Engineering Mathematics II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Khagarpur
Lecture 32
Derivative of Fourier Series

So, welcome back to lectures on Engineering Mathematics II and today we will discuss the derivation of the Fourier series, this is lecture number 32 on Fourier series and integral transforms.

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So, today we will discuss what are the piecewise continuous functions because that is required for constructing the Fourier series and second we will go for the derivation of the Fourier series.

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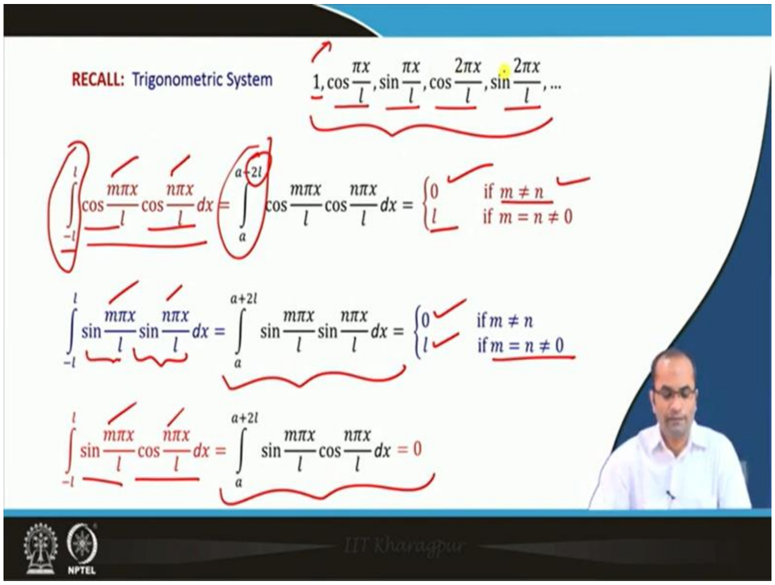

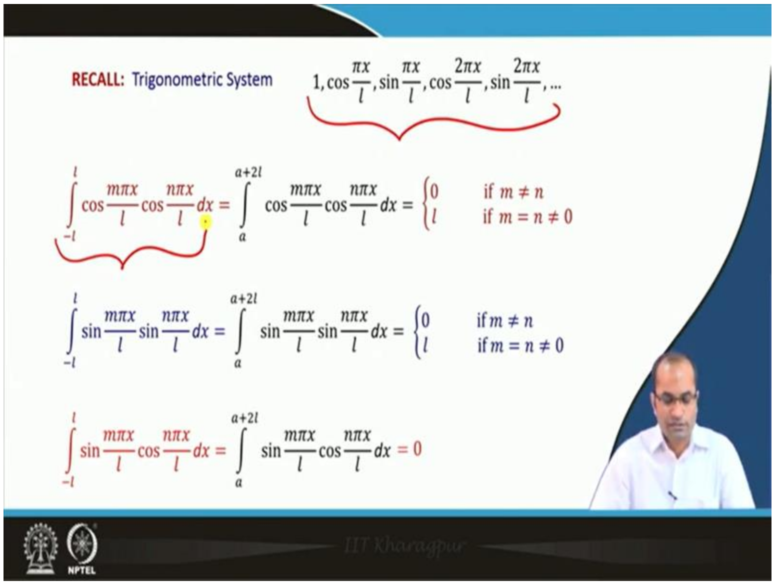

So, just to recall form the previous lecture what we have done that was the trigonometric system which was discussed in detailed. So, this is a general trigonometric system having this 1 and then cos pi x over l sin pix over l and then then 2 times then 2 times then 3 times etcetera. So, here the common period of all these functions is 2l this is what we have discussed.

And then we have also discussed various properties of such a trigonometric system. And most importantly we have if we integrate here from minus l to l the product of any 2 different member of this trigonometric system the value will be 0 when m is not equal to n. Because both are from cos family so this m this n, m and n are different in that case we have the value 0 of this integral.

And when m equal to n so basically they are the same member in that case this is coming as l, so this was derived also in the previous lecture. And similarly, we have the members from the sine family so again the same result if m and n are different the value is 0. So, this is what we call that the system is orthogonal and if the value is equal m is equal n then we have this l.

The another result when we have this 2 different members sine and cosine then whatever m and n are irrespective of that the value will be 0. And also we have discussed that whether once we have the periodic function as its integrant whether we integrate from minus l to l or from any point to 2l.

So, the covering the length of this period 2l so the value will be the same and we have also discussed that if with 1 also if we take any other member of the family again the result will be 0. So, this trigonometric system is orthogonal so this will be useful for evaluating directly such integrals while constructing Fourier series or deriving Fourier series in today’s lecture.

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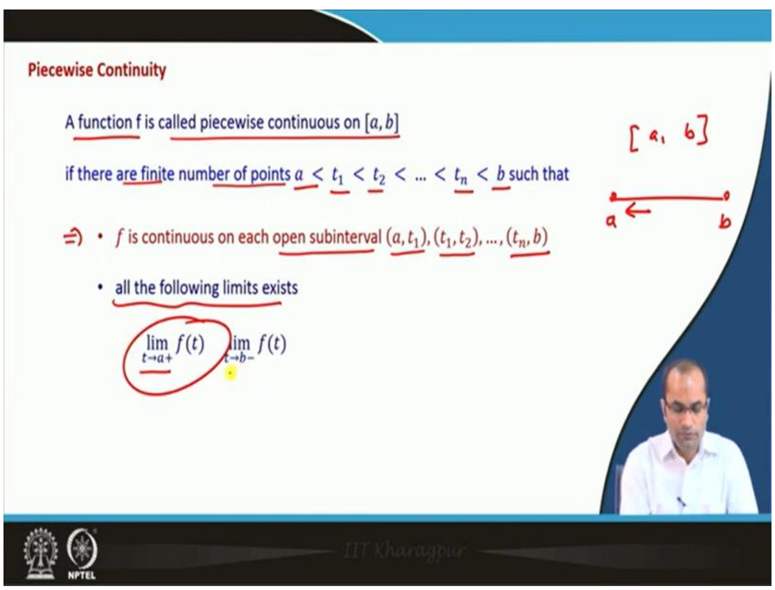

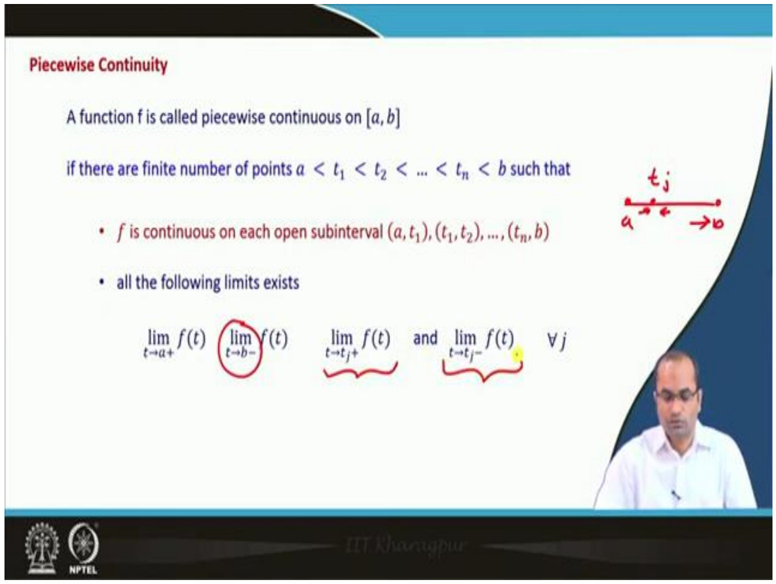
Piecewise Continuity
A function f is called piecewise continuous on $[a, b]$
if there are finite number of points $a < t_1 < t_2 < \dots < t_n < b$ such that
• f is continuous on each open subinterval $(a, t_1), (t_1, t_2), \dots, (t_n, b)$
• all the following limits exists
$\lim_{t \to a+} f(t)$ $\lim_{t \to b-} f(t)$ $\lim_{t \to t_j+} f(t)$ and $\lim_{t \to t_j-} f(t)$ $\forall j$
t_j
a
b
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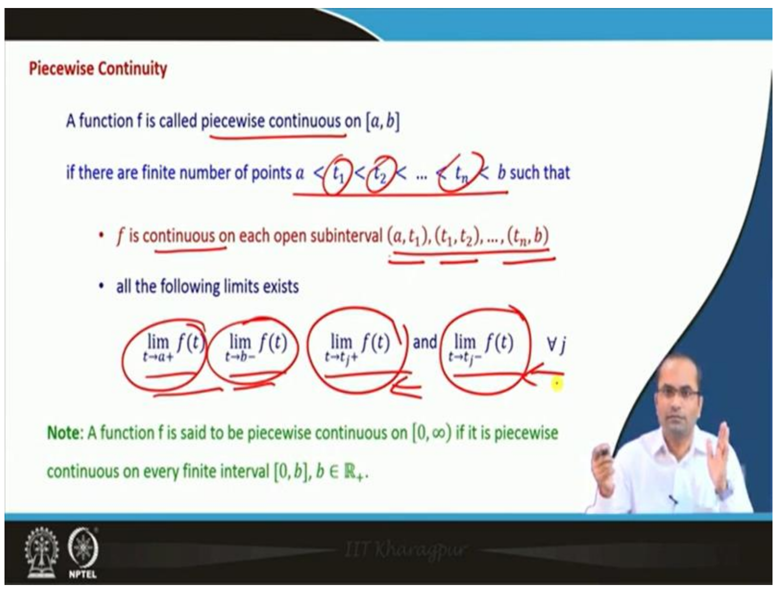
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Note: A function f is said to be piecewise continuous on $[0, \infty)$ if it is piecewise continuous on every finite interval $[0, b]$, $b \in \mathbb{R}_+$.
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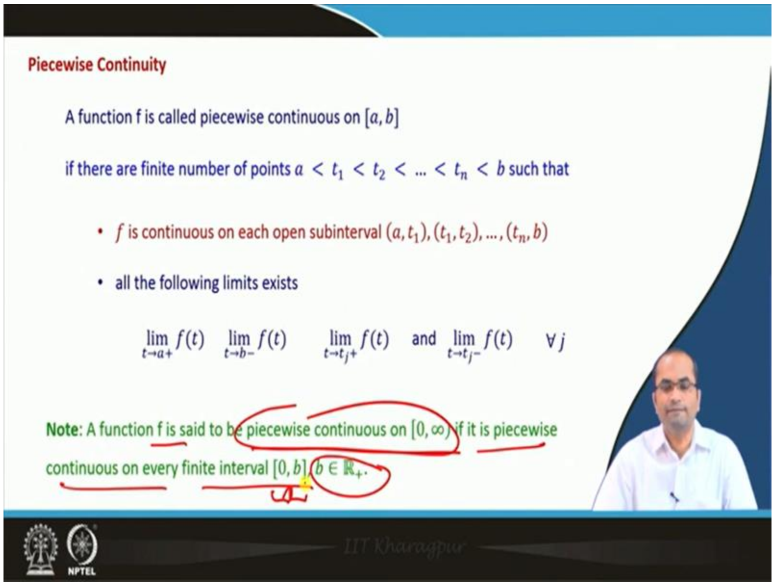

