Engineering Mathematics - II
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Lecture 34 - Convergence of Fourier Series – I

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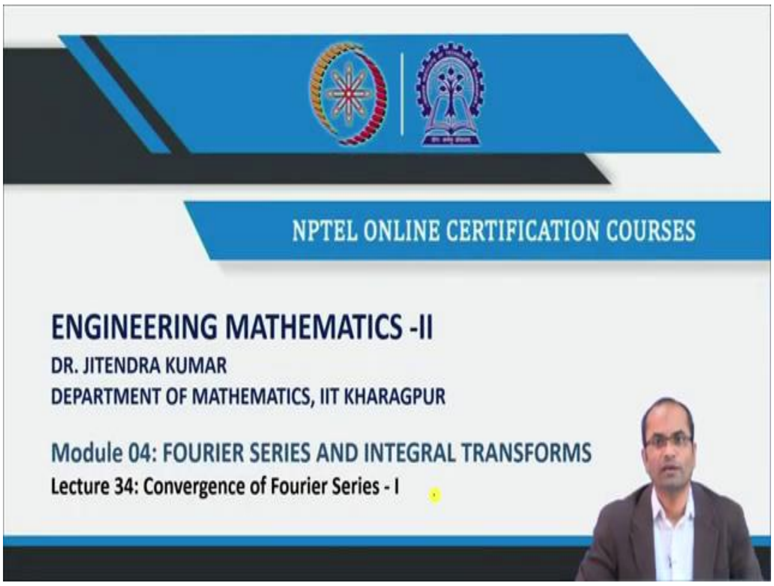

Okay, welcome back to lectures on Engineering Mathematics - II. So, this is lecture number 34 on Convergence of Fourier Series. So, in the last lecture, we have seen the construction of the Fourier series and during the construction we have observed that those coefficients we are getting by equating the function integral and the term by term integration of the series. So, naturally, we do not expect that the function value will be equal to the value of the series if at all it converges.

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So, in this lecture, we will be talking about these convergence issues and in particular. So, we will be covering the portion on convergence of the Fourier series and there is a famous result by the Dirichlet that these are the sufficient conditions for the convergence, under these conditions the series will converge and we will see that it will converge to what value related to the given function.

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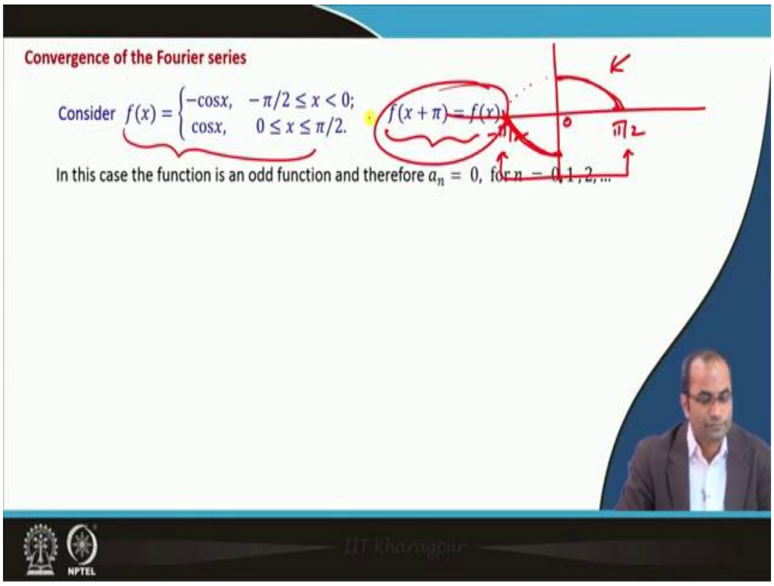

Convergence of the Fourier series

Consider $f(x) = \begin{cases} -\cos x, & -\pi/2 \le x < 0; \\ \cos x, & 0 \le x \le \pi/2. \end{cases}$ $\quad f(x+\pi) = f(x).$

In this case the function is an odd function and therefore $a_n = 0$, for $n = 0, 1, 2, \ldots$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos(2nx)\,dx = 0$$

odd, even → odd.

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We compute the Fourier coefficient b_n by

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin(2nx)dx = \frac{4}{\pi}\int_0^{\pi/2} \cos x \sin(2nx)\,dx$$

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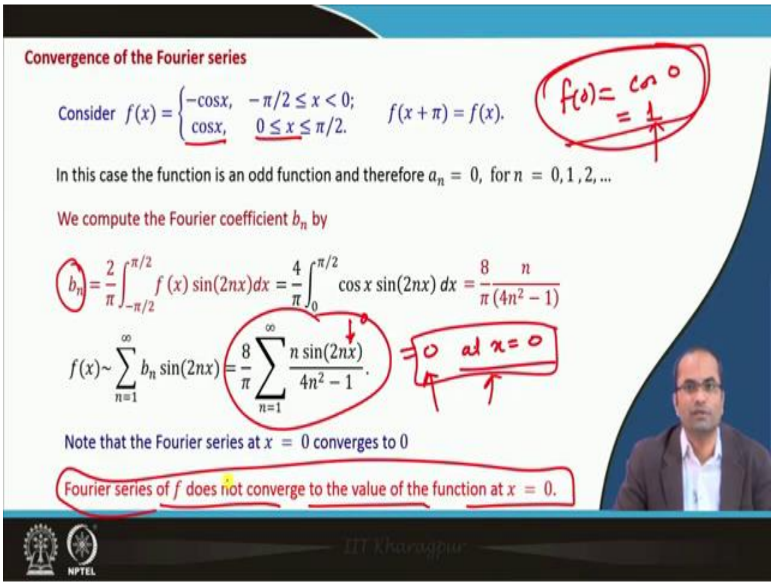
$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(2nx) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n\sin(2nx)}{4n^2-1}.$$

Note that the Fourier series at $x = 0$ converges to 0



So, again if we for instance, consider such a function f x, which is given as minus cos x in the region minus pi by 2 to 0 and then cos x in the region 0 to pi by 2. So, we have the situation that from 0 to pi by 2, if we take this as pi by 2 and this is minus pi by 2 then from 0 to pi by 2 it is cos x. So, this is the function and then in this region minus pi by 2 to 0, it is given as minus of cos x. So, the cos x is usually positive there and we are taking now the minus of it, so it will go in this way. So, we have this function f x given or defined in the region 0 to pi by 2 by cos x and in the region minus pi by 2 to 0 by minus cos x and then we can say that this is periodic because this period is here pi from minus pi by 2 to pi and then this value will be repeated.

