Engineering Mathematics - II
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Lecture 33
Fourier Series - Evaluation

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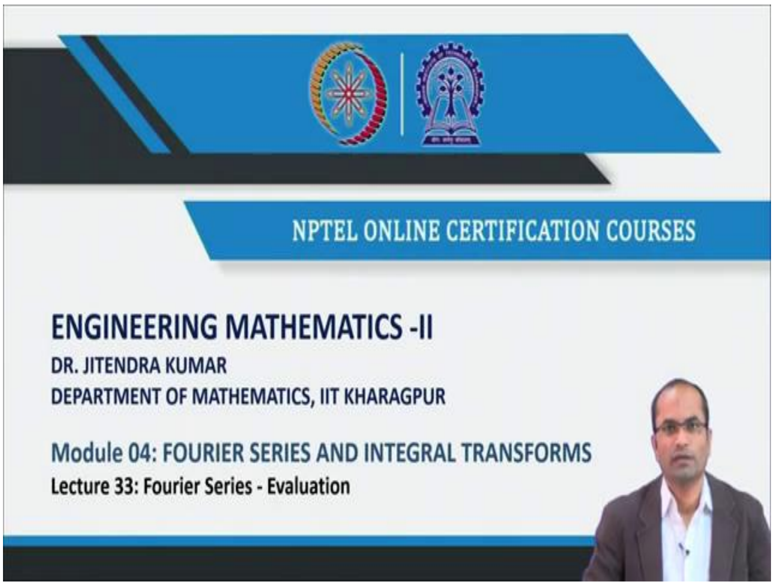

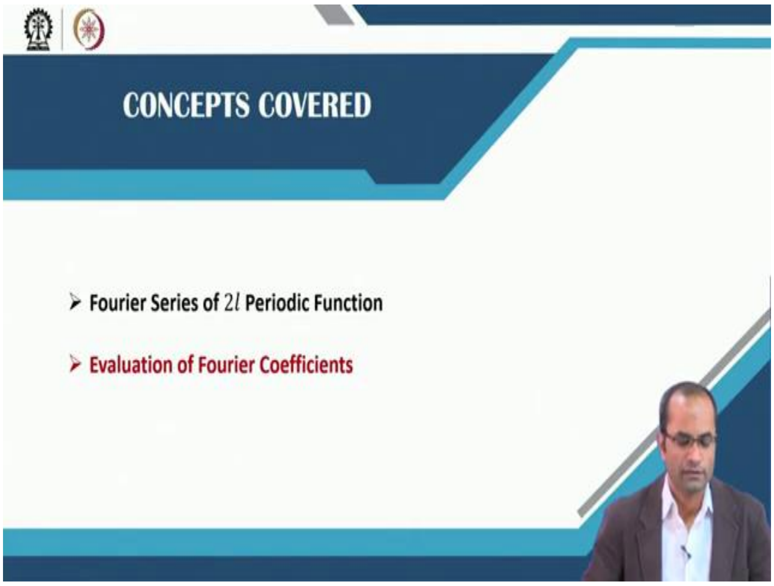

So, welcome back to lectures on Engineering Mathematics - II and this is lecture on Fourier series and we will evaluate for different functions. So, today we will cover the Fourier series of 12 periodic functions. So, in the previous lecture, we have already discussed the Fourier series of 2 pi periodic function. So, that is just a generalization of those 2 pi periodic function and then we will evaluate the Fourier coefficients and respectively the Fourier series for many functions.

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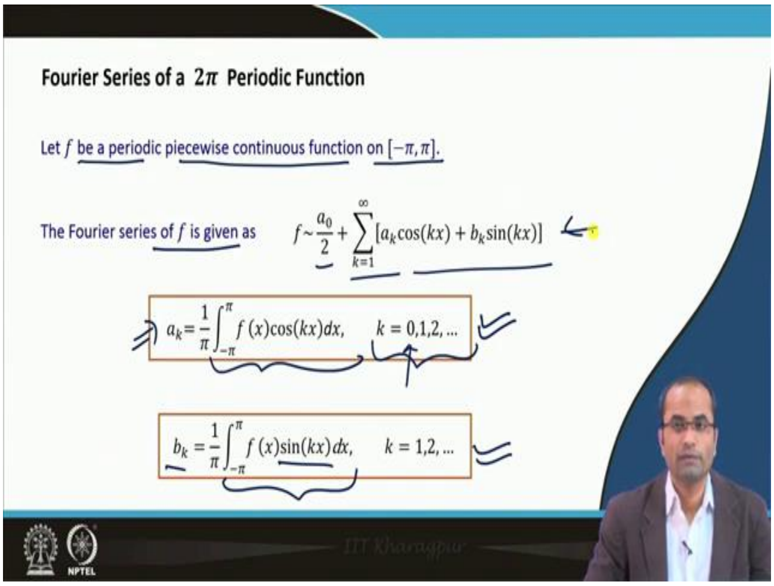

So, just to recall that Fourier series of a 2 pi periodic function we have discussed in the previous lecture and suppose this f is a periodic piecewise continuous function on the interval minus pi to pi. So, it is Fourier series is given by this trigonometric series a by 2 and then we have the summation over a k cos k x plus b k sine k x, where the coefficients, the so called Fourier coefficients a k, we can compute by this integral from minus pi to pi f x cos k x dx.

And similarly, the coefficients b k, we can compute again with the similar integral having this sin term here sin k x dx and this K goes from 1, 2, 3 and so on. Here, this a k was valid for a 0 as well. So, therefore, this case starts from 0 and then continues to 1 and 2. So, it is extension for 2l periodic function will take a similar structure and we have already discussed the trigonometric system which is 12 periodic in previous lectures.

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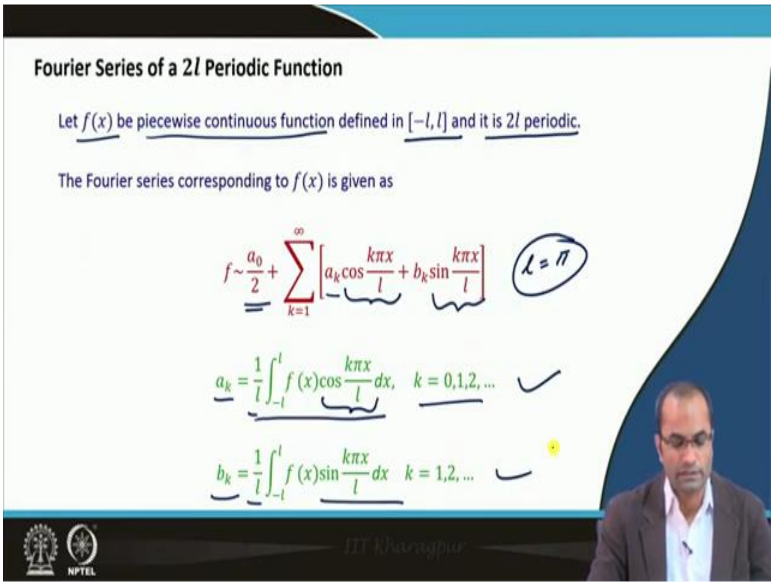

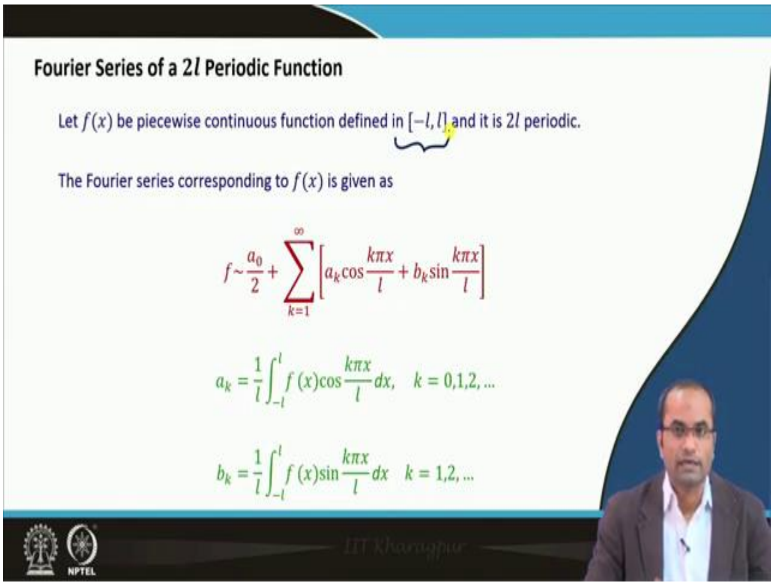

So, here for the 2l periodic function, let us suppose this f x piecewise continuous function defined in this interval minus l to l or from 0 to 2l, whatever ways and it is supposed 2l periodic then it is Fourier series will be given as, so the a naught by 2, a constant term and then we have this a k and instead of cos pi x.

Now, we have k pi x over l and here sin k pi x over l. So, for l is equal to pi this will reduce exactly to 2 pi periodic function which we have already discussed? And the coefficients are also similar now, so a k we have here 1 over 1 instead of 1 over pi, so it is 1 over 1 and then minus 1 to 1 f x and then here we have the cos term cos pi x over l and then k goes from 0, 1, 2, 3, et cetera.