So, now coming to the piecewise continuity of function, so suppose we have a function f and it is called piece wise continuous on interval a, b if there are finite number of points between these 2 points a and b and these finitely many points we are calling t1 t2 and tn. Suppose there are n points between a and b, what is the property of these points?

So, now the f is continuous on each sub open sub interval that means a to t1 or t1 to t2, t2 to t3 and the last interval here we have this tn to b. So, in each of these sub intervals the function is continuous, moreover if the following limit exist. So, what are these limits? One is the right hand limit at the left point so because we are talking about the close interval a b. So, here at this end when we are at a we are talking about the right hand limit and similarly when we are talking about b so again we have this a and we have b.

So, at this end, we are talking about the left limit here and then at all these points in between wherever we have let say this is tj. So, both the sides the limit should exist so the right hand limit and also the left hand limit should exist for all j there. So, again a function is called piecewise continuous if there are finitely many such points where the function is continuous on each sub interval open sub interval and this limit exist, then we call that the function is continuous. So, because there, these are the points on discontinuity is basically t1, t2, etcetera.

But in the open interval a to t1, t1 to t2 and tn to t tn to b the function is continuous and most importantly these limits should exist so at the end point the limit should exist and also at each of these points where the function is possibly discontinuous or it is not define. In that case these limit should exist. So, if this is the case we call such a function a piecewise continuous function.

So, this class of piecewise is continuous function is bigger than of course the continuous functions, because we are relaxing here to have a finitely many points where the function may be discontinuous or it may not be define at those points.

But the limit at each points from the right hand side as well as from the left hand side should exist and at the end points, so the right hand point the left limit should exist and the beginning point the left hand point a the right limit should exist. So, having these properties, we call such a function a piecewise continuous function.

And just a note that f is said to be piecewise continuous on this interval 0 to infinity closed at 0. If it is piecewise continuous on every finite interval 0 to b and b is any number the positive real number then we call that this function is piecewise continuous on 0 to infinity. If the function is piecewise continuous on 0 to b for any b from the set of positive real numbers.

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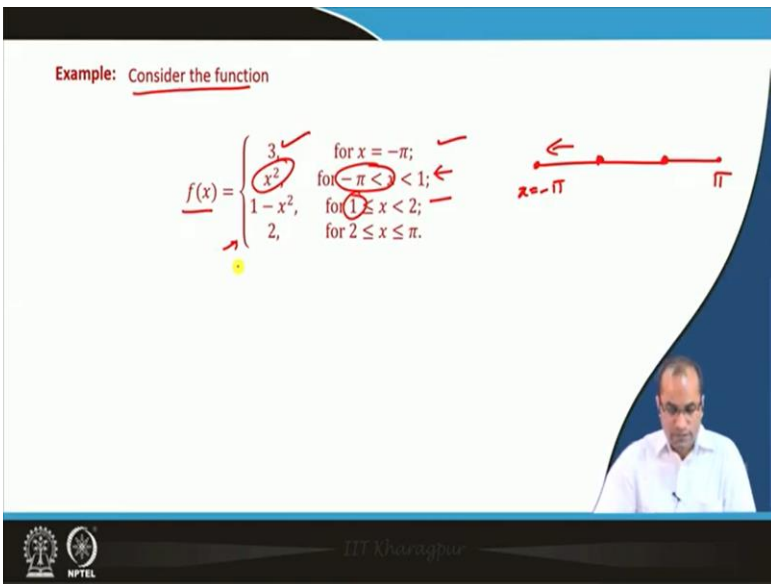

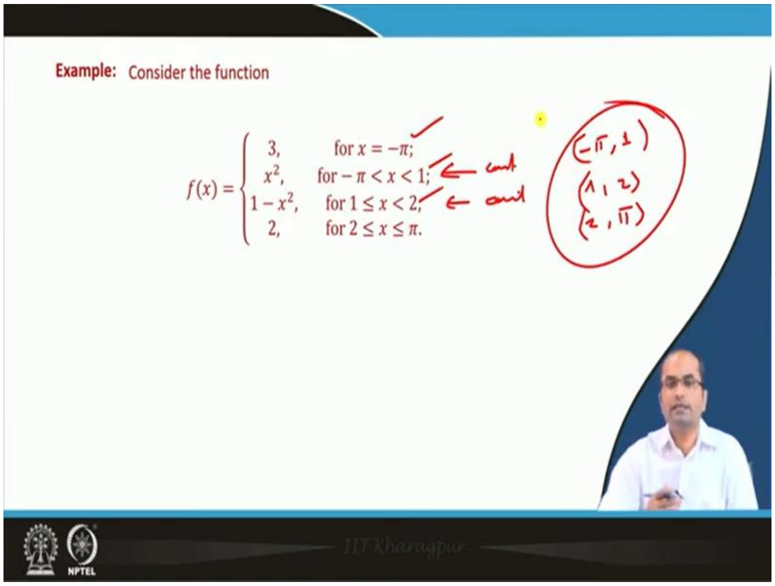

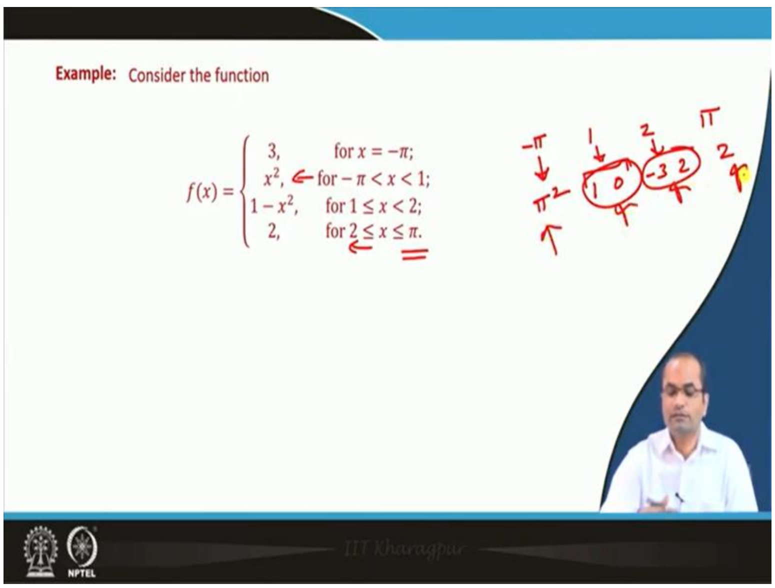

So, this is what we have the piecewise continuity, we will just discuss some examples on piecewise continuity and then we will come to the point of the derivation of the Fourier series. So, if we consider this function which fx is defined as 3 at the point this x is equal to minus pi and then x is equal to here pi the function is defined as 2.