So, it is a periodic function f x plus pi is equal to f x and in this case, the function is an odd function as we have seen in the picture it is an odd function. So, and therefore, this a n, if we compute a n, similar discussion we already have in previous lecture. So, if we compute this a n, in this case, this will be like 2 over pi and then we will integrate from minus pi by 2, 2 pi by 2 and the function f x and then for a n, so that is going to be cos 2 nx and we will have dx. So, this is a even function and this f x is an odd function. So, the product of even and odd will be odd function.

So, this integrant here for this integral minus pi by 2 to pi by 2 is an odd function and therefore, this integral value is going to be 0. So, all these a n’s these Fourier coefficients will be 0 for n 0, 1, 2, 3 and so on. Then we can compute the other Fourier coefficients. So, b n and we can use this formula 2 over pi minus pi by 2 to pi by 2 f x sin 2 nx dx. So, in a n, we had here cos and now we have sin that is the only difference. So, this again can be computed. So, we have then this is going to be odd and then odd, so even so, we can have 2 times.

So these two times the 4 by pi and then 0 to pi by 2 and in that case 0 to pi by 2 the value of the f x, the function is cos x and then sin 2 nx and dx. And this integral can be evaluated easily.

And, we are not showing all the steps here. So, finally, this value of this function will be 8 over pi and n over 4 n square minus 1. So, with this we can now write down the Fourier series. So, the Fourier series will be having only b n term, so b n sin 2 nx. So, here the period of the function is pi not 2 pi, therefore we are getting these 2n terms and here also this factor 2 over pi was coming. So, we have this Fourier series representation and then substituting the this value of b n as 8 over pi and then we have n sin 2 nx over 4 nx square, so we have this Fourier series, which somehow should represent the given function f x minus cos x in the range minus pi by 2 to 0 and then cos x, 0 to pi by 2.

What we notice now for this Fourier series, that if we substitute there x equal to 0 or we evaluate the series at x equal to 0 because of this sin function, so if we put here x equal to 0, what will happen, the sin will become 0 and everything here, the whole sum will become 0 at x equal to 0. So, this is what we call that the series converges to the value 0 at x equal to 0. However, if we note here the function, the function value at 0 can be evaluated from here that is cos 0 and that is going to be 1. So the function value is 1, however, the series clearly we can see that it converges to 0. So, there are some issues which now lead us to more discussion on this convergence of the Fourier series because for instance, in this case, the series converges to 0 at x equal to 0, whereas the function value is 1 at this x equal to 0.

So which is also was not expected because of the instruction and during the constructions these a n’s and b n’s were related to f x in the sense of the integrals were set equal to 0, not the function value of the series and was a set equal. So, naturally, there must be some issues on the convergence like we have seen here at this point itself that x equal 0, the series converges clearly to 0 whereas, the function value at 0 is 1. So, this we will discuss now, so, Fourier series of f does not converge to the value of the function at x equal to 0 for instance in this case and we can construct many more such cases where the Fourier series will not converge to the function value.

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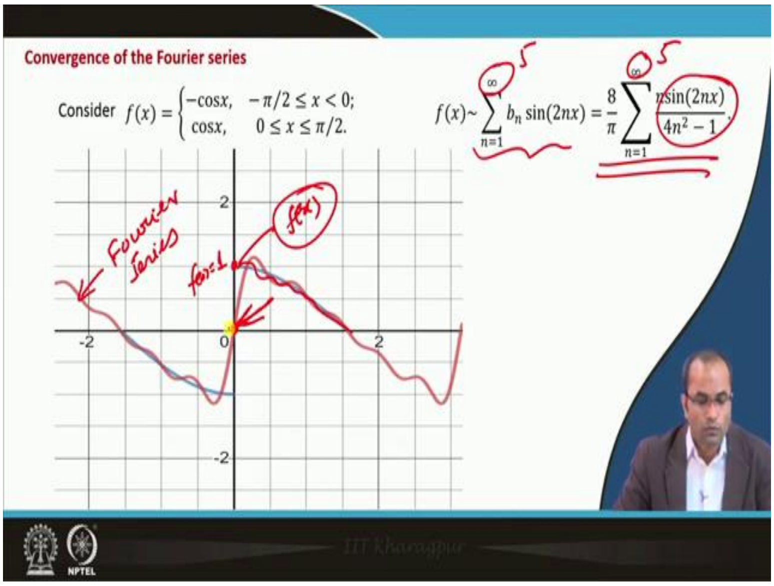

Just a plot for this function and for some value of not infinity naturally. So, we have taken here like just 5 terms and having these 5 terms we, this is the Fourier series. This is the Fourier series that represent the graph of the Fourier series and this was the cos function. So, this is the f x, the given function f x. So, as we can see that at 0, the Fourier series value is 0, we can increase the number of terms, but it will always pass through this point, 0 point which is obviously clear from the nature of this series, because sin is there and sin 0 will be 0.

So, it will always produce this 0 irrespective of how many terms we are taking in the series. The better match we can expect on this region, but it will pass always from this 0, so it will have this convergence issues. Because at 0, the function value here is 1, the f as 0 is 1 is but this will always pass from the origin.

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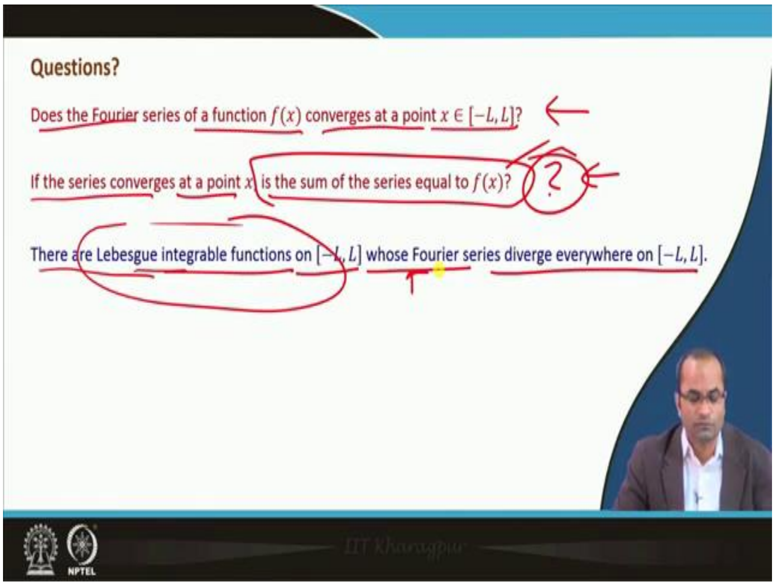