So, the b k is again 1 over l and then instead of cos kx we have sin pi k pi x over l and then k in this case varies from 1 to et cetera. So, this is a more general case now for any 2l periodic function or the function defined in this minus 1 to 1, we can have, we can write down the Fourier series.

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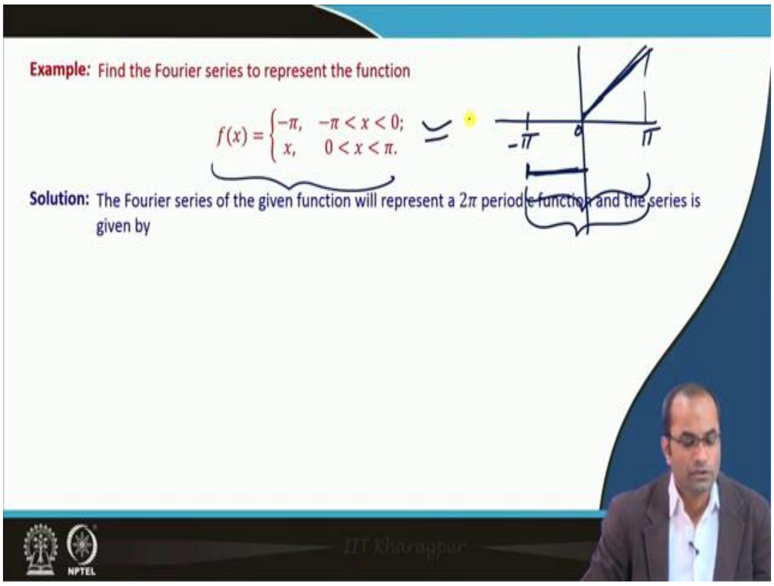

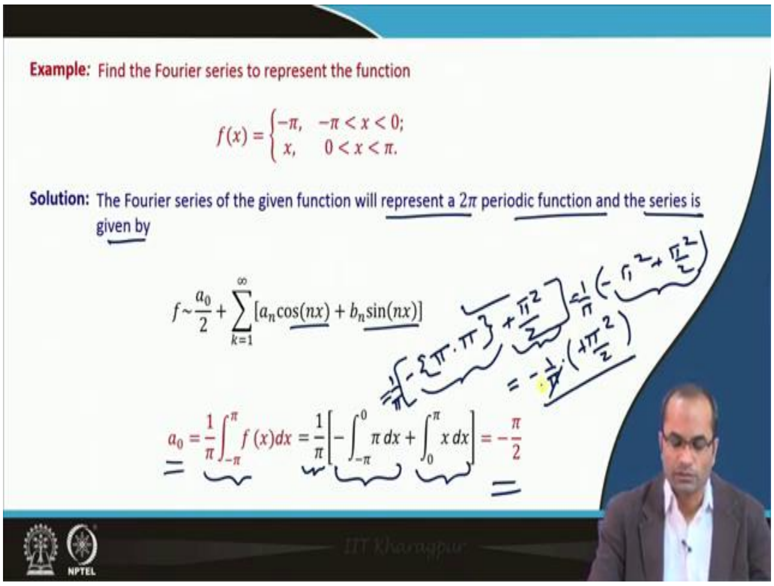

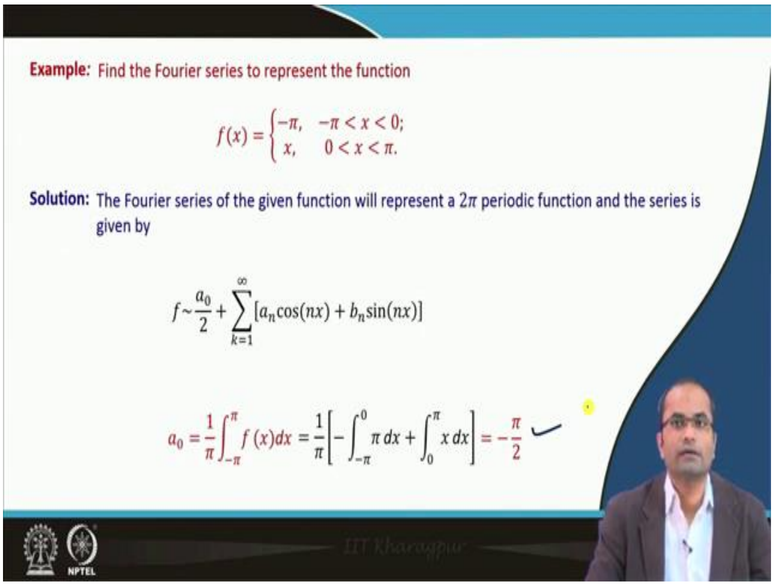

So, now this some examples, the first one we want to find the Fourier series to represent this function for instance the f x is given as minus pi. So, f x is given as minus pi when in the range minus pi to 0. So, this is the function and then 0 to pi, the function is given by x, so this is pi, this is minus pi.

So, that is the given function here from minus pi to 0, it is represented by minus pi and then the function is given by x in the range 0 to pi. So, we want to, so it is a piecewise continuous function and we want to write its Fourier series. So, its it will be a 2 pi periodic function to whom the Fourier series will converge. So, but just to write the Fourier series, we need a function defined in some intervals.

So, in this case it is given in the interval minus pi to pi and now we will write the Fourier series and that Fourier series if it converge, it will converge to this 2 pi periodic function where in this one period, the function will be represented by this given a function here. So, the Fourier series of the given function will represent a 2 pi periodic function and the series is given by this standard series because we are talking about 2 pi periodic function.

So, in this case, we have written this standard series a n cos nx and b n sin nx. So, the a naught we know already it is a 1 by pi n minus pi 2 pi f x dx. So, we can compute this integer, so we have 1 minus pi and then from minus pi to 0, the function was defined as minus pi. So, we have substituted this and 0 to pi, the function is defined as x.

So, if we compute this first integral for instance, so it is a pi n then we have x, so 0 minus and then minus x, so this is minus pi square and then here pi square by 2 will come from the second one. So, this is the and then 1 over pi is, was also sitting there. So, minus pi square and then plus pi square by 2 and then 1 over pi is there.

So, then here we have minus pi the square by 2. So, this pi pi get cancel and we will got, we will get just by pi by 2 with a minus sign because this was the minus here minus pi square and then we had this plus pi square by 2 in the bracket and the 1 over pi was sitting outside. So, it is a minus pi square by 2 is coming with 1 over pi, so we are getting this 1 over, no minus 1 by 2 into pi. So, that is the a naught and now we will compute a n and b n.

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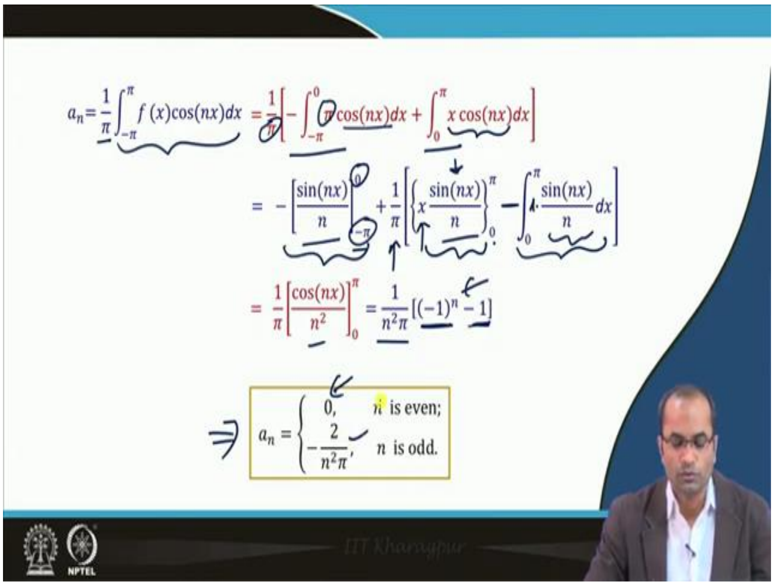

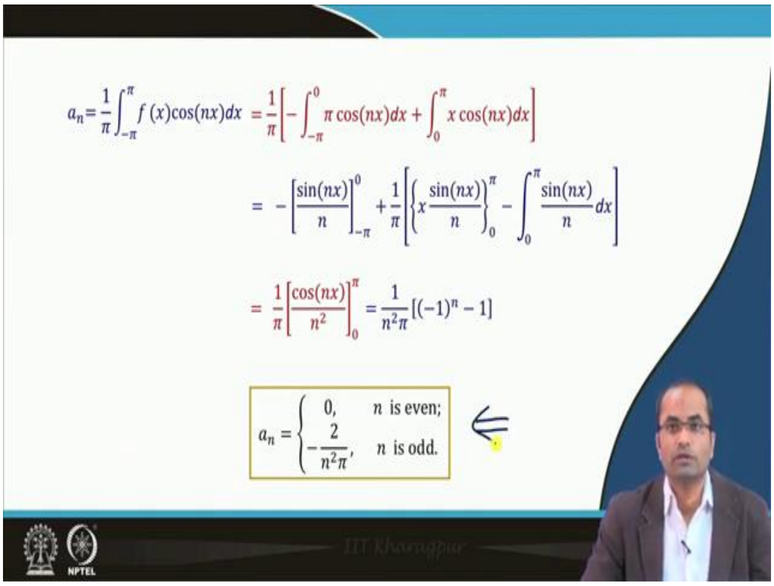

So, for a n we have this formula 1 over pi and this f x cos nx, so again f x is given as minus pi in the range minus pi to pi and 0 to pi it is given as x. So, we need to compute now these two integrals. So, for the first one, we have this pi will get cancel and so, we have not written here

1 over pi over or this pi, so this pi and this pi gets cancel for the first one. We have cos nx, so that will be sin x over n and limit is minus pi to 0 for the second one.