And in between also, from minus pi to 1 and then 1 to 2 these functions are define as x square and then 1 minus x square. So, if we look at for instance minus pi point the value of the function at minus pi is 3 there, but if we look at the limiting value from the right hand at this point this will be pi square. So, definitely the function is discontinuous at minus pi.

Similarly, at 1 also because if we look at from left hand side the value will be 1, but if we look at from the right hand side or this is exactly defined here at 0. And similarly at 2 also we have discontinuity. So, all these points here whether it is pi or it is 1 or 2 or all these points the functions is discontinuous.

So, we have to now check whether we can put this function in the category of piecewise continuous because piecewise continuous is not just that we have these points where the function is discontinuous. But, we have to also check the limits, so in each sub interval so for instance here minus pi to 1 its x square it is continuous here also it is continuous.

So, in each open interval whether it is from minus pi to 1 or it is 1 to 2 or it is 2 to pi the function is continuous. So, the first property is satisfied that it should be continuous in these open sub intervals. The second property is the limit so here we have to check at all these points whether it is a minus pi or it is 1 or it is 2 or pi that the limit should exist.

So, at this point we will look at the right hand limit because the function is define only towards the right from this point minus pi. So, if we look at this limit here this will be coming just pi square, so it is this exist at 1 the left hand limit will be 1 and the right hand limit will be 0. So, here we have both the limits at this 1 point.

Again similarly at the point 2 we have from the left hand it is 2, 4 square so that is minus 3 so 2 square so that is minus 4 plus 1 minus 3 and from the right hand side when we take this limit that is just 2 is a constant. So, here also we have the limit and at pi we will just consider from the right hand side or from the left hand side so that is 2. So, at all these points in the internal points both the limits the left hand and the right hand exist. And at the end points the right limit and the left limit exist.

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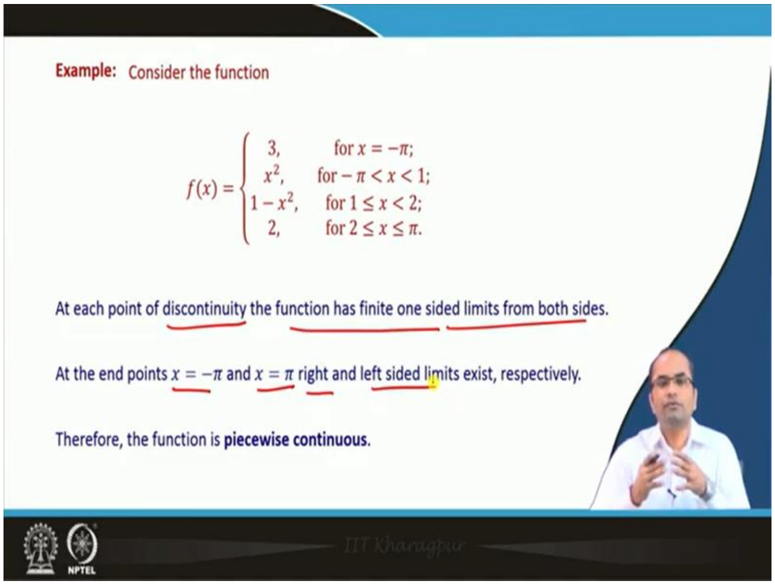

So, all these limit exist, therefore this function is piecewise continuous as written here. At each point of discontinuity the function has finite 1 sided limit from both form both sides and also at the end points. So, minus pi or pi the right and the left handed limit exist respectively and therefore the function piecewise continuous.

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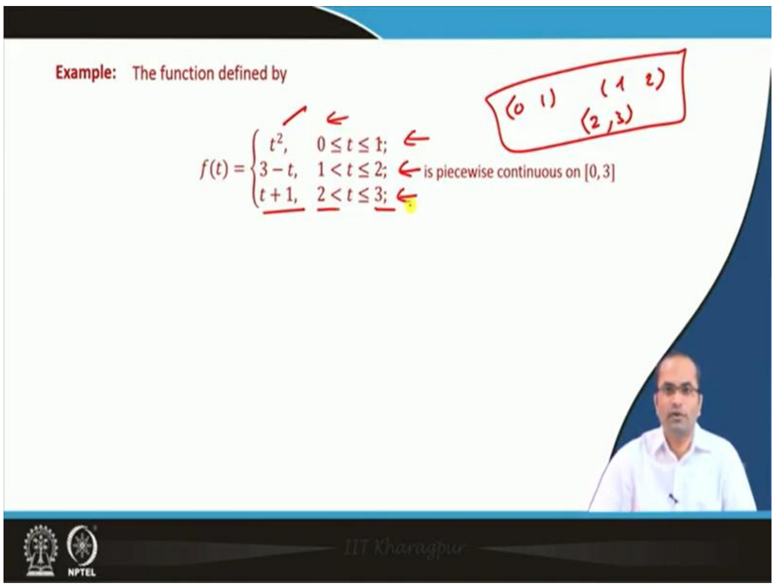

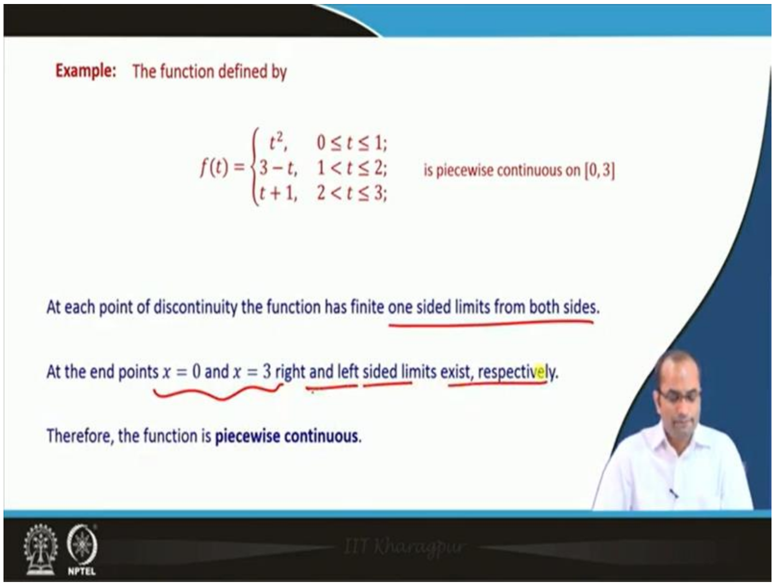

So, we will consider another function which is bit similar to what we have already discussed so here the t square in 0 to 1 and then we have 1 to 2, 3 minus t and then 2 to 3 the function is defined as t plus 1. So again, we have the situation that in each open sub interval whether it is a 0 to 1 or it 1 to 2 or it is 2 to 3 the function is continuous because it is defined by t square 3 minus t and t plus 1.

So, that first property of the piecewise continuous functions is satisfied for the second property we have to discuss the limits. So, again at 0 if we take the right limit at 0 this is going to be 0 at 1 we have at both the ends limit here it is 1 and that from the right hand side it is 2 and at 2 also this is going to be 1 from the left hand and then from the right it is going to be 3 and again at 3 we can compute the left hand limit and that will give 4. So, at all these points of discontinuity the right and the left limit exist. And therefore, the function is piecewise continuous. Also these end points the right and the left handed limit exist, so again this function is piecewise continuous function.

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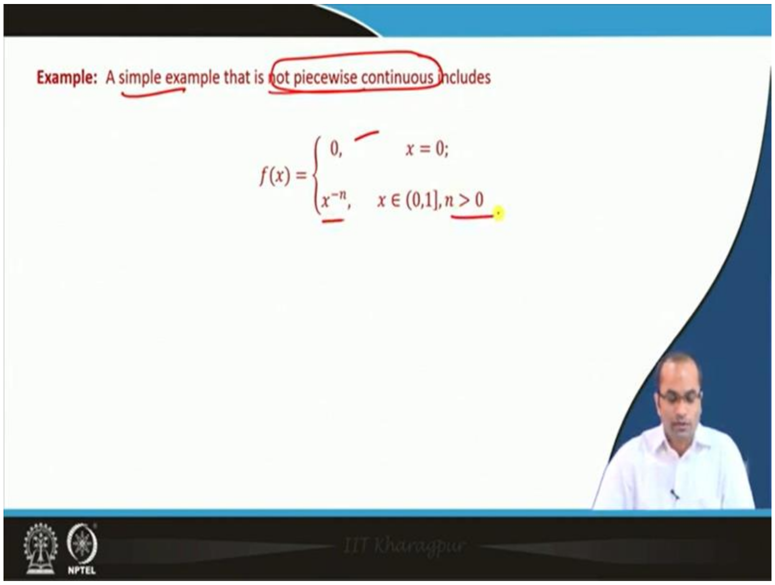