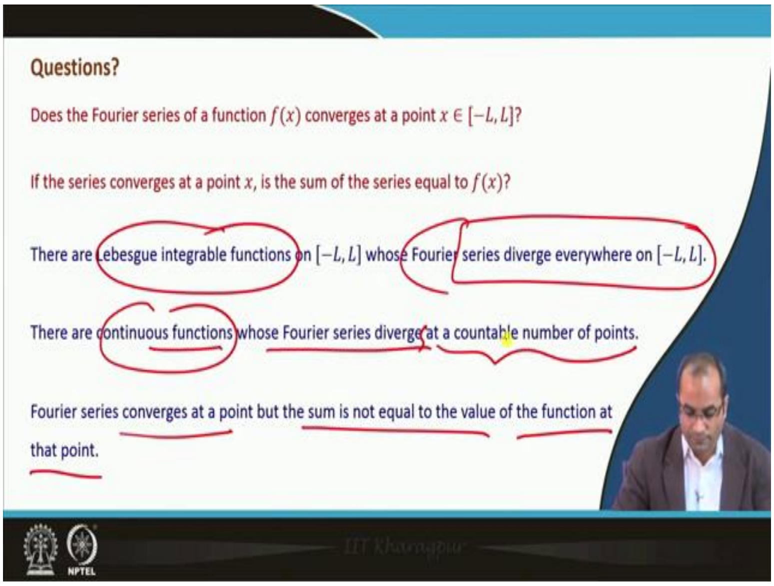

Well, so, now the questions here again, we thus the Fourier series of a function f x converges at a point x, I mean, the first question is whether the Fourier series always converges for any x in the interval minus L to L. Again, if the series converges, so there are 2 questions, one about the convergence itself, is it does the Fourier series always converges for x from the interval or if it converges, is the sum of the series equal to f x? And this is what we have seen that this is not equal to f x in the earlier example itself. At some point we have checked at x equal to 0 and the sum was not equal to the function value which was 1 in our example.

So, definitely the answer is no to this question and also the answer is no to the first question also, because there are functions whose Fourier series does not converge at all, because what we have realized already in the previous lecture that given a function which is piecewise smoothness that is sufficient to write the Fourier series. So, for a given piecewise smooth function, we can always construct a Furrier series, but the question is whether that series will converge or not. And if it converges, whether it will converge to the function whose series we have written, so all these questions, so we will try to answer in this lecture.

So, the first just a remark, say there are integrable functions the Lebesgue integrable function on this interval minus L to L and these such functions exist in the literature whose Fourier series diverges everywhere in the interval minus L to L. So, any point we choose and that the sum of the Fourier series diverges, it goes to infinity. And there are such examples, we are not going all into the details that what kind of because they are not trivial examples, but their Fourier series does not converge at all, it diverges everywhere in the interval.

Also, there are continuous functions, so even the nice functions we have whose Fourier series diverge at a countable number of points. So, there are such examples as well exist in the literature and the Fourier series converges at a point, but the sum is not equal to the value of the function at that point. So, this example anyway we have seen, the last one we have seen that there are examples where the series converges, but the sum is not equal to the function at that point. But the interesting point is that there are integrable functions, we can write down the Fourier series. But at the end, we will realize that this Fourier series diverges everywhere.

And there are indeed continuous functions. So that is again interesting that we have nice functions and we can write down their Fourier series. And this Fourier series diverges at countable number of points. So, everything is possible in this case, so then we should have some conditions on the function under which the Fourier series converges and converges to what value that also has to be discussed.

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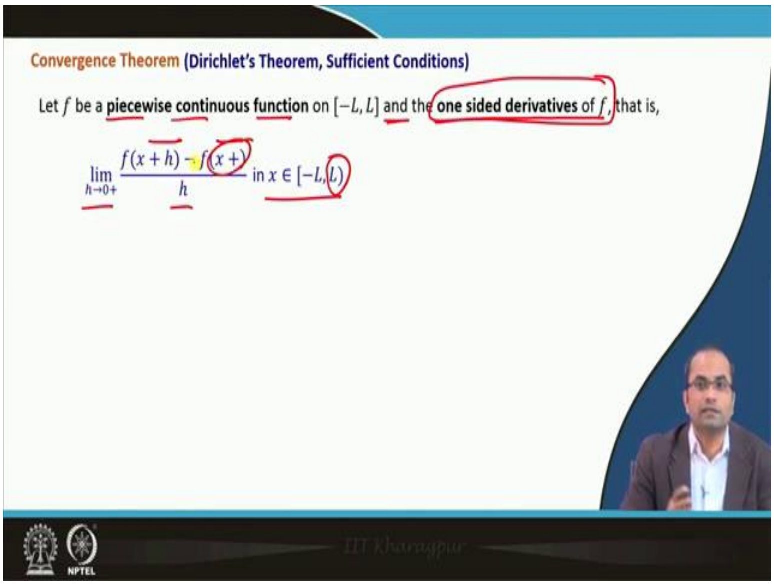

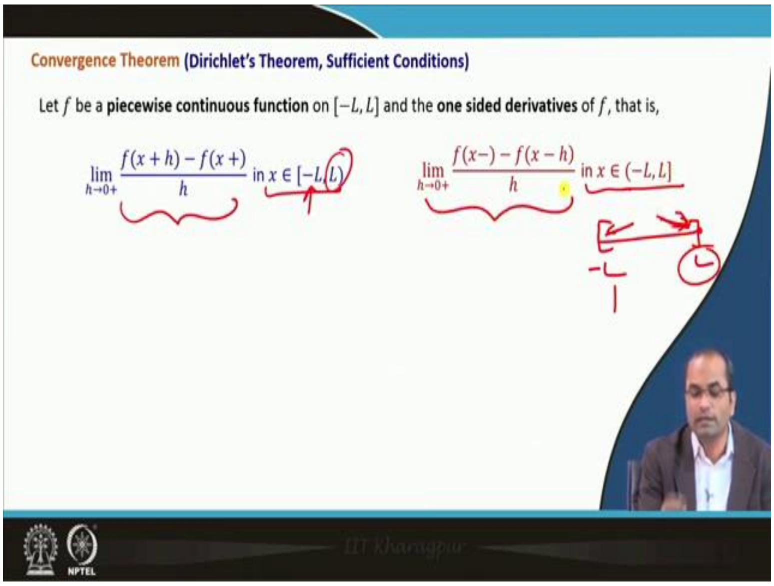

So, there is a famous convergence theorem and these are indeed the sufficient conditions that means if these conditions are met, definitely the series will converge and it will converge to what value that we will see in this result and this is called Dirichlet's theorem. So, if f is a piecewise continuous function, so the, that is obviously important to write down the Fourier series itself. We have already seen that this is sufficient for to write the Fourier series, so the function must be piecewise continuous in the interval minus L to L for instance, just for a simplicity we choose minus L to L but we can also talk about 0 to L or 0 to 2L whatever. So, here we have the function which is piecewise continuous function and the second condition is more important, one sided derivatives of f.

