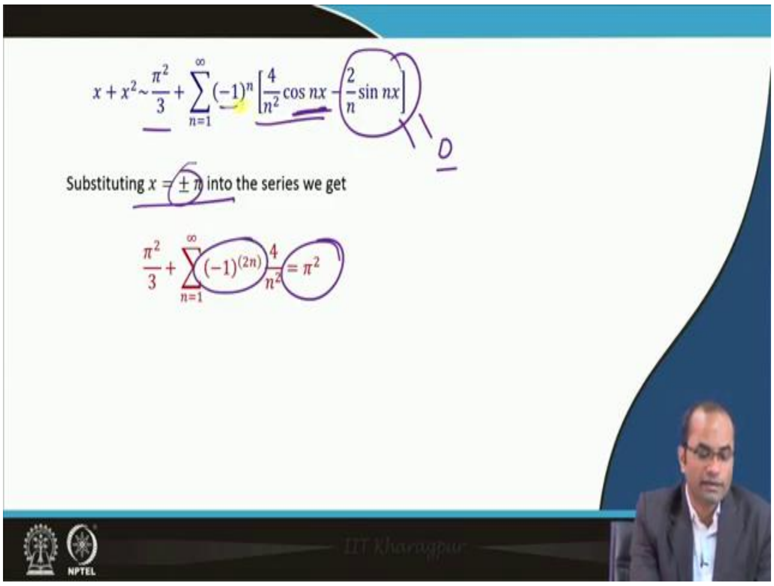

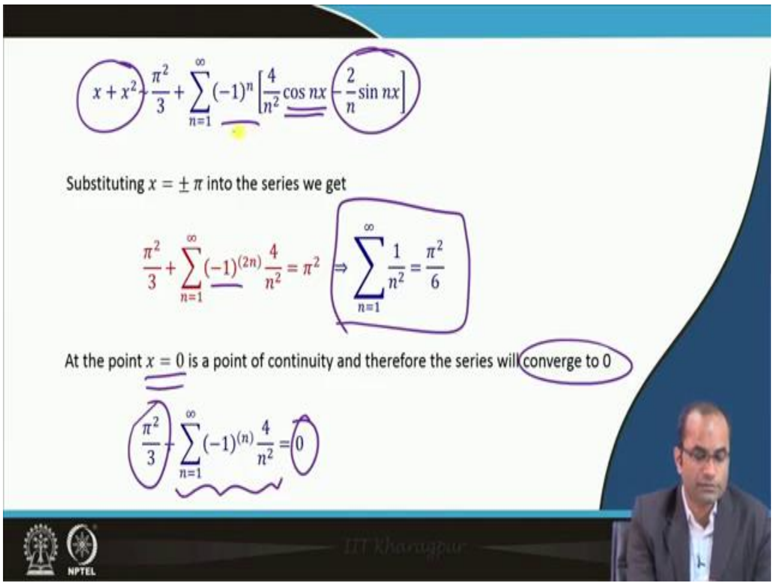

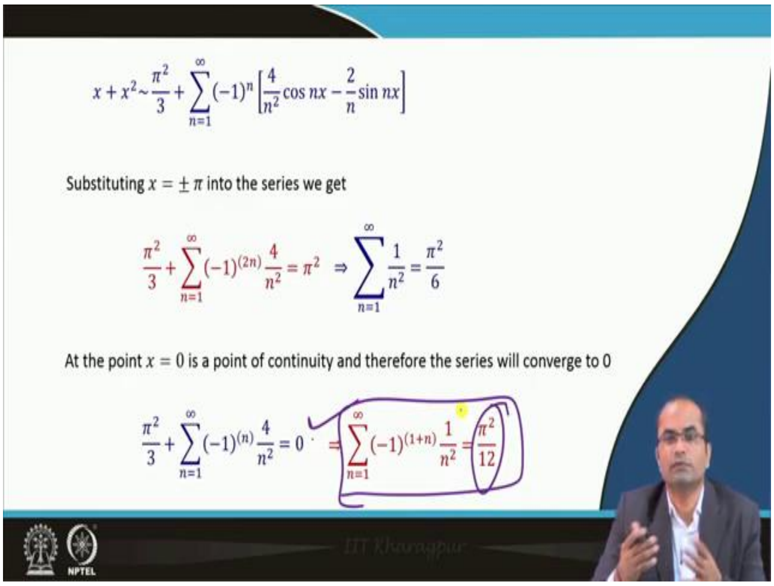

So, we have this series, now at this here what we get the right hand side means the sum is pi square and then here we will put that x equal to plus minus pi. So, the because of the sign this will become 0 and here cos n pi whether plus or minus pi does not matter. So, here minus 1 power 2n, power n will come and then minus 1 power n is there. So, minus 1 power 2n, you will get here and out of this we can easily figure out that our series 1 over square that value will be pi square by 6, so one of the series which was asked in the question.

Another one when we put x equal to 0, so we will get plus minus as we were getting in this desired series. So, this will again go to 0, here this will become cos 0 as 1 and then we have to see the convergence there. So, that is it will converge to 0. So here, we have the pi square by 3 and then when we have substituted here x equal 0, everything is set to 0 and there we will get this series which is precisely the series which was asked in the question. So, here the pi square by 12. So, by this convergence theorem, we can find the value of series, the sum of the series for many series.

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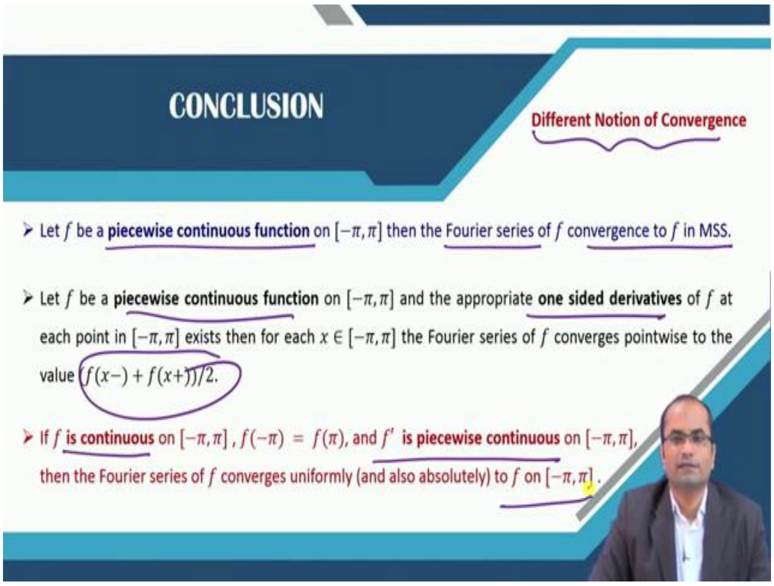

So, these are the references we have used for preparing the lecture. And then just to conclude that today we have discussed different notion of convergence, the one was the, if f is piecewise continuous, then the Fourier series converges in the mean square sense. And if piecewise continuous plus one sided derivatives exist, this is the Dirichlet theorem and the Fourier series converges to this average value. And if f is continuous on the top and then f prime is piecewise continuous, then the Fourier series converges uniformly on this interval minus pi to pi. So, that is all on convergence in this lecture and I thank you for your attention.