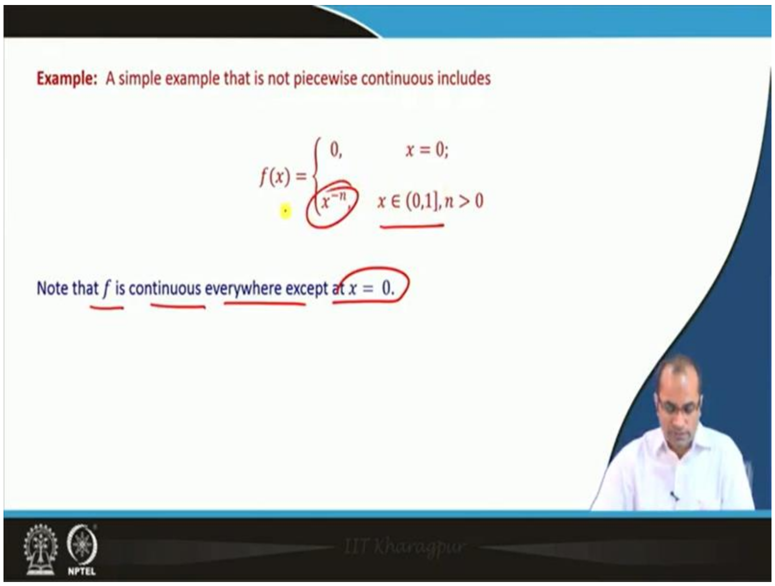

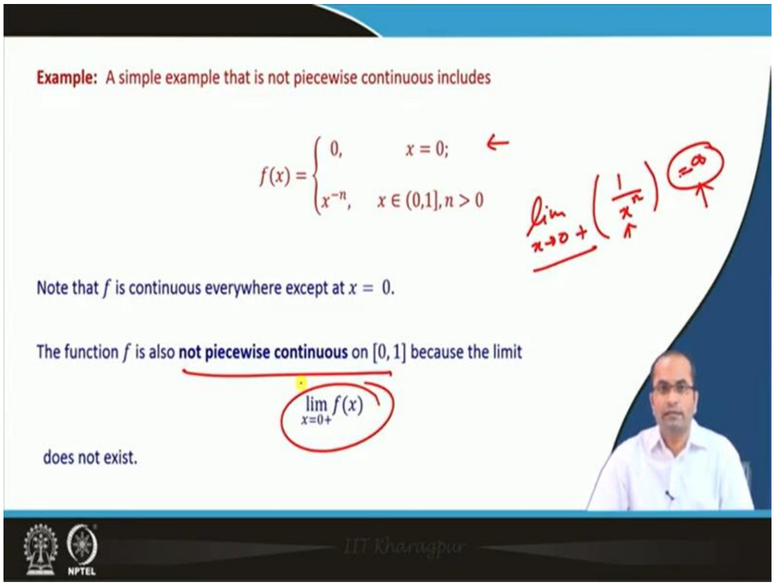

This is a simple example where we will see that this is not piecewise this is not piecewise continuous and this function is 0 and here we have when x is between 0 and 1. It is x power minus n and n is some positive real number. So, in this case this function is not piecewise continuous and the reason is clear because the function is continuous everywhere except at x is equal to 0.

So, the problem is at x is equal to 0 because when x is between 0 and 1 the function is continuous there is no problem. The problem is at x is equal to 0, so at x is equal to 0 what is the problem if we take the limit here at x equal to 0. Naturally from the right hand we have to take because the function is given in 0 and 1.

So, at this boundary the left boundary at 0 we have to take the limit which is like x over n and n is a positive number. So, when x goes to when x goes to 0 this limit goes to infinity, so we do not have finite limit here we have this function the limit is going to infinity and that is the reason that this is not piecewise continuous. Because the definition says that all these limits should exist finitely, so here it is not piecewise continuous because this limit here x 0 plus fx this does not exist. And hence, this function is not piecewise continuous.

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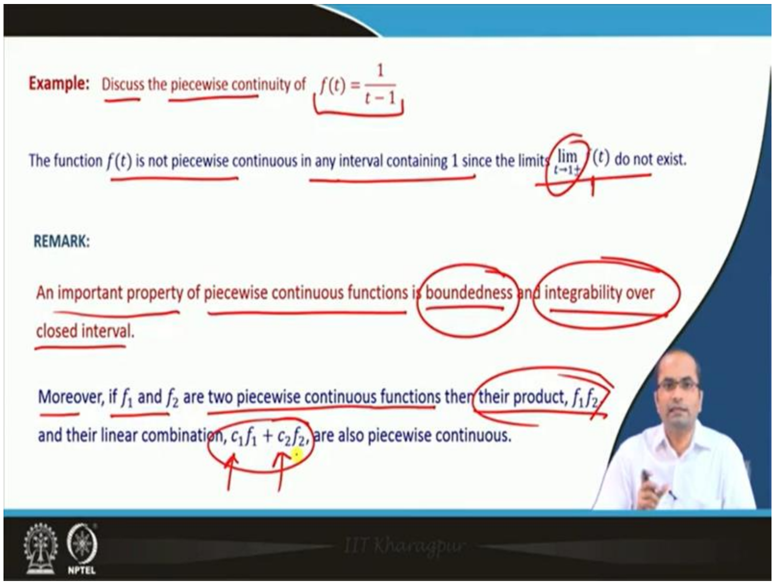

Another example we will discuss the piecewise continuity of this function which is ft equal to 1 over t minus 1. And in this case also we have a similar situation that the problem in this function is at t equal to 1 other than t equal to 1 the function is continuous indeed which is definitely piecewise continuous.

But if we take any interval which contains this point 1 then the left and the right hand limit at as we approach to t equal to 1 that does not exist again. So, here we have the problem at 1, so any interval containing 1 the function is not piecewise continuous because this limit here from plus from right hand and the left hand as t approaches to 1 this do not exist both the limits do not exist here.

And just a remark a quick remark on what is the importance of these piecewise continuous function, so an important property of the piecewise continuous function is the boundedness. So, what we have notice that the function is bounded because at each points we make sure that the limit exist from both the ends.

And also the function was continuous in each open interval so the boundedness is guaranteed and the integrability over the close interval is also guaranteed as a result of this boundedness. So, if a function is piecewise continuous than the function will be bounded in that interval as well as it will be integrable in that interval. So, these 2 properties are the important properties and we are looking for constructing the Fourier series based on this nice property.

Indeed all these properties are also valid when we are talking about the continuous function in a in a close interval but that we are putting too much restriction to have only continuous function. So, this is where we relaxed to have to include the piecewise continuous functions as well because they also have this property of integrability and the boundedness.

Moreover there is a 1 more nice property that if f1 f2 are 2 piecewise continuous functions then their product so if we make a product of 2 piecewise continuous function f1 and f2 they will be also piecewise continuous, the product will be also piecewise continuous. Not only the product but also their linear combination that means if we multiply by some real number c1 f1 plus another real number c2 f2, than this product this combination is also piecewise continuous.

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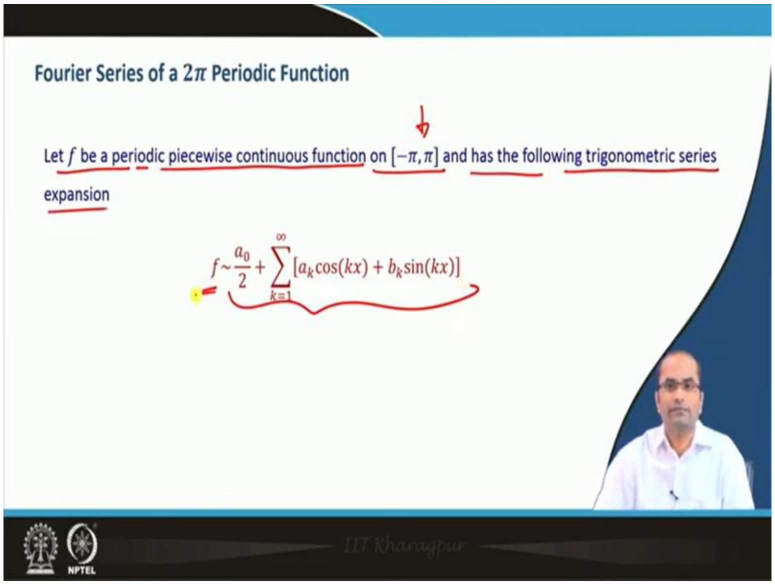