So, not only just piecewise continuity is enough but we also need one sided derivatives of f, what are the one-sided derivative? The 1 is here, the right hand derivative. So, limit h goes to 0 and it is taken to be positive, f x plus h and f x plus. So, this is called the right hand limit and divide by h. So, if this right hand derivative exist in this range minus L to 2L, we are not considering this L for instance here because, we are talking about the right hand derivative. So, at this point if we close it, it does not make sense the x plus, so here we have avoided that boundary points.

So, at all these points, the right hand derivative should exist and on the other hand in this interval minus L open into the close L, the left hand derivative should exist because, if our interval is up to L and here we have minus L, so at this point, we can talk about we cannot talk about the right derivative, we can only talk about the left derivative. Whereas here in this case, we can only talk about the right derivative. So, therefore, we have included here minus L and open at the other end, then we are talking about the right derivative.

Whereas, in this case, we have open here in this end and the close at this end, then we are talking about the left derivative.

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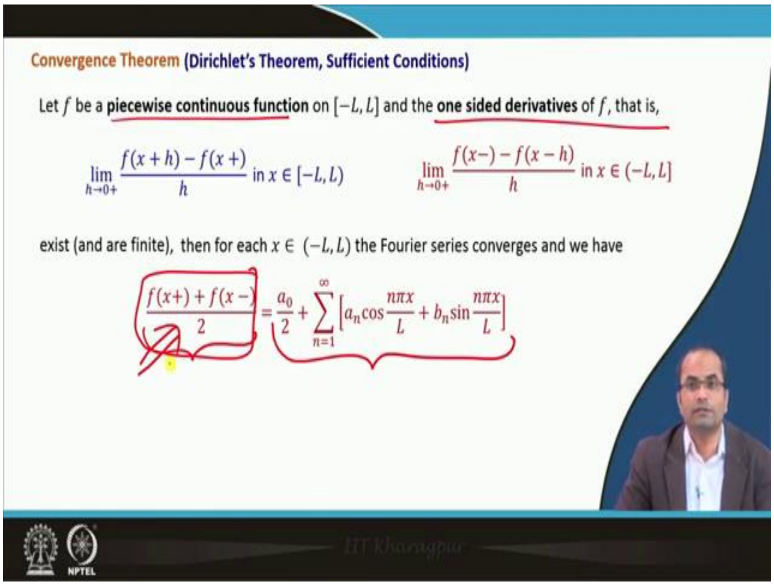

So, if the function is piecewise continuous and one sided derivatives that means, these two numbers, these two limits exist, for each L in these respective intervals and they are finite. So, then for each x, so we can take any x in this open interval minus L to L except the boundary points, we will also discuss the boundary points. The Fourier series converges and it converges. So, this is the Fourier series, the right hand side is the Fourier series of the given function f and it will converge to this average value.

So, the function was piecewise continuous, so it may not be continuous at a given point here x in the interval, but what we have this result that it will converge to this average value, so f x plus is the right hand limit of f at x and this is the left hand limit of f, not the derivative, these are just the limiting values divided by 2 and remember that these limits exist because we are talking about the piecewise continuous function and for piecewise continuous function these limits will definitely exist. So, here we have this nice convergence result that this series will converge not to directly to f x but to the average value of f at that point x.

And what are the conditions now? The piecewise continuity and one sided derivatives. So, under these two conditions, we have the result that this Fourier series will be equal to or it will converge and it will converge to this value, this is the result, the main result of this lecture today.

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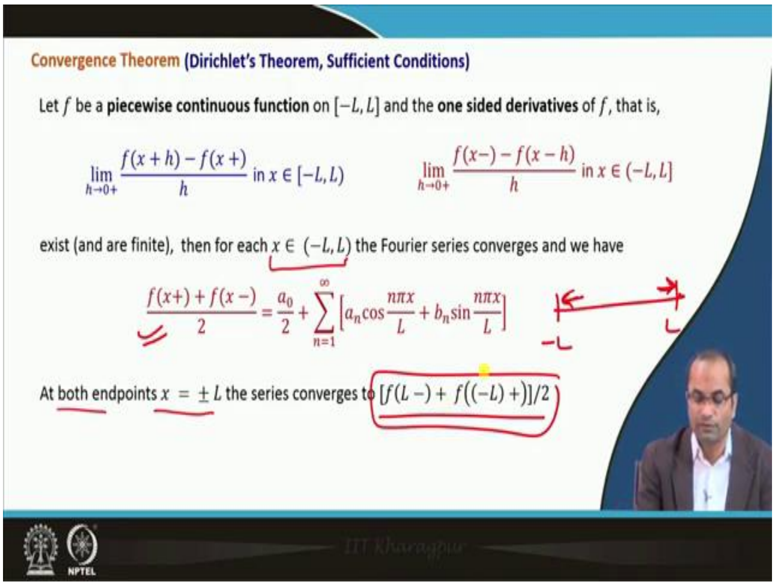

Convergence Theorem (Dirichlet's Theorem, Sufficient Conditions)

Let f be a **piecewise continuous function** on $[-L, L]$ and the **one sided derivatives** of f, that is,

$$\lim_{h\to 0+} \frac{f(x+h)-f(x+)}{h} \text{ in } x \in [-L, L) \qquad \lim_{h\to 0+} \frac{f(x-)-f(x-h)}{h} \text{ in } x \in (-L, L]$$

exist (and are finite), then for each $x \in (-L, L)$ the Fourier series converges and we have

$$\frac{f(x+)+f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

At both endpoints $x = \pm L$ the series converges to $[f(L-) + f((-L)+)]/2$

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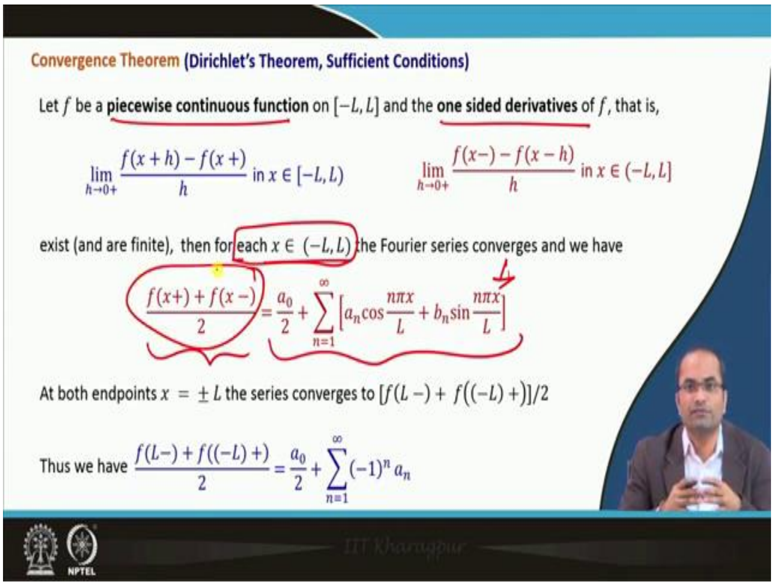

And so, here we have taken all these inner points of the interval and at the end points. So, we have two end points, one is plus L and then the minus L and the series at this these two points will converge to again to these average values, what are these average values? So, suppose our interval was minus L to this L, so, this f L minus, so here we are talking about the left limit and then here minus L, we are talking about the right limit. So, we take these two limits and then take the average.