So, we have 1 over this pie outside and then we have to apply this partial fractions, sorry the partial integration rule. So, here the x is there and then cos nx will from the cos nx, we will get the sin nx by n and the integral the limits will be 0 to pi and then minus differentiation of this x will be 1 and then again we have sin nx over n and then limit 0 to pi.

So, the first one when we have the upper limit here, sin will become 0 and then for pi also this will become 0. So, the from the first term, we are not getting anything, it is a 0. From this second term also when the because of this sin here pi n 0, so this will again become 0. And the last term will contribute.

So, we have one over pi and the sin again will be integrated to get this cos. So, we have the cos nx over n square, 0 to pi and this minus is adjusted for this integral of the sin nx. So, we have 1 over pi, cos nx over n square and 0 to pi. So, when substituting these limits, we have the cos n pi which is minus 1 power n and minus this cos 0 that is 1 and then 1 over n squared pi just sitting outside.

So, this a n, the co-efficient these Fourier coefficients a n are now computed as when n is an even number, so this will be positive and this is negative. So, this will get cancel and we will get 0 and when n is odd, so this will be minus and this is minus 1. So, we have minus 2 divided by n square over pi, so this we have got the Fourier coefficients the a n.

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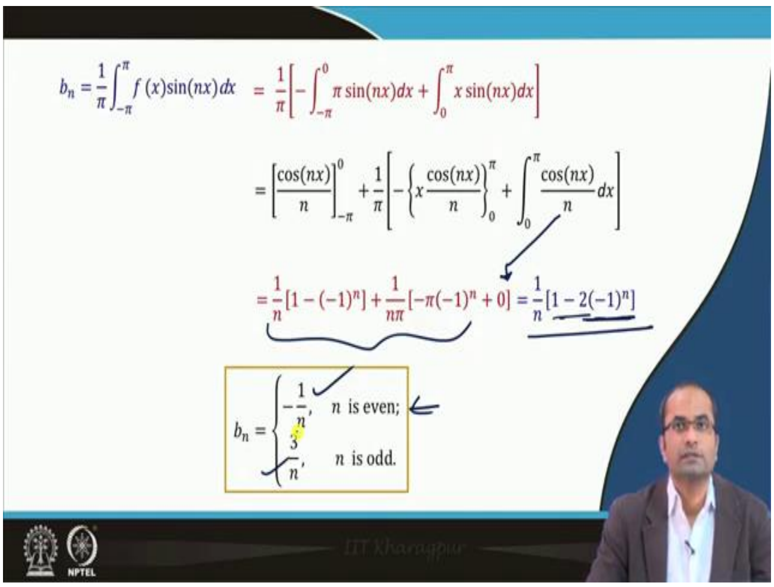

And now similarly, we can also compute b n, so in b n we have here the sin function. So, again so minus pi to 0 we have pi, the value of the function and 0 to pi we have the value of the function as x. So, here the first one this pi pi get canceled, sin will be cos nx over n and this minus will be adjust for the integration and then minus pi to 0 and then we have 1 over pi sitting outside and then again we have integrated this x sin and x. So, x as it is and sin nx became this cos nx with the minus sign and then minus minus plus again, the cos nx over n dx.

So, this time this term will not vanish when the 0, we have this 1 and then minus pi, we will also get cos n pi which is minus 1 power n. And then in this second case, we have 1 over n pi and here we will get with pi and then cos nx will give minus 1 power n. And in this case when we integrate this, this will become sin nx and divide by n square, but when we substitute the limits, so the upper limit or the lower limit because of the sine function, this will become 0. So, that is the contribution of the last integral as 0. And now we can simplify this.

So, we are getting 1 over n and 1 minus 2 minus 1 power n or in this b n for n even because when n is even here, we will get this minus 2, so 1 minus 2 the minus 1 over n will come and when n is odd, so this minus 1 power n will be minus, so these two will be added. And so, we will get this 3 over n. So, that is the co-efficient b n we have.

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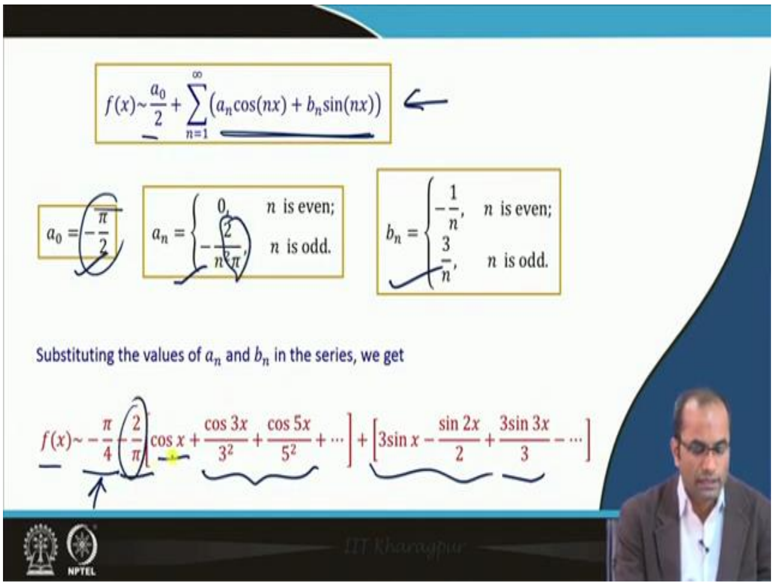

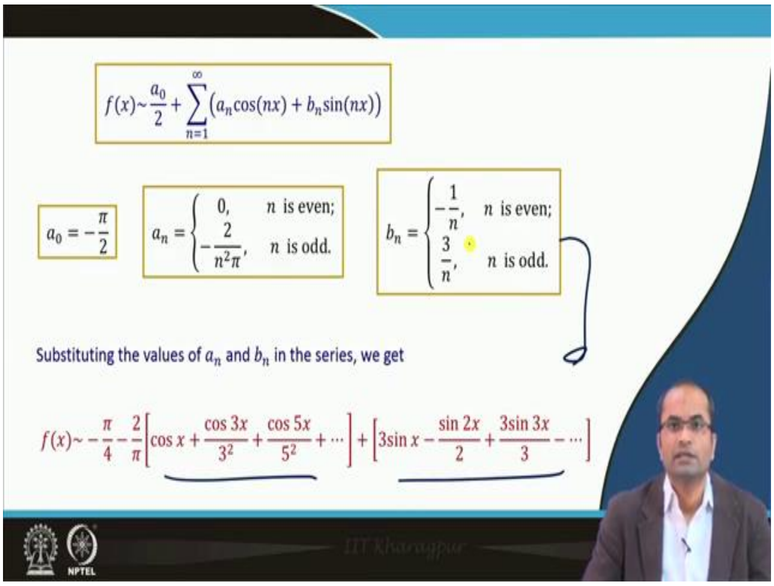

Now, coming to the Fourier series, so we can write down its Fourier series now, so a by 2 and then this cos nx and b n sin nx term where a is 0, a n and b n, we have just computed, so we can substitute these coefficients in the Fourier series and we can obtain the Fourier series as follows.

So, f x we have minus pi by 4, so a by 2 this will be minus pi by 2 and then we have minus 2 over pi. That is the term will come from when we substitute this a n and b n terms. So, this cos, we have just written in this expanded form. So, this is coming because of this 2 over pi was here in the cos, so 2 over pi is here and then we will get these cos terms and then in the sin terms will be coming exactly from the b n.

So, in this way we can obtain the Fourier series for a given function, the function was piecewise continuous. The piecewise continuity in the last lecture, we have already discussed that ensures the integrability of the function in the in this range where we are integrating for the period. So, the a n and b n definitely be computed if these the given function is piecewise continuous.