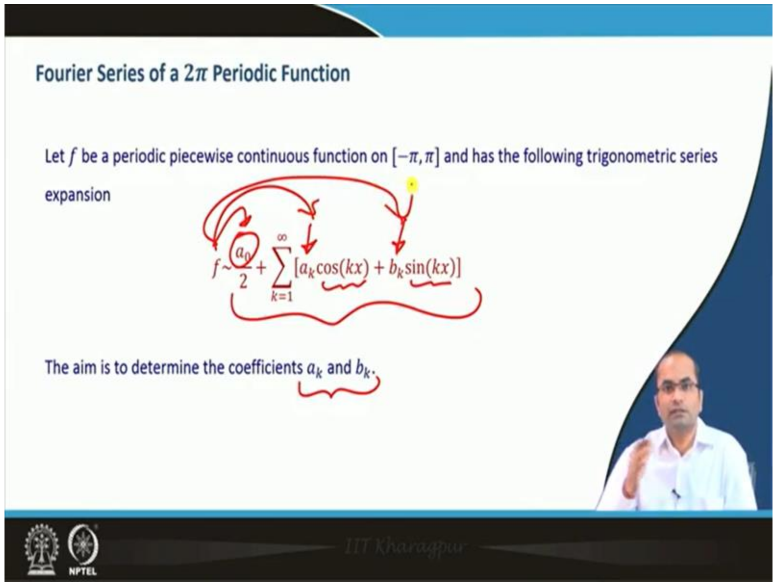
Fourier Series of a 2π Periodic Function
Let f be a periodic piecewise continuous function on $[-\pi, \pi]$ and has the following trigonometric series expansion
$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$
The aim is to determine the coefficients a_k and b_k.
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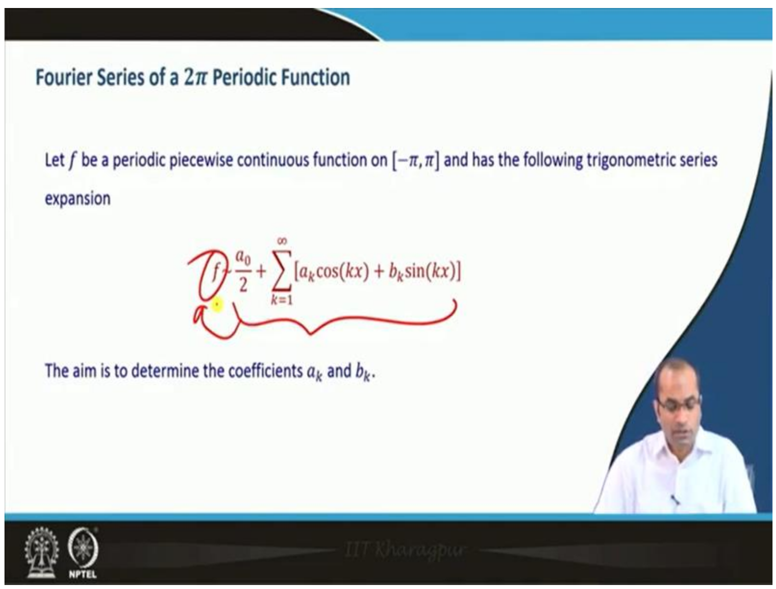
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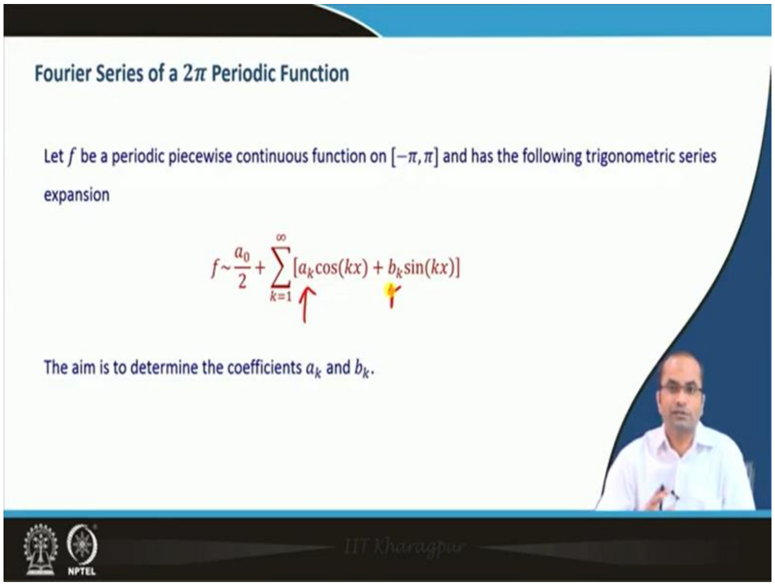

So, the Fourier series now we are ready to construct the Fourier series having the knowledge from the previous lecture about this orthogonality of the trigonometric series and then having the knowledge of what is a piecewise continuous function we can go for a constructing the Fourier series.

So, how to construct this Fourier series let me just begin with. So, suppose f is a piecewise continuous function that is a reason we have define the continuous function. Here will be talking on piecewise continuous function. So, we have a periodic function which is piecewise continuous the periodic function overall we defined in previous lecture. So, define on this interval minus pi to pi and has the following trigonometric series expansion. So, we will look into that what kind of what is the relation between this function and its trigonometric series expansion etcetera.

So, suppose at the movement we just suppose that this f has some kind such expansion which is trigonometric series here. And now we will try to relate that how this series is related to f such that in some sense this series should represent this function f. And this is what the objective now is that that what are these coefficient here because in this series what is unknown or what can be changed here.

These are these coefficient because cos function is given sine function is given so what is to be related with the function this f we have to relate these coefficients, we have to relate these coefficients to the given function f. So, that this series can represent in some sense and in what sense that will be discuss later this function f. So now, the idea is that how these coefficients are related now to the given function, so to construct the relation to establish the relation what we will do now.

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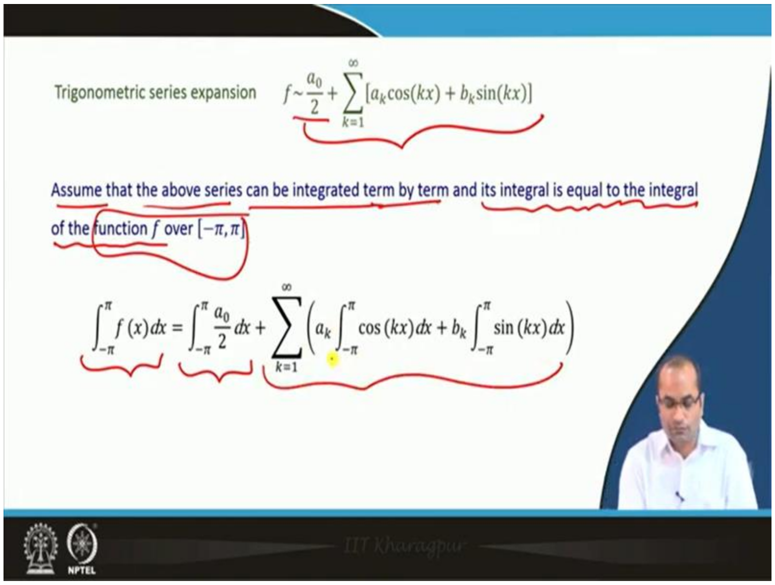

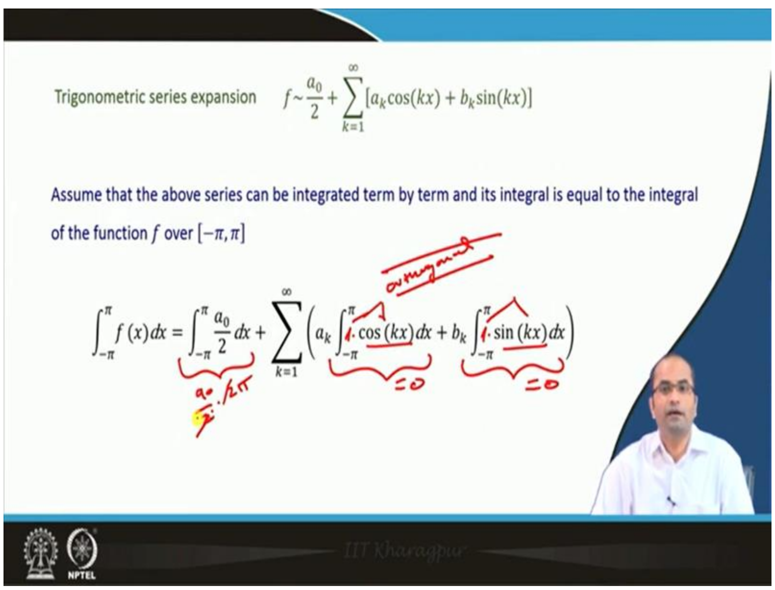

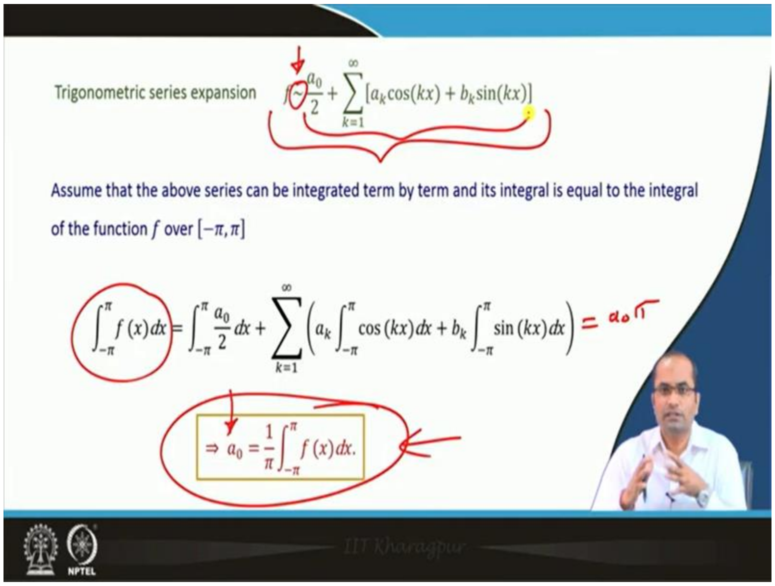

We assume that the above series can be integrated term by term and most important its integral equal to the integral of the function f in over this minus pi to pi. So, what we are assuming that the integral of this series, when we integrate this term by term, that value that the value of that integral is equal to the integral of f over this range minus pi to pi. So, with this assumption we will get a relation between the function f and the coefficient of these this series.