So, it is the same situation what we have here indeed. So, we will take the two end points, the limiting values obviously here the left, n values, here we will take the right hand values, value and then we will take again the average. So, in a nutshell, at any point, whether it is n point or it is a point somewhere in between, it will always converge the Fourier series will converge to the average value, why at the end point? We are talking about the other end also because we have the periodic function and this function will repeat again at that point.

So, suppose we have a function here, which is given for example, this minus L to L and then since it is periodic, so it will repeat again or let us take a better example where we have actually, no continuity at that point. So, suppose, we have this function which we have also discussed before, but here again I mean now, this will be repeated. So, here this value will go along and then again we have to have this again there. So, therefore, these endpoints we are talking about one limit to the left limit from this end and then right limit from the other end. So, it is actually again the average value at the end points as well if we consider the periodicity of the function.

So, and thus we have at these end points because the end points, the series will be simplified. So, when we put this end points there, this will be sin n pi, so it will become 0 and then here will become minus 1 power n and then we have these a n's there. So, this is equal to the average value of the function again considering its periodicity.

Well, so, this was the convergence theorem or the Dirichlet theorem and these two are the sufficient condition that is again important to note that the piecewise continuity and this existence of one sided derivative, these two conditions are sufficient to make sure that this series converges for each x and this each x is also important that we are fixing x here. So, for each value of x, the value of the Fourier series, value of this trigonometric series will be equal to the average value.

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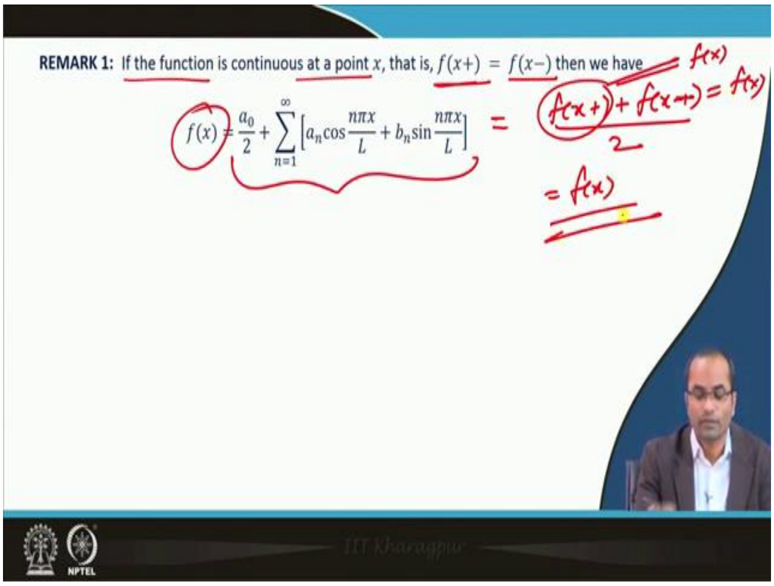

Well, so some remarks, if the function is continuous at a point x, so if it happens to if it happens that the function is continuous at that point where we are talking about the convergence. In that case, these limits f x plus will be equal to f x minus because the function is continuous, the two limits should be equal to the function value if at all it exists there. So, if it is continuous, naturally, these two should be equal limits and in that case this series will converge not to the average value because this was actually the average value f x plus and plus f x minus divided by 2. But if the function is continuous, so this will be also equal to f x and this will be also equal to f x. So, we have 2 times f x divided by 2, it is simply f x.

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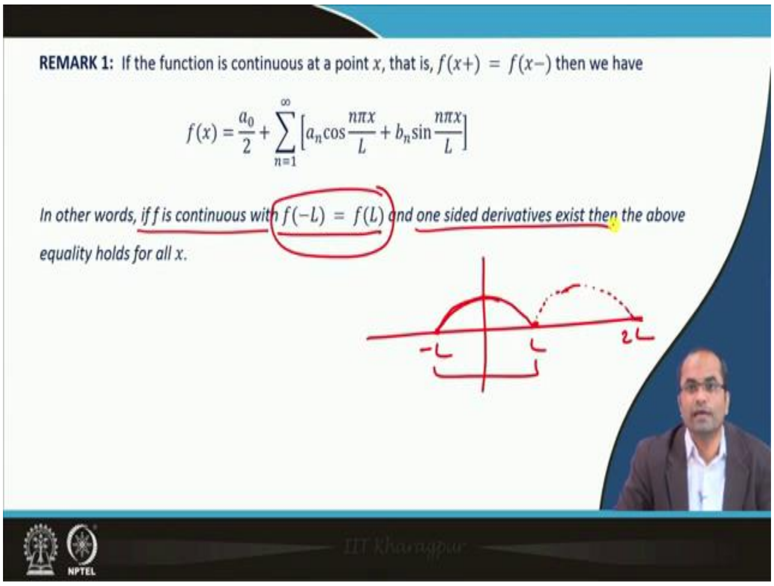