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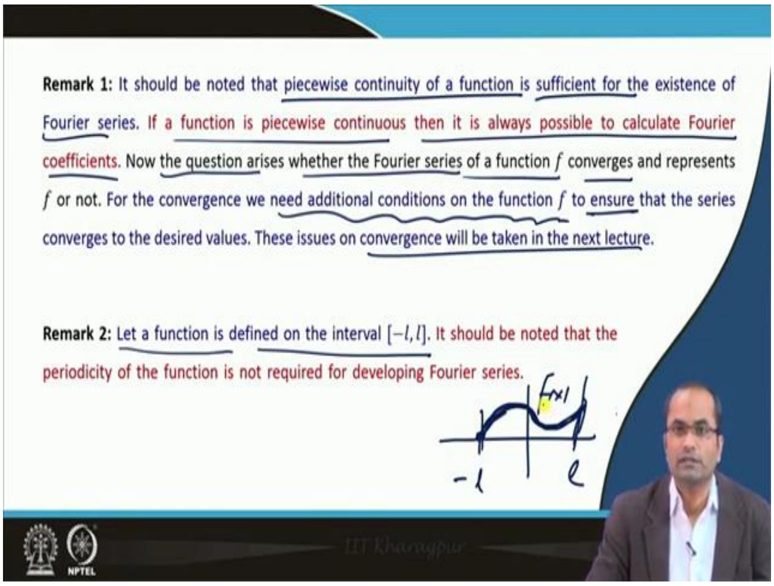

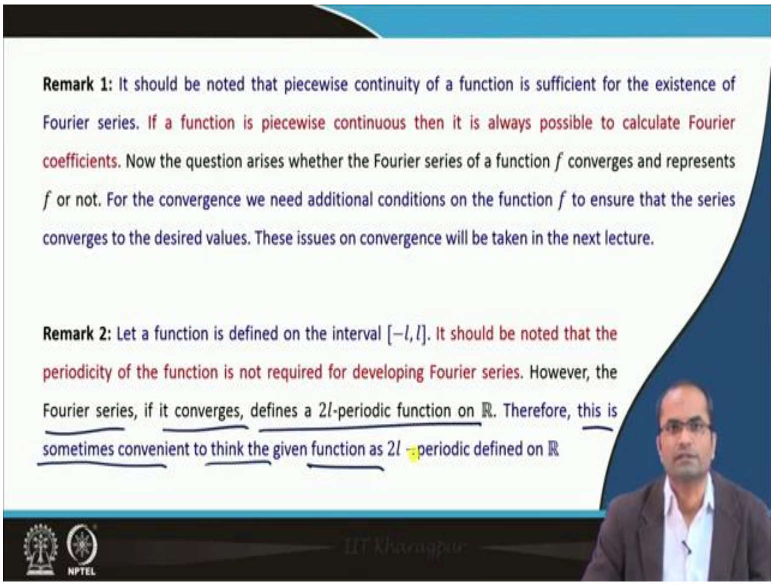

So, just a remark that it should be noted that the piecewise continuity of a function is sufficient. So, that is sufficient exactly what I was discussing here for the existence of the Fourier series. So, for the existence of the Fourier series, what do we need? We need piecewise continue of the function.

Because if the function is piecewise continuous, we can get a n and b n because the function will be integrable piecewise continuous functions are integral in that close interval. So, that is sufficient for the existence of the Fourier series. So, if a function is piecewise continuous then this is always possible to calculate Fourier coefficients. That is very clear.

Now, the question arises whether the Fourier series of a function converges and represent f 1 naught, that is the question which has to be discussed and we will discuss in the next lecture.

And there we will see that we need some additional conditions on the function, not just the peace piecewise continuity but we need some more conditions to ensure that the series converges to the function to the desired values. So, these issues we will take in the in the next lecture.

Now, another remark let a function be defined in this minus l to l. So, we have a function which is defined minus l to l that is all and periodicity of the function is not required as such to develop the Fourier series because we need a function defined minus l to l or minus pi to pi for the special case.

And then we can find out a n and b n and we can write down the corresponding Fourier series. However, if this Fourier series converges, that will converge to a periodic function with the period that too well. But at a beginning, we have a function which is just defined from minus l to l.

And if it is not periodic or we do not have any other information, just the function is defined from this minus l to l or from 0 to 2l, whatever. So, and then we can write down its Fourier series, so the function is defined and that is all we do not have any other information, but we can write down its Fourier series.

And then the Fourier series if it converge, if it converges, it will converge to a periodic function and in one period, that function will look like exactly this affects this given function to whom we have written the Fourier series. So, I think that point is clear. However the Fourier series if it converges it will define it to pi periodic function on R.

And therefore, this is sometimes convenient just to think always that the given function is a to 2l periodic defined on the whole R but as it is not required a function is given in some domain we can write down its Fourier series and if this Fourier series converges, it will converge to a periodic function in 1 period, the function will be exactly the same which we have used for constructing the Fourier series.

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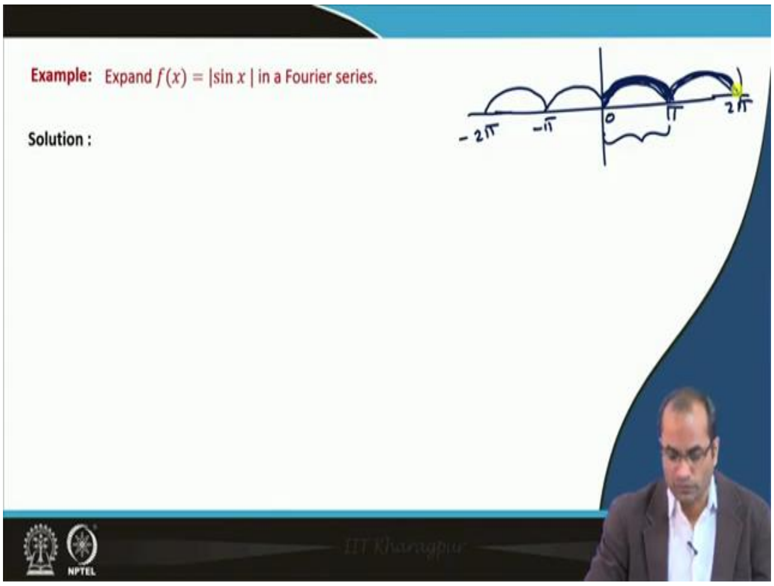

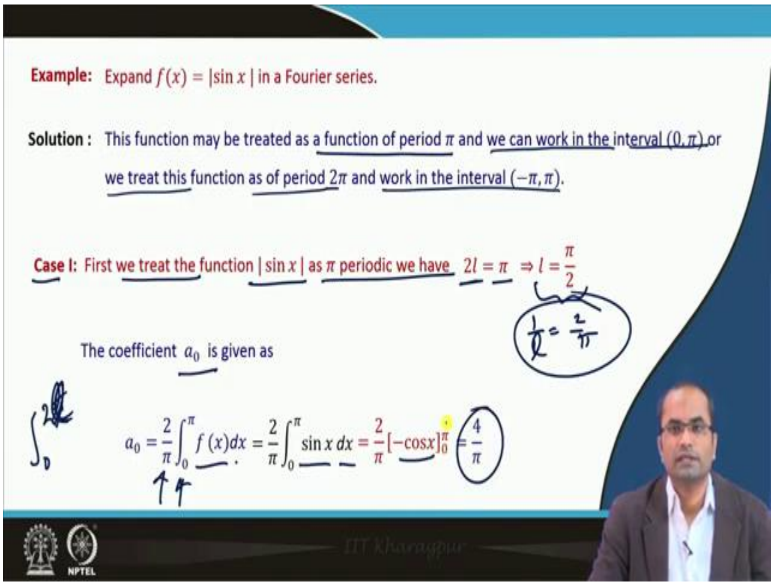

So, we have the another example where we will expand this f x is equal to sin x in a Fourier series. So, this sin x function if we just take a look, it is with a modulus sin x, so it is a 0 here and then at pi it will become again 0 and then since the positive, so it will never go to the negative portion.

So, this will be the graph of this function, so 0 to pi and then 2 pi et cetera and then we have minus pi minus 2 pi and so on. So, here there are, there could be many ways, but we are just considering 2 cases just to show that the period actually does not matter. For example, here the period of this function is pi. So, if we take period pi that means a function given in this range here and write down the Fourier series over we take a function from the 0 to 2 pi and then write down the Fourier series.

Both the Fourier series will be same and that we will demonstrate using this example for instance. So, this function may be treated as a function of period pi. And we can work in the interval 0 to pi or we can treat this function as a function of period 2 pi and we can work in the interval minus pi to pi. So, we will take 2 cases here, in the first case, we will treat this function as a function of period pi which is actually the period, the fundamental period of this function.

And in the second case, we will multiply by 2 this fundamental period pi that means we will take a 2 pi, we will treat this function as 2 pi periodic function and then construct its Fourier series and at the end we will realize that it is actually the same whether we take a period pi or take period 2 pi or 3 pi, et cetera.

It does not matter for the Fourier series, we will have the same Fourier series. So, in the first case, we will treat this function sin x as by periodic and then we have so for pi periodic that is the general periods not a standard minus pi to pi we take it is a pi periodic, that means a 2l is pi and then we have this l as pi by 2, so in general version of this Fourier series we are going to use now.

So, the coefficient a naught as per the definition we have this 1 over l, so 1 over l will become 1 over l will become 2 over pi. So, this is 1 over l outside is 2 over pi. And then we will integrate from 0 to 2l or from minus l to 2l. So, in this case, we are taking 0 to 2l, so 2l is pi, so 0 to pi f x dx. It is 2 over pi, the f x is from 0 to pi is just sin x. It is a positive, so we can remove the mod here, dx and the sin x will be cos x will minus sin 0 to pi, so that we will get 4 over pi. So, that is the first coefficient a naught 4 over pi.