So, what we are doing now? We are integrating here minus pi to pi fx dx and then the right hand side also we are integrating, so the first term here is integrated minus pi to pi a by 2 dx, and then the second term also. So, all other terms we have integrated here from minus pi to pi. So, now we should realize the property of this trigonometric system, the orthogonality.

And here we are having one and the product with this cos ax here also we have the same situation one with the cos product with the product with the sin kx. Since one and this are orthogonal here also one and the sine are orthogonal, we have discussed in previous lecture. So, this is going to be 0 and this is also going to be 0 and what we have here now, a by 2 and then the integral will be just 2 pi, so 2 get cancel here we have a simply a not pi.

So, the right hand side is just a not pi. So, from here we can compute this a not now we can relate this a not to the given function. So, this a naught will be 1 over pi and this integral over this fx in the range minus pi to pi. So, by this assumption that the value of this integral the value of the integral of fx is equal to the integral of this (())(22:00) series we have got the relation at least to compute this a naught with the given f.

What one should note here that, I will discuss again this, we are not putting that f equal to this series here, we are just telling that this is a representation of the f. Exact relation and all we are not talking about, we are talking about that this is an expansion some kind of representation of this f and what kind of representation and all that question is still open that we will discuss in next lectures.

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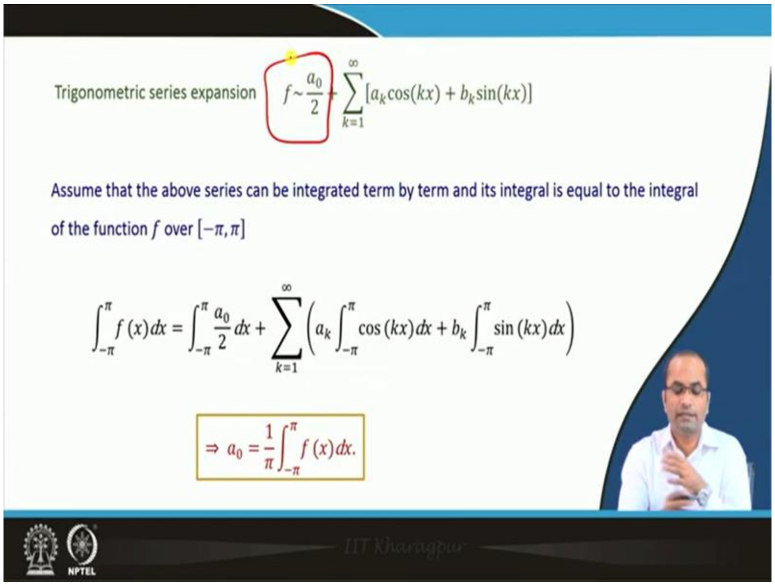

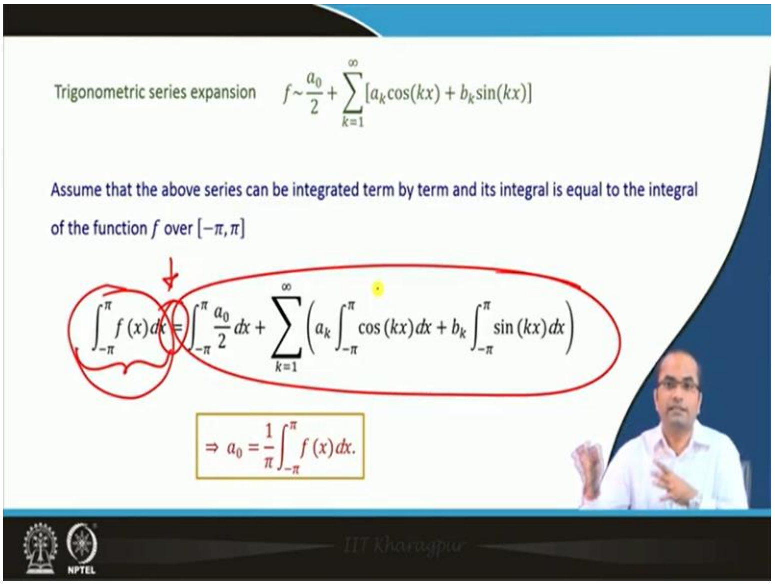

But it should be clear that we are not putting here any equality or anything between the series and the function f. But, this should be noted here that this is our assumption for the computation of these a not for instance, that this is equal, the integral of fx is equal to the integral of the series.

So, the first equality we have here which is now relating to the series to the function and its expansion. Now, we will move further with more assumption so this was the first one that if we integrate the function and the series assuming that these two are equal, we got one relation of a not to the function.

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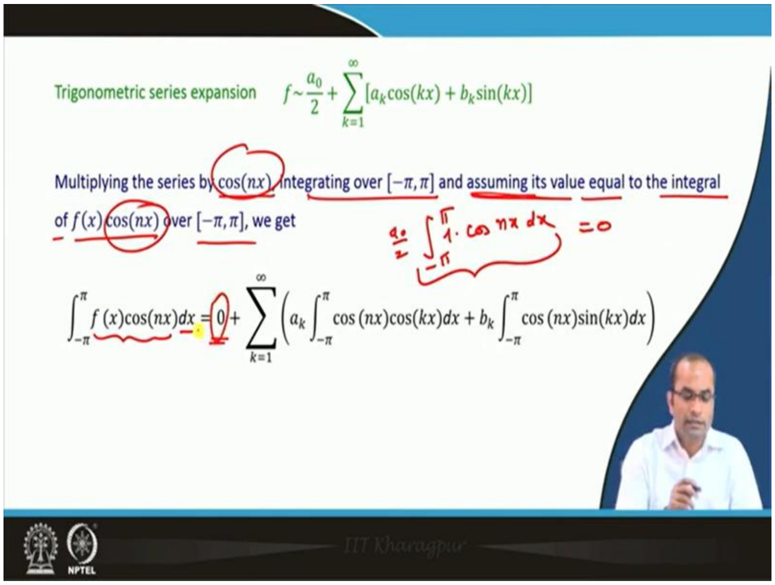

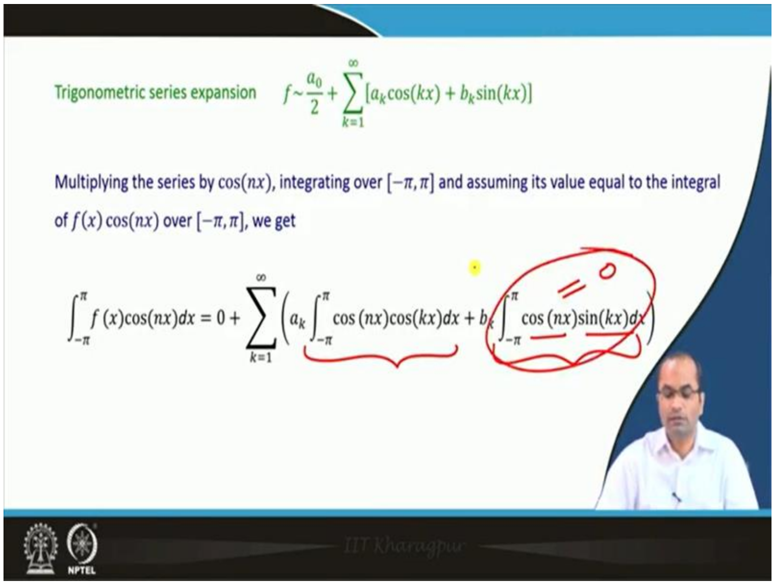

Trigonometric series expansion $f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$

Multiplying the series by $\cos(nx)$, integrating over $[-\pi, \pi]$ and assuming its value equal to the integral of $f(x)\cos(nx)$ over $[-\pi, \pi]$, we get

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0 + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(nx)\cos(kx)dx + b_k \int_{-\pi}^{\pi} \cos(nx)\sin(kx)dx \right)$$

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$$\Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n = 1, 2, \ldots$$