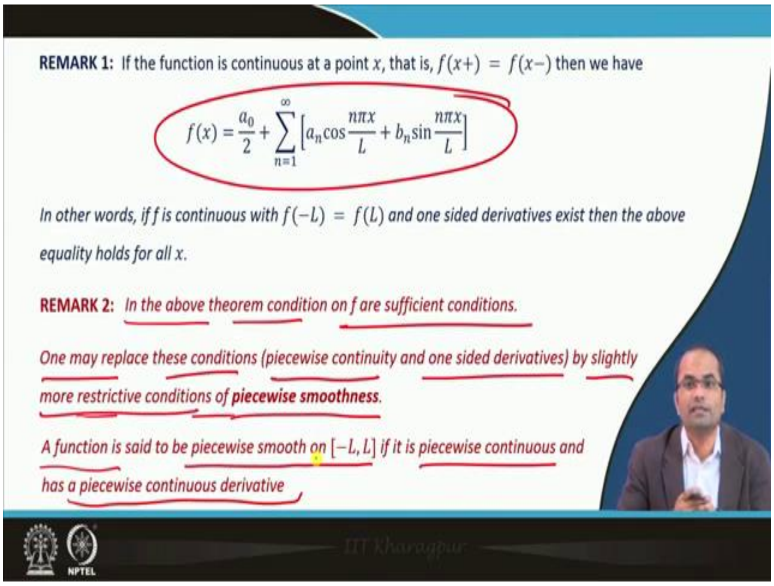

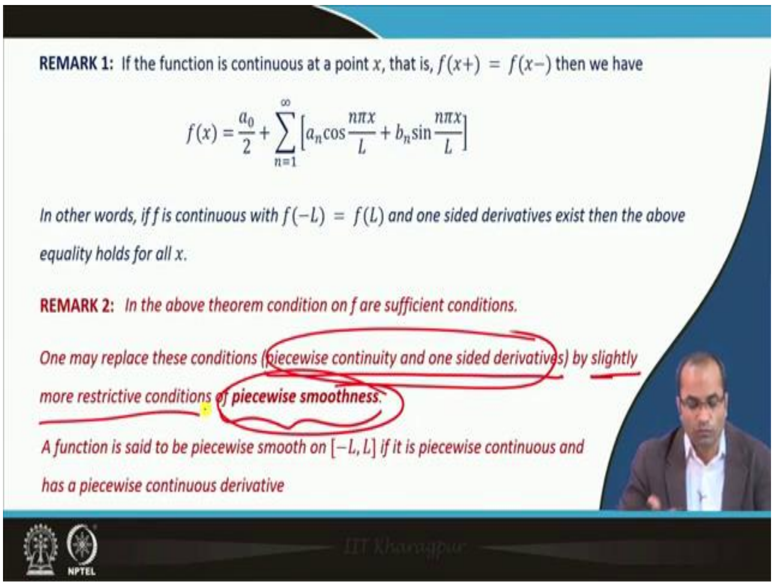

So in that case, if x is a point of continuity, in this case, the series will converge directly to f x. So, that is another result we have that at the point of continuity of the series of the function, the series will converge directly to f x under the case, when those left and right hand derivatives exists, so those are that is the important condition and those are sufficient conditions indeed. So, in other words, if f is continuous, for instance, if the completely the f is continuous that means at the end point also, f is continuous and also at the end point they matches. So, in that case, one sided derivatives exist and then above equality hold for all x.

So, what do we have now, that if f is continuous also the endpoints are same that means, when we take the periodicity of the function, the overall the function will become continuous like in the example I have just discussed before that for instance this is the case here.

So, if we have this continuous function from minus L to L and the end points also matches and the reason is because if we consider this periodicity of the function, then it becomes continuous now everywhere. So, the function is continuous everywhere basically, so if f is continuous in the interval minus L to L and also at the end points the values are equal. So, this was minus L and then this was plus L. So, this is equal and this one sided derivatives, which we have discussed before they exist, in that case the equality holds for all x, then you can take any x and this equality will hold. Another remark, in the above theorem, these conditions are sufficient conditions. So, the conditions we have discussed, the one sided derivatives and piecewise continuity, these are sufficient conditions not necessary conditions.

So, if those conditions are not met, we cannot say that it will not converge but they are just sufficient, if those conditions are met definitely, this will converge. So, one may replace these conditions, the piecewise continuity and one sided derivatives and in many textbook you will find by slightly more restrictive conditions of piecewise smoothness. So, instead of saying piecewise continuity and one sided derivative, we can replace this by slightly more restrictive condition of piecewise smoothness, what is the piecewise smoothness? We will also discuss in brief here, so a function is said to be piecewise smooth in this interval minus L to L, if it is piecewise continuous in that interval and we have already discussed what do we mean by piecewise continuity and it has a piecewise continuous derivative.

So, if its derivative also in this region where it is a continuous, we can find the derivative and if it happens that or the derivative of this function is also piecewise continuous and the function is piecewise continuous, then we call that the function is piecewise smooth.

So, we have now, instead of this piecewise continuity and one sided derivative, we can replace in this earlier theorem by piecewise smoothness. So, we can say if the function is piecewise smooth then this series will converge to that average value. Now, what is the connection between this piecewise continuity and one sided derivative to the piecewise smoothness? We have written here that it is a slightly more restrictive putting the condition piecewise smoothness, it is more restrictive and why it is more restrictive this will be clear now with the help of the following example.

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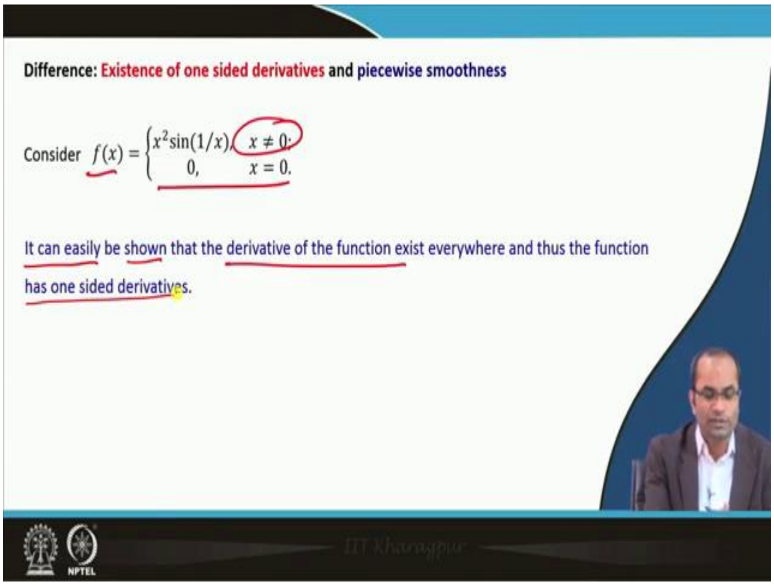