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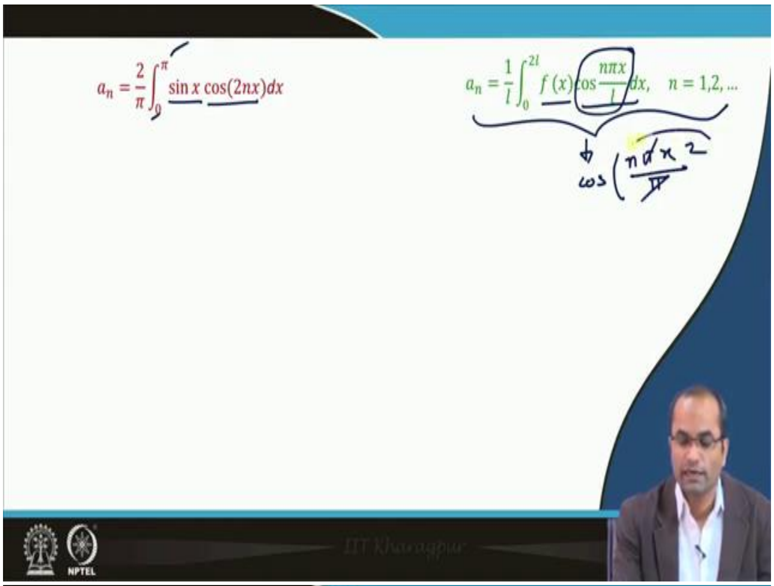

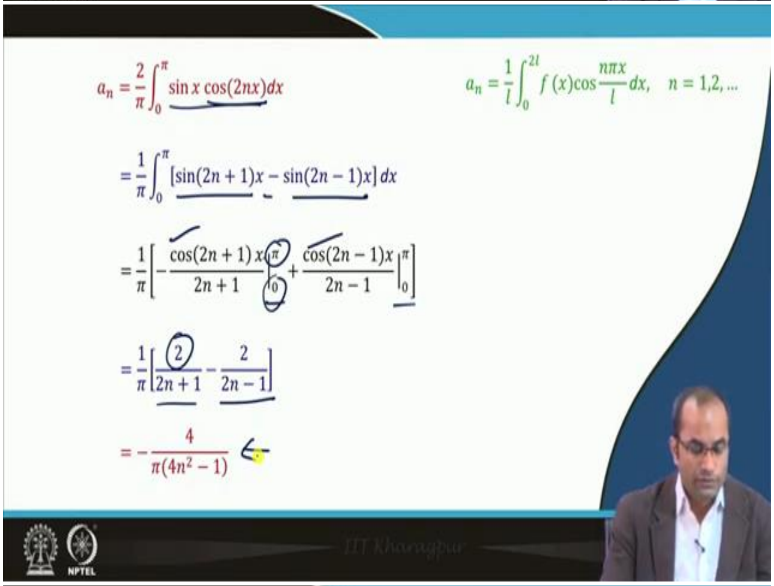

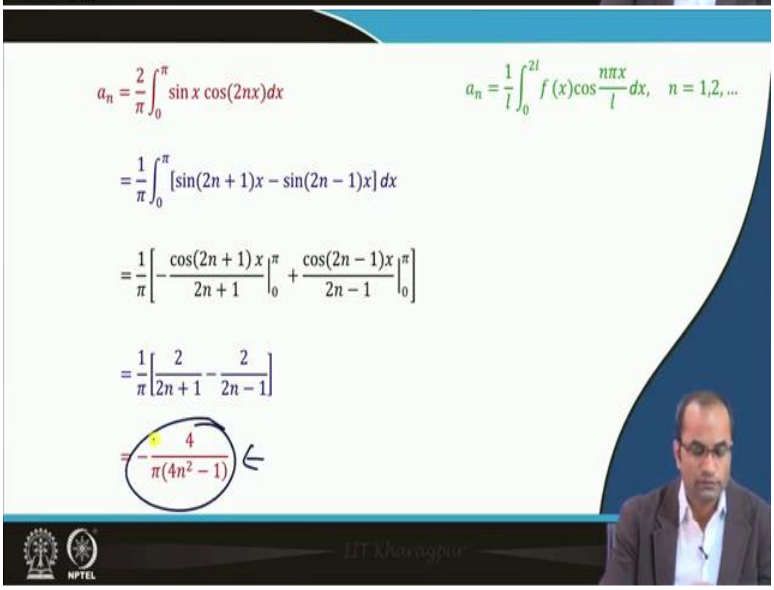

Now, we will compute a n, so the formula is given here 1 over l, 0 to 2l f x cos n pi x l and pi x over l dx. So, here we will take this f x. So, 0 to 2l that means 0 to pi, the function is positive in 0 to pi, so we have a sin x and then we have cos 2nx. So, cos 2nx, how we are getting cos 2nx? The cos and then n pi x and l is pi by 2. So, this pi gets cancel and we have 2nx.

So, this is 2nx and we can integrate now this, so 2 times sin a cos b, formula we will apply. So, to sin this, the sum will come 2n plus 1x and minus the sin will come with 2n minus 1x and then we can integrate it having this cos at both the places and then limit 0 to pi. Substituting this pi there we have 2n plus 1, so we will get this value 2 there and here also we have cos then 2n plus 1 again.

So, because we have the 0 also and the pi, so this is going to be 1 and then again this is going to be when we substitute 0, 1. So, we will get this 2 there and the value after simplification, we will get minus 4 over pi, 4n square and minus 1. Well, so, what we will get now? So, this is the coefficient for a n and n is 1, 2, 3. So, we got the value already.

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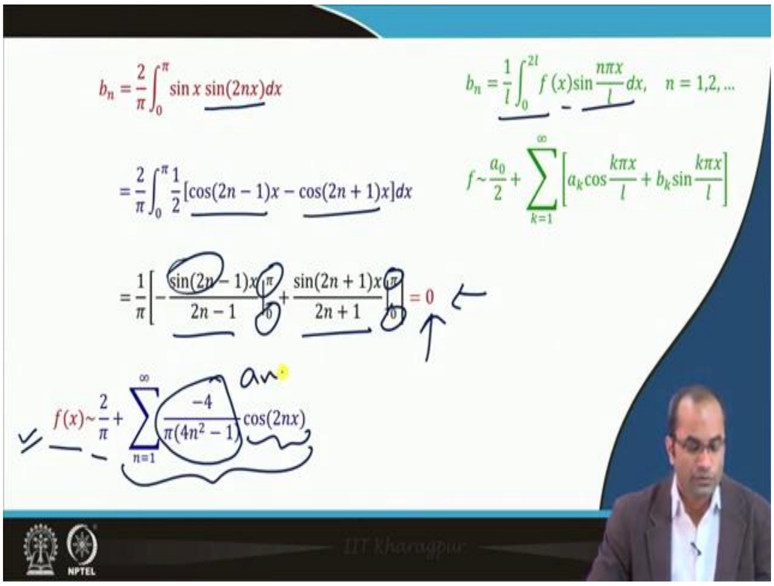