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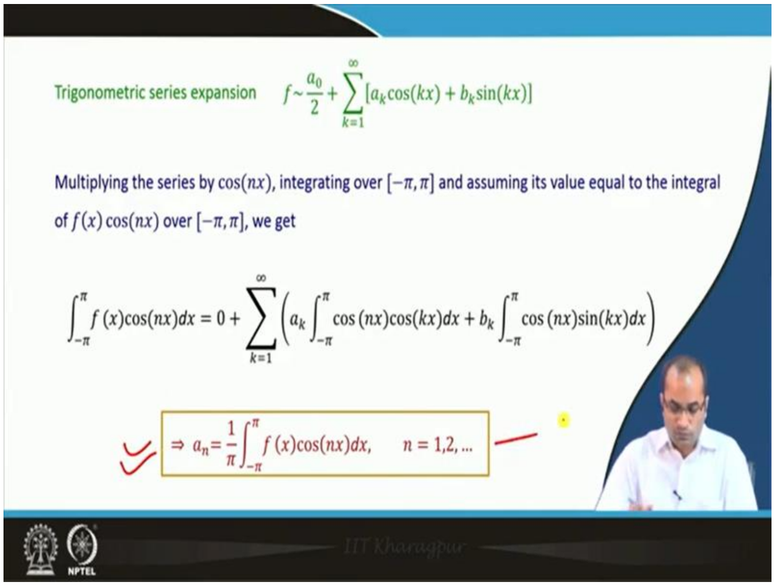

Next what we do that if we multiply the series by cos nx and then integrate and assuming again some assumption that its value is equal to the integral of again the fx is multiplied by same function cos nx over this minus pi to pi. So, the first, in the previous slide we had just integrated f and we have integrated the series. Now, what we are doing? We are multiplying the series by cos nx, also we are multiplying the function by cos nx and then integrating from minus pi to pi and to have this aim to get other relations for other aks and bks, so let us see what will happen now in this case.

So, we have fx and cos nx dx it is integrated, then the first one has become 0 because that is a by 2 a not by 2 integral minus pi to pi and then we have cos nx dx. And again the same reasoning with orthogonality we can say that this is 0. So, the first integral is going to be 0 and then we have a little more here, so we have these this integral and we have this integral. Clearly the second integral we have the cos function and the sine function and we know that these are orthogonal functions, so this is going to be 0.

So, this second integral will disappear in the first one, so this is 0 here. In the first one, this k varies here the k varies from 1, 2, 3, and so on. And if to recall the result which the very beginning of this lecture I summarized that we have here cos nx and for some k here when k will be n, so this will be also the cos nx dx.

This term will survive, all the terms will again with the argument of the orthogonality will make this integral 0, except the case when we have this cos nx and together with this cos nx, because k varies from 1 to infinity, so we are talking about the same n what is already there cos nx. So, when we have this product of the same function the value was pi in that case, so this integral will reduced to just a and pi. So again, and the second one is 0, so from here we can get this an equal to 1 over pi and integral minus pi to pi fx cos nx dx and n can go 1, 2, 3, and so on. So, we got another relation of the function to the coefficient an's and n can be nay number 1, 2, 3, and so on.

So, we got all the 2 relations. One was a not, which was which we got just by integrating the function and the series. And we got another relation here by multiplying the series by cos nx and then integrating over minus pi to pi.

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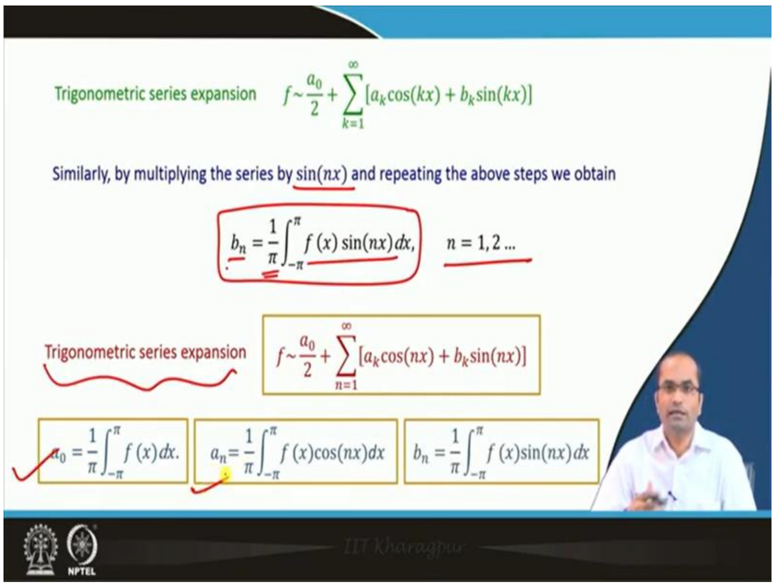

So, now what we will do? We will multiply by sin nx instead of cos nx, so by doing so what we can get with the similar same argument which is done earlier we can get bn. In this case the first integral will vanish and from the second one we will get a coefficient here bn which is fx sin nx dx.

So we have related now all these coefficients the bn, an and ak with the function with the given function. So, in some sense our representation is approximating the function in some sense because their integrals are equal when you multiply by cos nx again their integrals are equal and when we multiply by sin nx again the integral of f and the other side with the series that is equal so they are related now that sense we have just deriving these coefficients.

Well so, just to summarize we have the trigonometric series expansion and we got the a naught we got an, and we got this bn by assuming that their integrals are equal or after multiplication of cos nx and sin nx they are equal.

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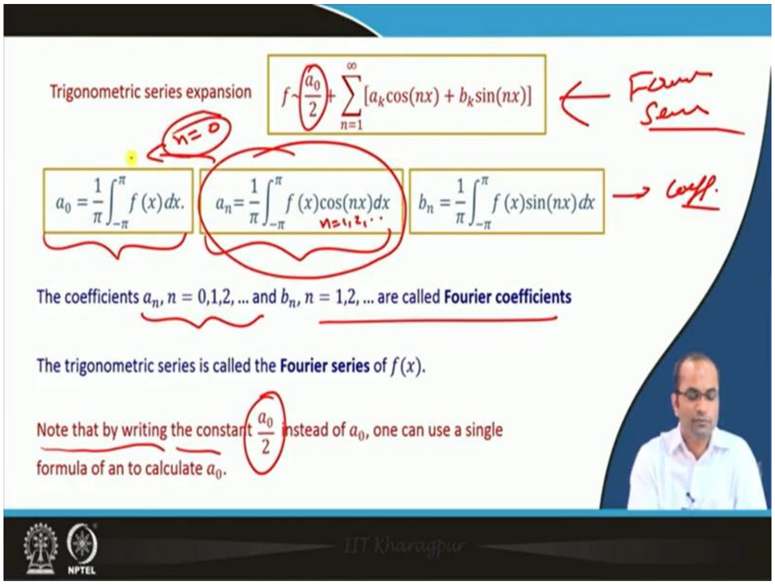

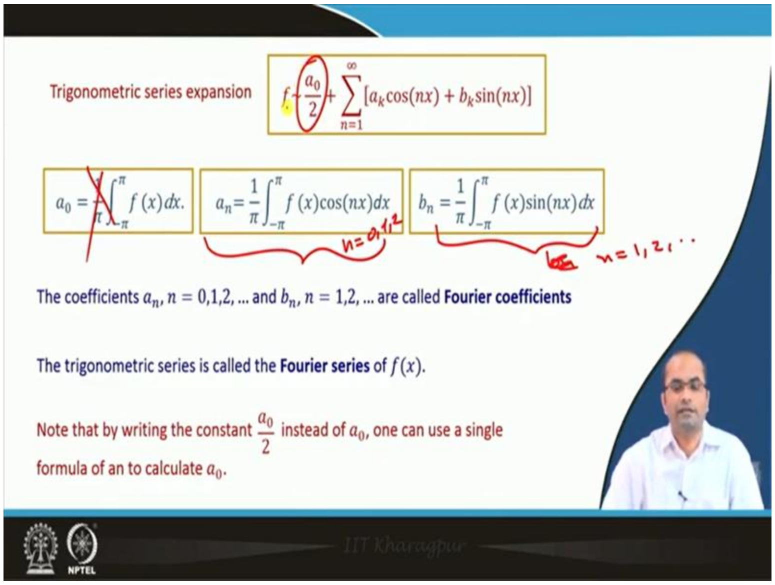

So, having this series, now these coefficients an, a0, a1, a2, a3, etcetera. They are called the Fourier coefficients and this series here is called the Fourier series, this is called Fourier series and these are called Fourier coefficients. Just one more point which must be clear now that instead of this having some constant we have started with a naught by 2, and the aim was to have this a naught by 2 exactly from here.

If we look at these two formulas of an which was valid for n1, 2, 3, and so on and this is a0 was separately derived but at the end what we realized that this formula is more general, it is covering this a not as well because if n is 0 this cos0 is 1 and we have exactly this formula for n equal to 0.

So, we do not have to have these three formulas, we can have just this an which is valid for 0, 1, 2, and so on, and this bn which is valid for eligible to 1, 2, 3, and so on. So, this we do not need now and that series and we have to started with a naught by 2, if in take just here some constant a naught then that a not formula will be different than the rest.