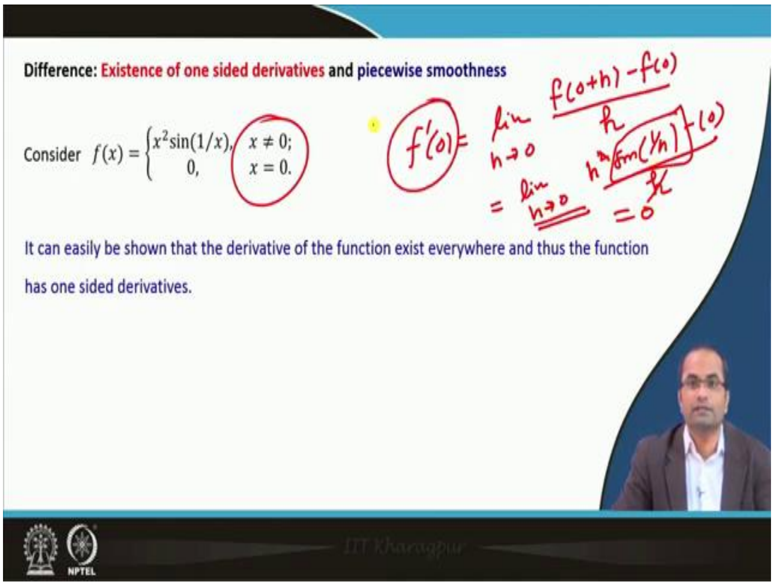

Difference: Existence of one sided derivatives and piecewise smoothness

Consider $f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$

It can easily be shown that the derivative of the function exist everywhere and thus the function has one sided derivatives.

However the function is not piecewise smooth because the $\lim_{x\to 0} f'(x)$ does not exist as

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

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Consider $f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$

It can easily be shown that the derivative of the function exist everywhere and thus the function has one sided derivatives.

However the function is not piecewise smooth because the $\lim_{x\to 0} f'(x)$ does not exist as

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Not p.c.

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So, if we consider for instance, this example, x square sin 1 over x and when x is not equal to 0 and 0 otherwise. In that case, we can easily show that the derivative of the function exist everywhere and thus the function has one sided derivatives. See, if the function has the derivative everywhere, definitely, it has the left hand and the right hand derivative because that is also the definition. If the left hand derivative exists, right hand derivative exists and they are equal then the derivative exists. So, here for this function, the problem is at x equal to 0 but we can show that the derivative at 0 also exists because we can show with the definition that limit h goes to 0, f 0 plus h minus this f 0 divided by h is equal to the limit h goes to 0 whether from positive or negative side does not matter.

So, here f h that means we have this h square and sin 1 over h and minus f 0 that is given as 0, and then we have h, so this h and h will get cancelled, and as h goes to 0, so this is something bounded here, h goes to 0, this will go to 0. So this limit exists. That means the function has derivative at 0 as well. And at all other points, we can easily just differentiate this and we can find out the derivative. So, this function has derivative everywhere, that means the left hand and the right hand derivative exists for this function. However, what we will observe that this is not piecewise smooth and as we said before that the piecewise smoothness is putting more restrictions on the function.

So, this is one example where we can easily visualize this that this function is not piecewise smooth, though it has left hand and the right hand derivative at all points. Indeed, at all this is a differentiable function, the derivative exist everywhere. But it is not piecewise smooth because this limit here f prime x, the f prime x is not piecewise smooth. So, if you compute here f prime x, if you compute f prime x that means at x equal to 0, the derivative is 0 we have just evaluated. And when x is not equal to 0, we can differentiate this there is no issue, you can directly differentiate. So, we got the derivative everywhere here at prime x when x is not equal to 0 given by this when x equal to 0, it will be given by it is just 0.

So, in this case, this limit does not exist when we consider the limit of this function as x approaches to 0. So, from here, we have to compute when x approaches to 0, this will go to 0 plus and this cos 1 over x does not exist. So, basically this limit does not exist, when x goes to 0. This f prime x, x goes to 0 does not exist and hence this is not piecewise smooth function because the derivative must be piecewise continuous function but this is not. This is not piecewise not piecewise continuous, this is not piecewise continuous, the f prime is not piecewise continuous. Hence it is not piecewise smooth, the f is not piecewise smooth.

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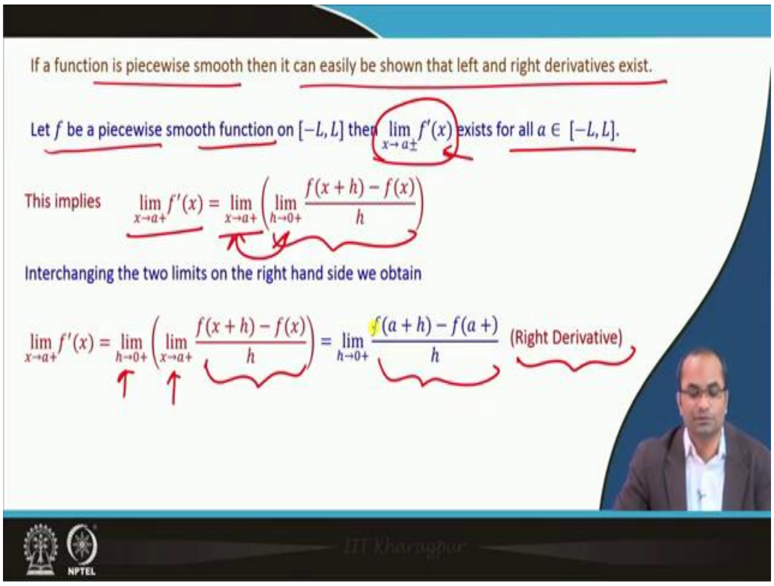

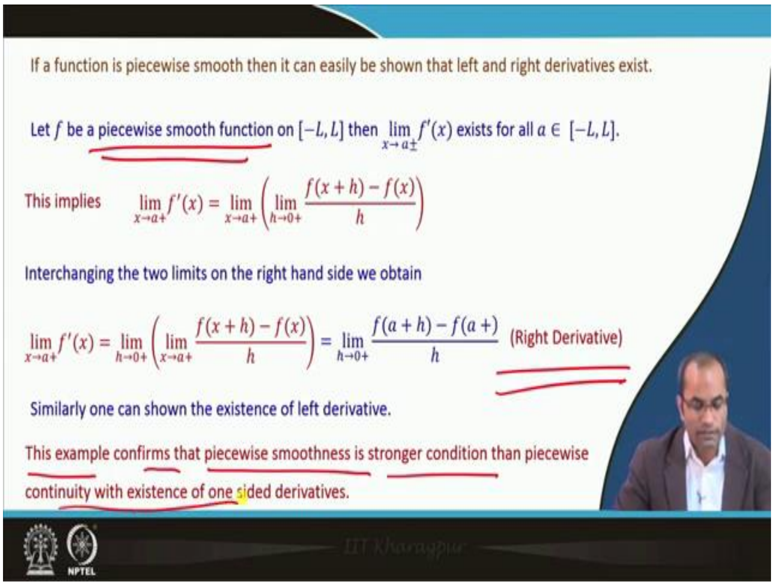

However, if a function is piecewise smooth, it can be easily shown that left and right hand derivatives exist. The other way around, we can show if the function is piecewise smooth we can show that left and right derivative exists. And for that, we can just consider that f is a piecewise smooth function, that means all these limits exist, the derivative is piecewise continuous and this limit must exist for all points a in this interval. And this implies that if this limit exists, what do we mean by this limit? Limit x goes to a plus and the derivative f prime x we have just written here.