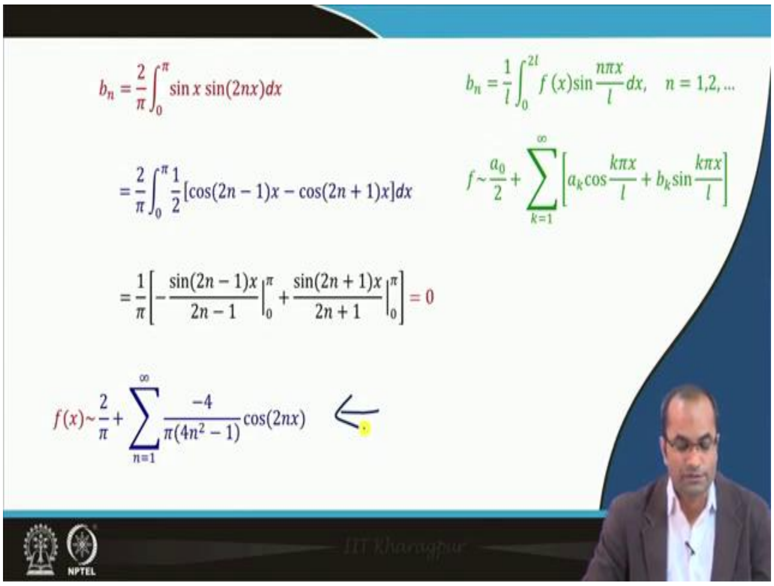

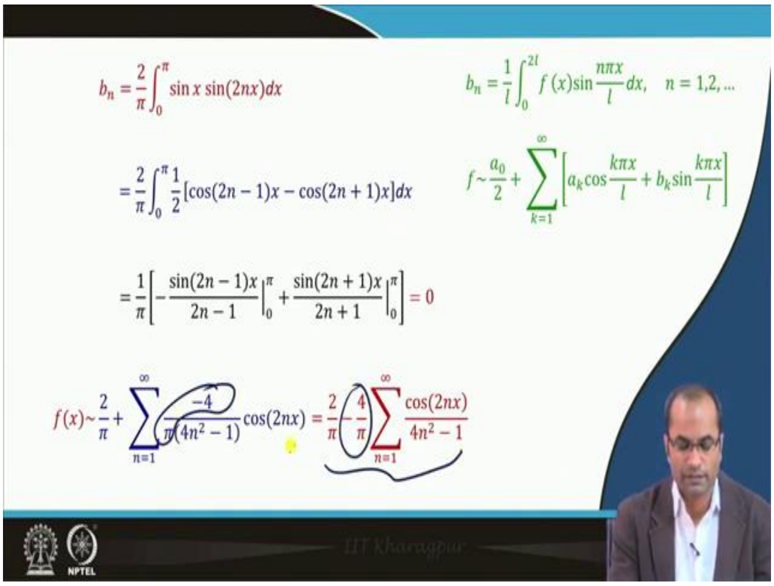

For b n again, we have the similar result 1 over n f x and then we have the sin here for b n. So, we can compute now, so with the sin 2nx will come with the sin and then again apply 2 sin a, sin b formula, sin 2n minus 1 cos 2n plus 1 and then cos will be sin and again this cos will become sin there. And since the sin is here, whether you put pi or 0 or here also pi or 0 this will be 0. So, at the end we realized that this b n is 0. And this fx we can write down as the a n which was pi by 2, so 2 over pi a n by 2.

And then we have this cos from only because the b n terms are 0, so there will be no sin term with cos 2nx we have this. This was the coefficient a n we have just evaluated before. So, we got this Fourier series when we have treated the function as a pi periodic function or we can just simplify a little more. So, here minus this 4 over pi, we have taken this outside and then we got this series here.

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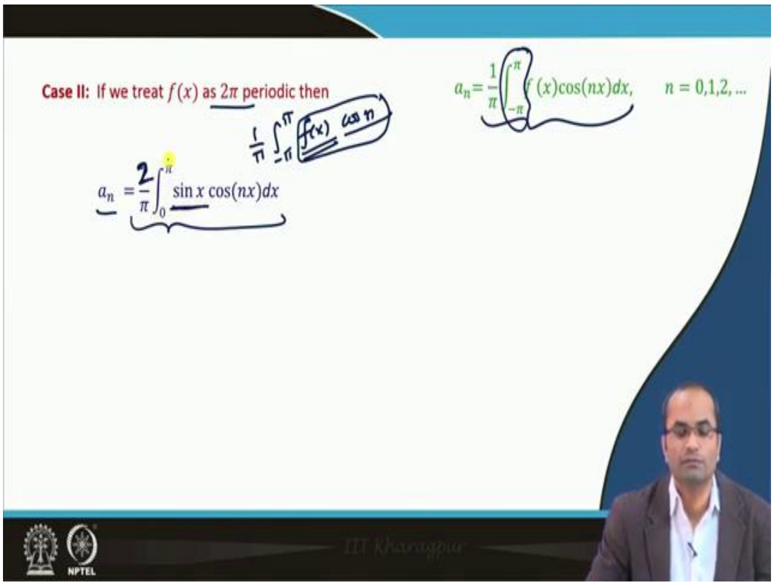

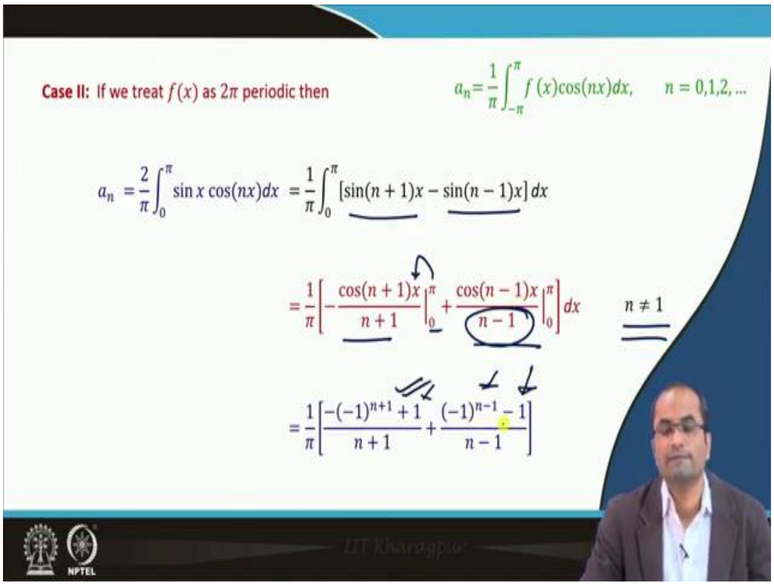

Well, so now in the second case, we will treat this f x as a 2 pi periodic function because 2 pi is not a fundamental period, but it is also a period of the function. So, you can consider this 2 pi periodic function and then we have the standard result that a n will be this 1 over pi and then minus pi to pi.

So, the result is a for n is 1 over pi and then we have minus pi to pi and then this f x and then we have cos, its written here already. So, this minus pi to pi and when we have we are talking about f x, f x is even function and then cos x is also even function, so the integrant is even. And then we can write down 2 times that this 2 times and then 0 to pi and then we have the function here but now in 0 to pi, the function is positive, so we have removed the mod, we have sin x and then we have this cos nx dx.

So, now we can apply this formula 2 sin x cos nx. So, sin n plus 1 and sin n minus 1x and we can integrate now this sin, so we will get with minus sin, this cosine and this cosine and again the same argument So, we will put first the pie there sin n plus 1 pi n then with minus this cos 0. So, what we will get? First we have to assume that n is not equal to minus 1 otherwise this will break down. So, we will get this formula minus 1 power n plus 1 and then for 0, we will getting 1 the here and similarly, minus 1 power n minus 1 and for 0, we will get 1 there.

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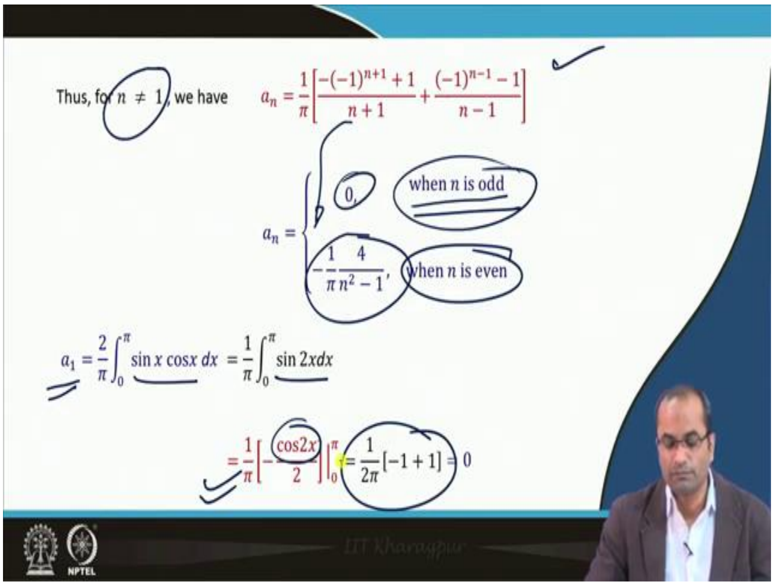

So, this for n not equal to 1, what we have, we have this formula, which can be further written in a more simplified form. So, when n is odd, this will become 0 and when n is even, this can be simplified to this minus 1 over pi and 4 over n square minus 1. So, a n we need to compute now separately because the formula.

The general one was not valid for n equal to one, so that we can separately do this cos nx cos x and sin x cos x. which is sin 2x and then we can integrate this to have this cos and then we have cos 2 pi and then minus 0 cos 2 pi is 1. And then minus 1 is also 1, so we will get simply this 0. So this a 1 is 0 a n have computed.