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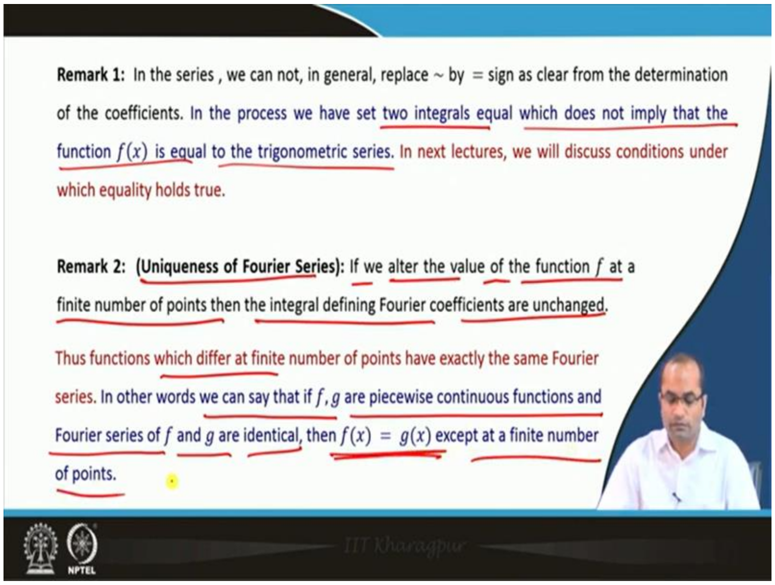

Just few remarks so that in the series we cannot in general replace this equivalent sign by this equal sign, which is clear from the determination of constant, because this constants are determined by equating integrals, the three integrals, one was direct integral other was while multiplied by cos nx and the sine nx.

So, they obviously we cannot just have equality there but their integrals are equal. And in the processes as I have set the two integrals are equal which does not imply that the function is equal to the trigonometric series. In the next lecture we will discuss under what condition we can have the equality between the function and its trigonometric or the Fourier series.

Second on the uniqueness of the Fourier series so if we alter the value of the function at finite number of points because we are talking about while computing these coefficients say and Fourier coefficients an and bns we have the integrals and the integrals value will not read the function a defined at just they differ by value at finitely many points.

So, if we alter the value of the function at finite number of point and the integral defining Fourier coefficients are unchanged and thus the functions which differ at finitely many points have exactly the same Fourier coefficients. So, if we have or respective the Fourier series, so here two functions which differ at finitely many points then their Fourier series or their Fourier coefficients will be the same. Or in other words, we can say that if we have two functions which are piecewise continuous and the Fourier series of f and g are identical, then they must be equal except at a finite number of points.

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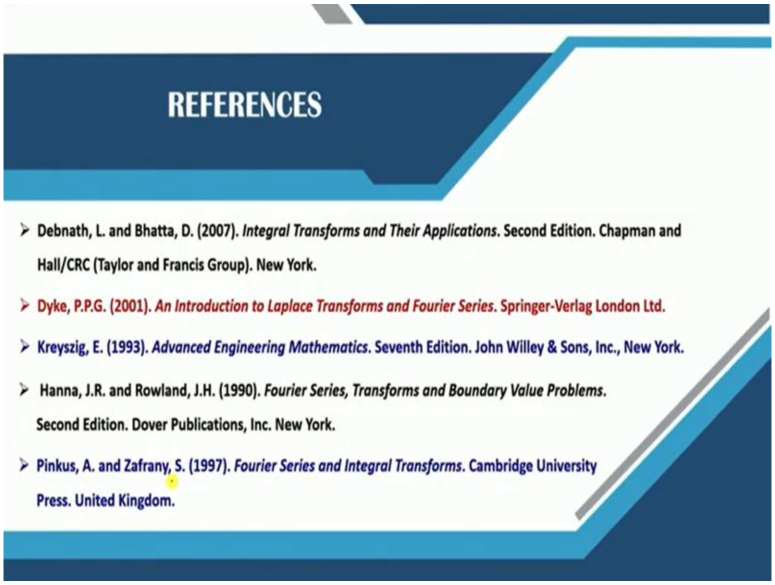

So that was a remark, these are the references we have used for preparing lectures.

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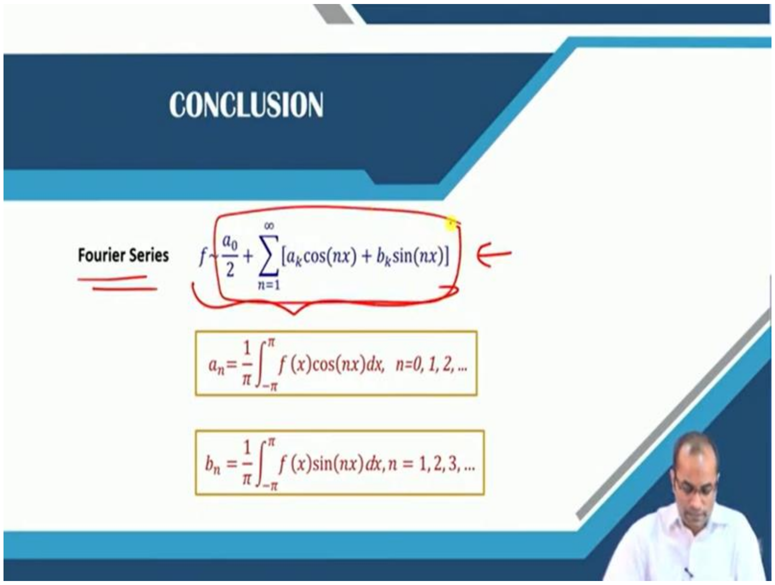

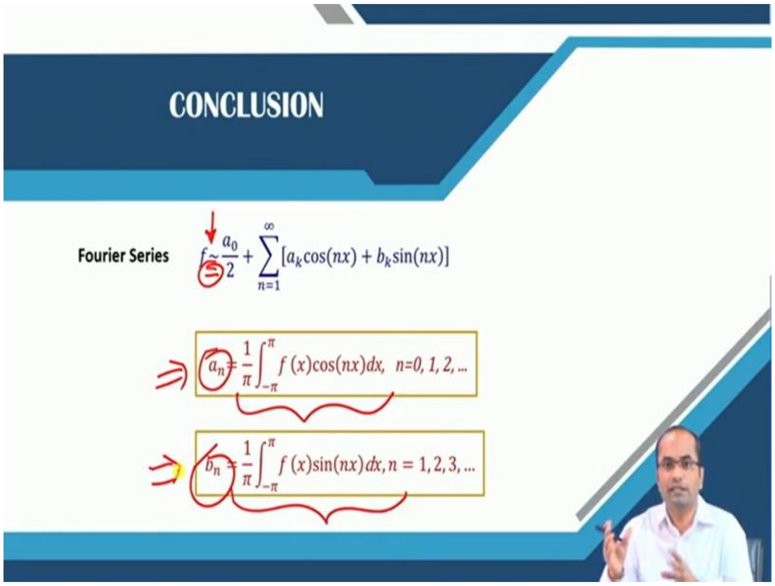

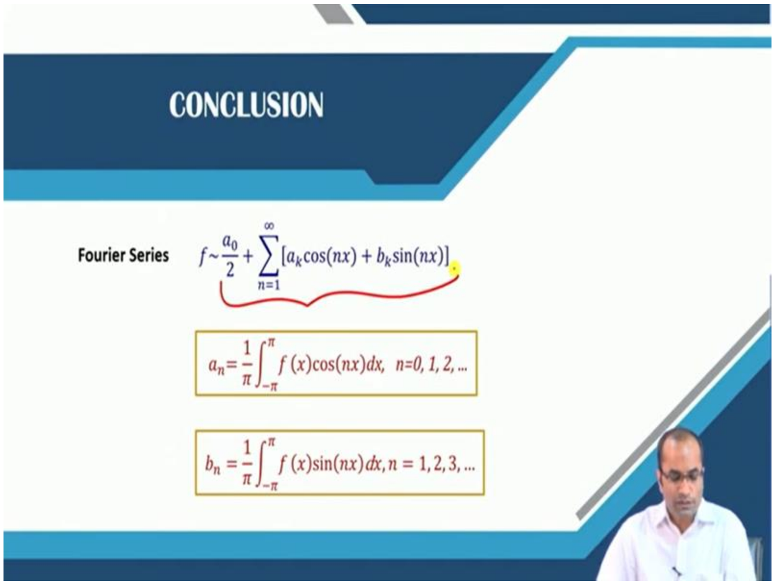

So, what we have discussed in this lecture that the Fourier series of a function which is defining minus pi to pi or 0 to 2 pi and it is periodic in with period 2 pi than it can be represented by this Fourier series and these coefficients so called the Fourier coefficients can be evaluated with the help of these integrals.

And the one important point which we should again mention here that we do not have the equality here, because these construction of these coefficients is done based on the equality of the integrals not direct equality of the function with the series here. So, with this we end I end this lecture and then I thank you for your attention.