And now we can interchange these 2 limits and what will happen, so if we interchange, so first we are having here h goes to 0 and then the limit x goes to a of this quotient here and here the limit x goes to a will be just f a plus h minus f a plus divided by h and this is exactly the right hand derivative at a. So, if the function is piecewise smooth, if the function is piecewise smooth then the right derivative exists. And similarly, we can show that the left derivative also exists.

So, what we have seen in this example confirms that piecewise smoothness is stronger than piecewise continuity with existence of one sided derivative.

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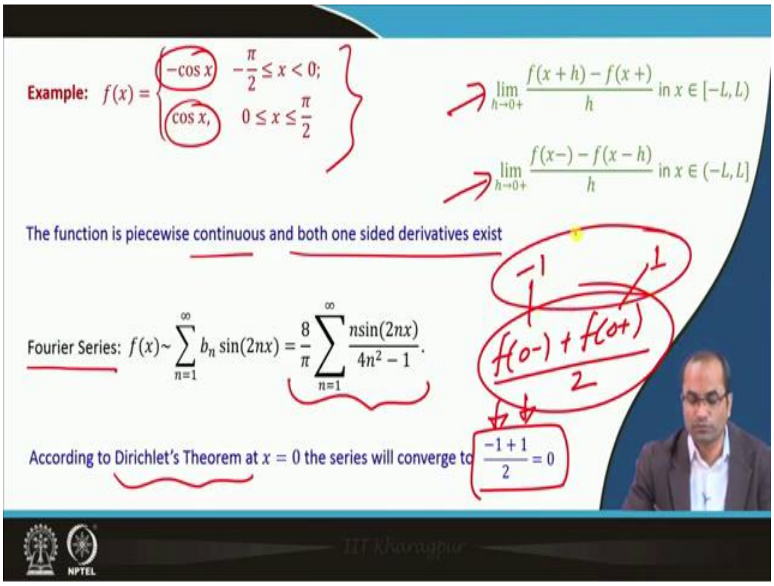

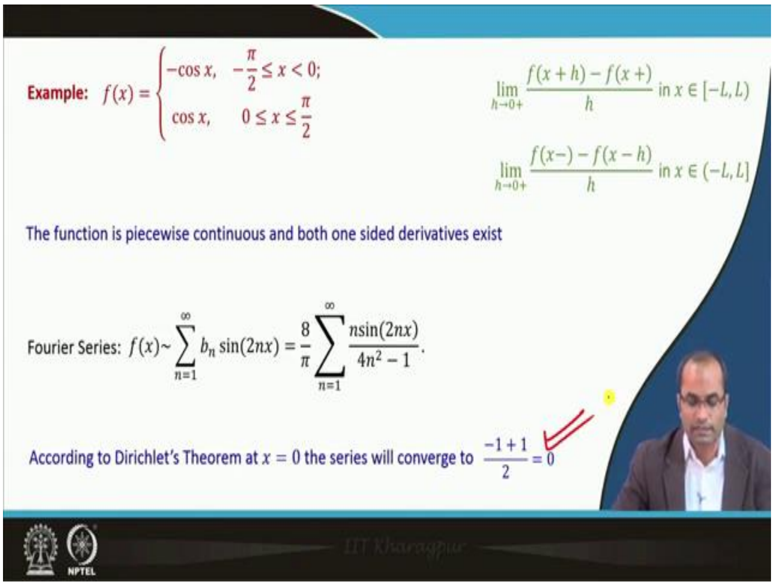

So, just an example to apply this theorem, so we have for instance this example, which was discussed at the beginning of this lecture and the function is piecewise continuous because both, one sided derivative exists. So, this we can evaluate and this also we can evaluate and both of them exist in this case. So, the Fourier series now which was given by this, according to this Dirichlet's theorem, according to this theorem what we have just discussed, the series must converge to the average value because at x equal to 0, we should take the average value. At one side we have here at x equal to 0, the value is 1. And the other side, when we take the limit, it is minus cos going to 0, so it is minus 1.

So, we have minus 1 and plus we have 1. So, f 0 minus plus f 0 plus we have to consider and divide by 2. So, the 0 plus is 1 and the 0 minus is minus 1 and which adds to 0. So therefore, we get this value 0, which was also visible from the plot, which we have just discussed at the beginning of this lecture there.

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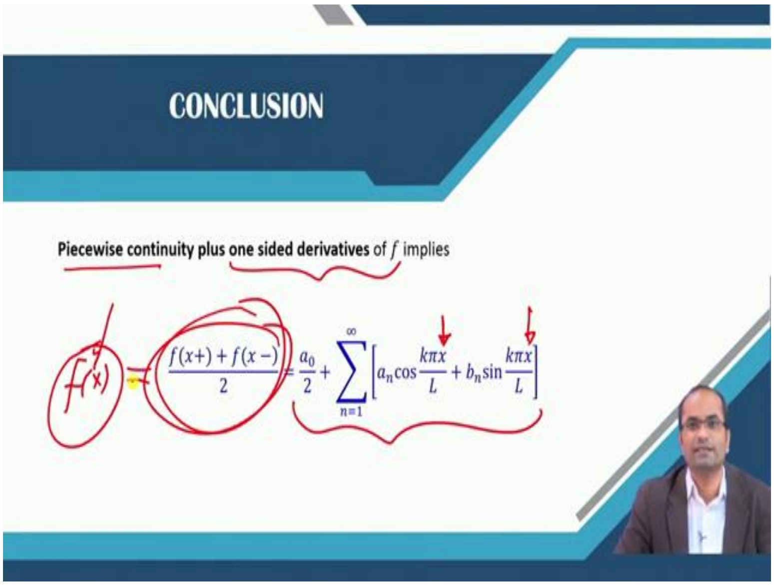

Well, so these are the references we have used for preparing this lecture. And just to conclude, now here we have discussed that piecewise continuity and one sided existence of one sided of one sided derivative. So, of f implies the convergence and the convergence is says that the value of this series, the sum of the series will be equal to the average value and if at that x the function is continuous, then it will be simply the functional value at that particular point x. So, this was for this lecture where we have considered that for each x for each value of x we are getting the sum as f x or equal to this average value.

But there are some more notions of the convergence which we will be discussed in the next lecture. So, that is all for this lecture. And thank you for your attention.