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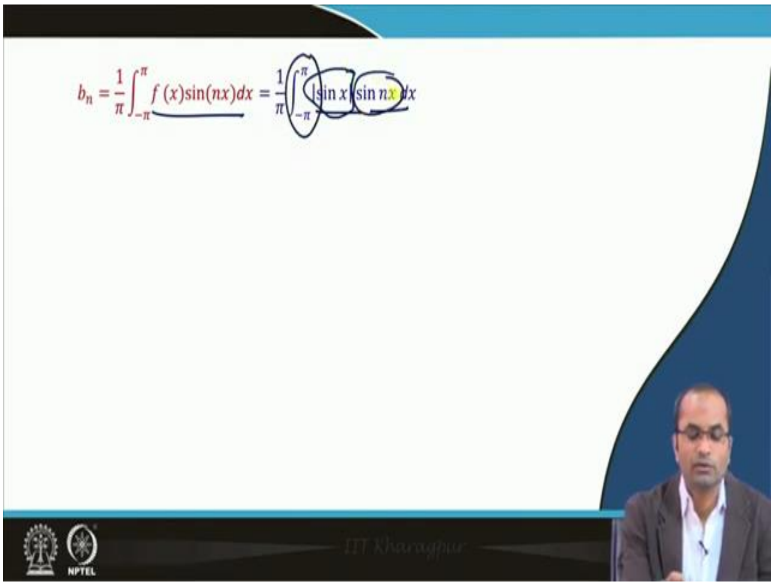

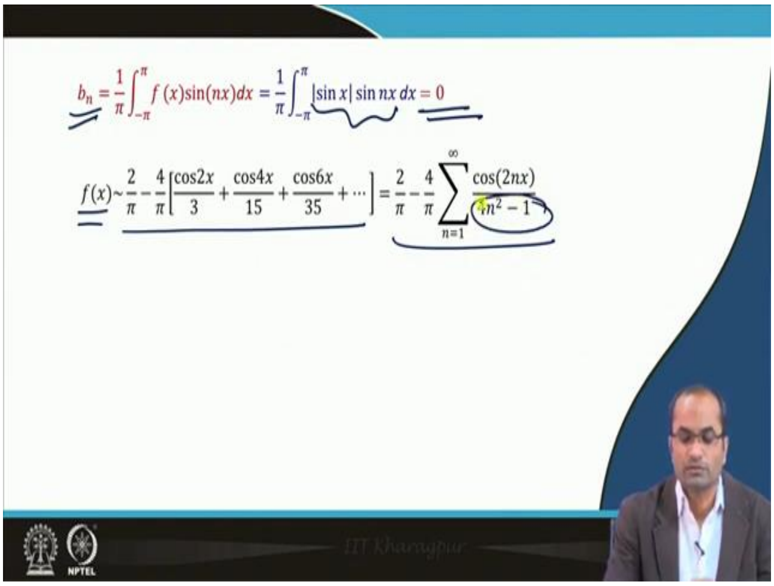

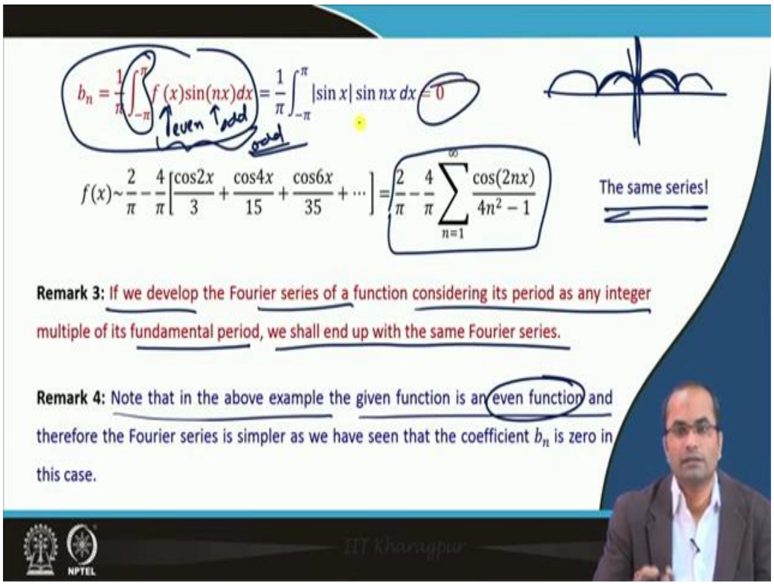

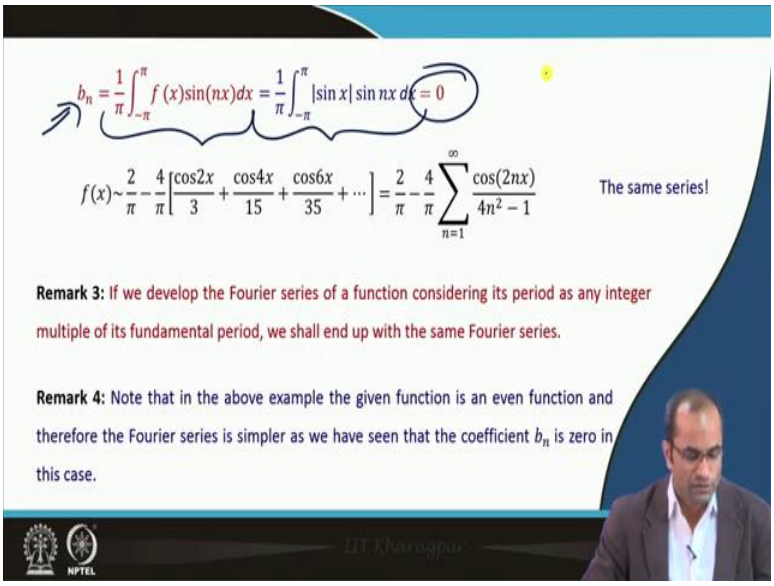

And for b n, if we write down these Fourier coefficients b n, so we have this sin x sin nx. And since this is the even one and then we have odd here, so minus pi to pi, with this odd function as a integral, this will become 0 straight forward without calculation. So, the b n, these coefficients are 0. And therefore our f x now after substituting these a ns and b ns can be obtained in this form, which can be again written in a more compact form having this 4n square minus 1 and this cos 2n x term So, this was exactly the series which we got assuming it as pi periodic.

So, we have consider 2 cases here, once we have taken the function as pi periodic and then we have taken this function as 2 pi periodic and at the end, we have observed that we are getting the same Fourier series which is very much expected because the changing this period is not changing the behavior of the function, it is just taking the extra length with having this repeated behavior of the function. So, if you develop the Fourier series of a function considering its period as any integer multiple of its fundamental period, we shall end up with the same Fourier series that is the conclusion and we have demonstrated from this example,

Also not that in the above example given function is an even function. So, another observation which we should keep in mind and later on, we will discuss in detail that this sin function was an even functions so the sin function here with absolute value, it was an even function. So, around this 0 here this has the same value as in the negative values there.

So, it was an even function and for the even function when we are computing the b n, so b n is f x and sin. So, sin is odd function and f x is a even function, so then the product is an even function, sorry the product is an odd function because we have the even and odd so the product will be odd. So, the b n coefficients when we computed here, this b n coefficient, so if this f x is even, and this is odd, so the product of these two, even odd is odd.

So, if the integrant is odd and we are integrating from minus pi to pi, then this integral will be 0, so that is another way of describing this and not to compute the 0 if this f x is an even function, naturally this b n will disappear or the vice versa. If we are talking about the odd function, then a n will become 0 and we have to just compute b n, so if the function is odd or even we can simplify the calculations and on either a n will survive or b n will survive.

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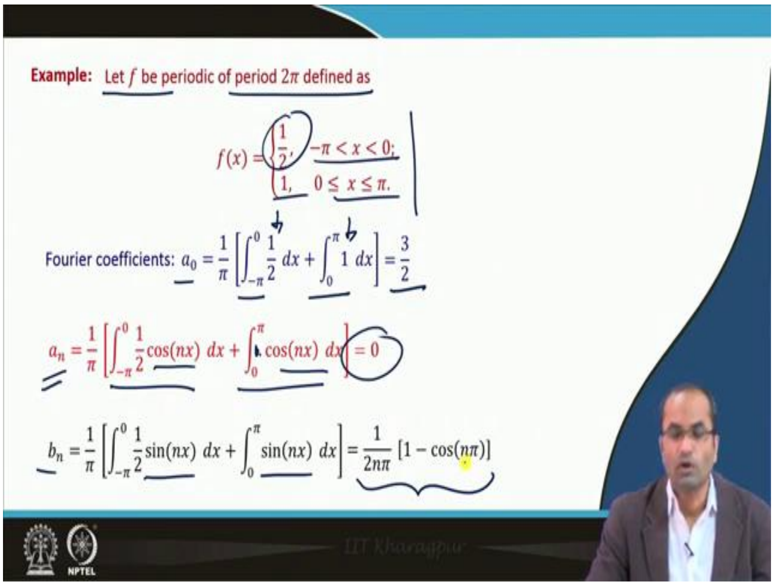

So, this is the last example we will discuss here. Let f be a periodic function with period 2 pi and defined in this way, so half in minus pi to 0, and then 1 from 0 to pi. So, it is Fourier coefficients, if you compute first a naught from minus pi to 0 and 0 to pi respective values half and then 1 there, so we can get this 3 by 2.

And then if we compute the a n again, we have to break from minus pi to 0 and 0 to pi having this cos nx, cos nx and half and then 1 there. So, again, what we will realize that this is 0 and this is interesting, because this is not 0 because of this even and odd structure, but this is just appearing, this is just coming to be 0. So, the b n if you compute is coming sin there and then again we can compute these simple examples, simple integrals and we will get 1 over 2n pi 1 minus cos n pi.

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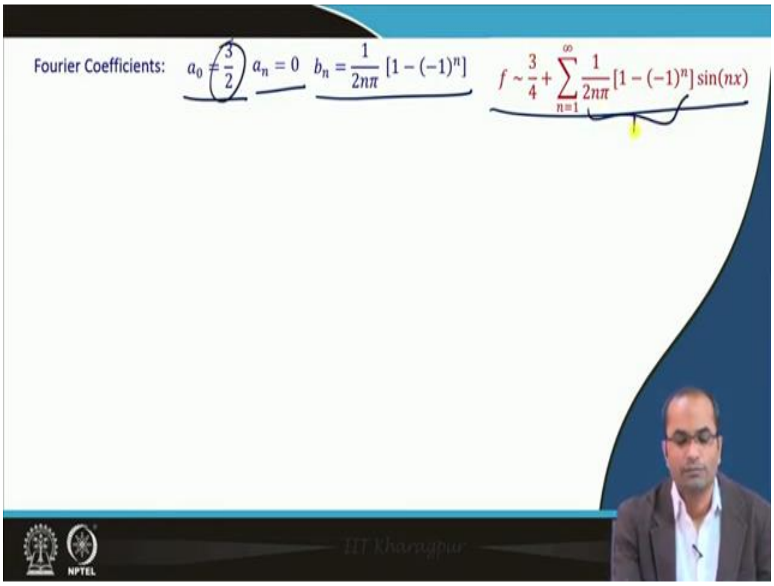

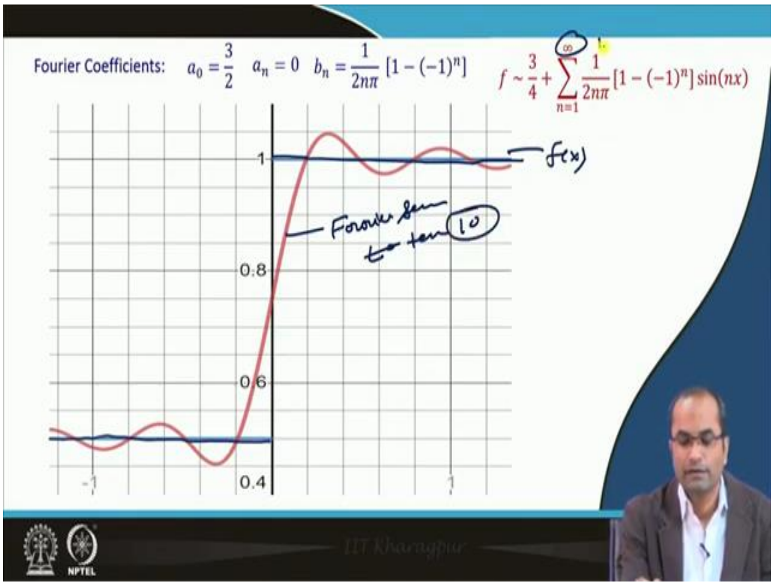

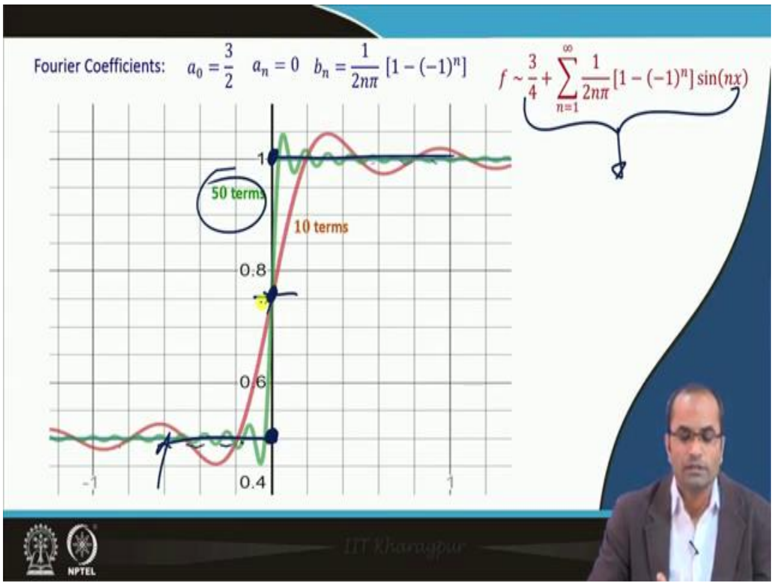

So, having these Fourier coefficients, the a naught was 3 by 2, a n, 0 and then we have the b n we can write down its Fourier series. So, 3 by 4, this a by 2 and then we have this b n and then we have sin x, so this is the Fourier series of this given function. So, if you just get the idea how this Fourier series look like, so just recall the function was given here from minus pi to pi as minus half minus pi to 0 was half and then the value was 1.

So, this was our function fx, and then this was a function f x and then this is the Fourier series we have plotted Fourier series, we have plotted for about 10 terms. So, the terms here we have taken 10, so this (())(31:41) of infinity, we have taken for instance 10 and we have plotted this graph and then you can see because we have taken only 10 terms, the approximation is not very good.

It is not matching with for example, the given function, it is not conversing or seems to converge at this for just for 10 to the given function. But what do we do if we take more, if we take more terms for example, if we take 50 terms here, now you can see it is quite close to the function here as well as here. But there was a point here that is continuity where the point where the function value in this site was half and then we have this one value. It is a problematic and it is always crossing at this middle value here between this 1 and one half.

And that exactly, we will observe in the next lecture theoretically that what is the convergence of this series or in other words to which function this series converges because from here as we can see that this is not converging exactly to the given function f x and mainly on at this point 0 for instance, because if we take more terms from 0 onward, it will match and from less than 0, it will also match to the given function. But what will happen at this 0 point there, which is definitely it is not matching here, it is always crossing this point in the middle.

So, that is what we will see in the next lecture that we will discuss the convergence of the Fourier series under what condition it will converge. And it will converge to the given function or it will converge to some other function and what will be that function, all these discussion will take place in the next lecture.

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So, here we have the references use for preparing this lecture.

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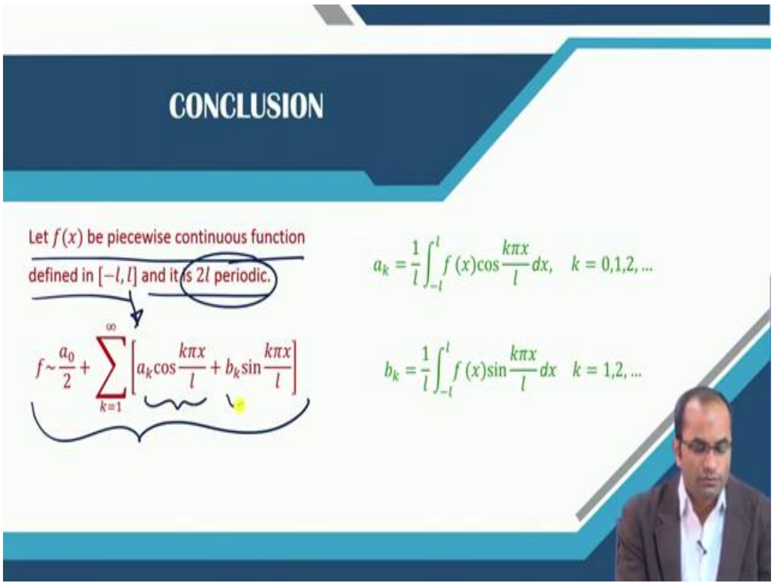

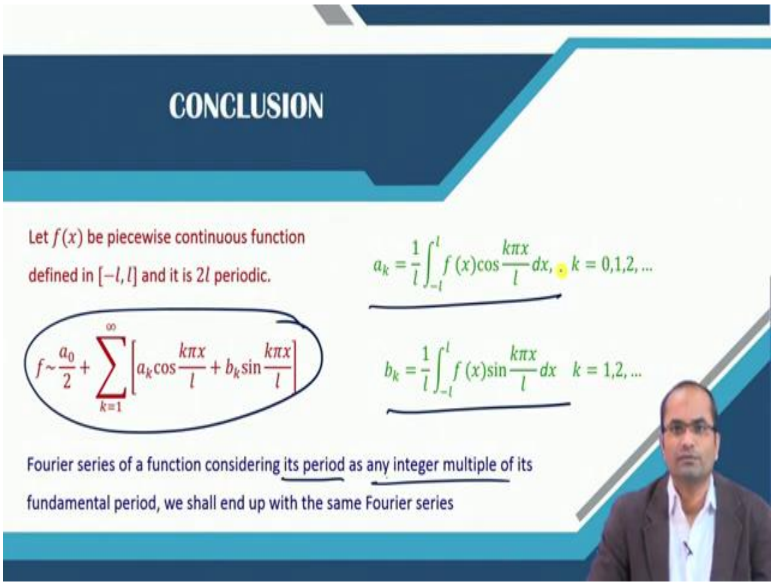

And just to conclude, so we have discussed that if f x is piecewise continuous define on this interval or it may not be 2 periodic or it is 2l periodic does not matter. We can write down the Fourier series and this Fourier series will converge to, if it converges to it will converge to a 2l periodic function. So, this was the more general form of the of Fourier series, where these coefficients and Fourier coefficients are computed in this way.

And we have also seen that if we take its period at any integer multiple of its fundamental period, the Fourier series will remain the same, the Fourier series will not change. And we have also demonstrated with the help of one example. So, that is all for this lecture and I thank you for your attention.