Engineering Mathematics II
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Lecture 31
Trigonometric Polynomials and Series

So, welcome back to lectures on Engineering Mathematics 2, so this is module number 4 on Fourier series and Integral transforms. Lecture number 31 and today we will discuss on trigonometric polynomials and trigonometric series.

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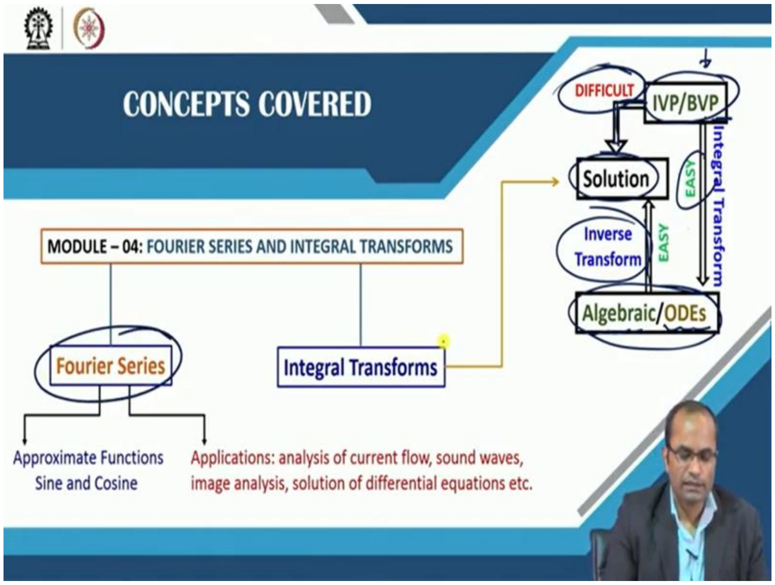

So, just first, let me tell you what we will be covering in this module four, so it's Fourier series on and this integral transforms. So, there will be naturally 2 different sections so 1 will be on Fourier series and other 1 on integral transform. So, Fourier series mainly we will their its used in a nut shell to approximate the function using sin and cosine functions which are simple and then also periodic.

So, and the second portion the other applications so we will be also talking about some applications so the Fourier series has several applications including this analysis of this current flow sound waves, in image analysis, and solution of many differential equations including partial differential equations etcetera.

The second portion will we be on integral transform and we will observe there is a connection from this Fourier series to the integral transforming particular the Fourier transform. So, that will be also studied and we will be talking about in general two types of integral transform 1 will be on Fourier transform, so that is exactly related to the Fourier series. And we will see the transformation from the Fourier series to the integral transform. There will be one more integral transform, the Laplace transform will be discussing in this lecture in this module.

And concerning this integral transform so we can explain in this way basically the motivation behind this integral transform, so will be covering this portion that the initial and boundary value problem and we try to solve directly. So, this getting the solution is very difficult for in this initial value and the boundary value problem. So, they are the differential equations associated with some kind of conditions to have unique solutions. So, those portion you will be covering in those particular lectures.

So, what we do usually we apply the integral transform to the initial value and the boundary value problem which is normally easy. And it converts the problem to a simple problem either algebraic problem or we are talking about the partial differential equations there than this integral transform will convert to the ordinary differential equations.

And in either both of them are easy to solve and then we apply the integral transform to get the solution again. So, this is a kind of technique which will be demonstrated in our lectures to get the solution of the differential equations using this integral transform. And directly if we try to get those solutions there will be very difficult but with the help of the integral transforms they will become very easy.

So, we will start with the Fourier series there will be about 10 lectures on Fourier series and followed by the 10 lectures on Fourier transform and then about 10 lectures again on Laplace transform.

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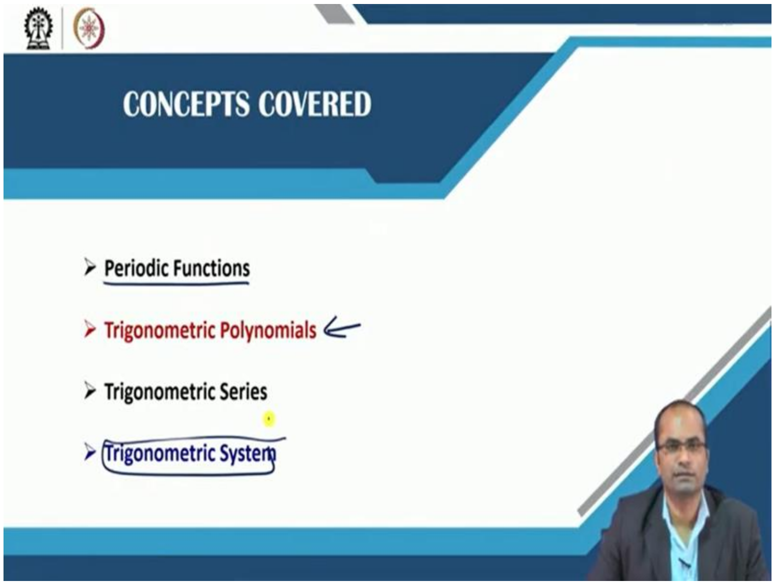

So, in this todays lecture we will be discussing what are the periodic functions because there is a connection of the Fourier series to periodic function. So, we will start with what are the periodic functions and than this is a very basic introduction to get prepared for discussing the Fourier series or to construct the Fourier series in the next lecture. So, we will be talking about the trigonometric polynomials and trigonometric series, and further as a whole we will be also using this term trigonometric system in today's lecture.

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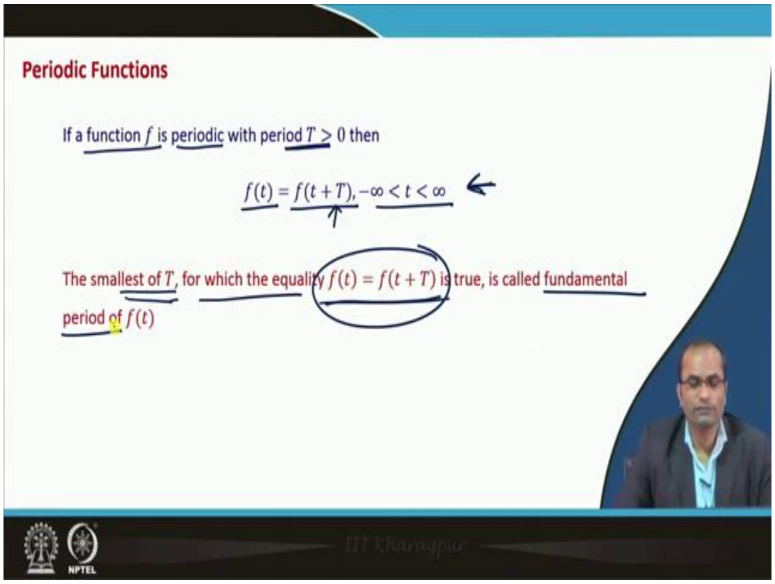

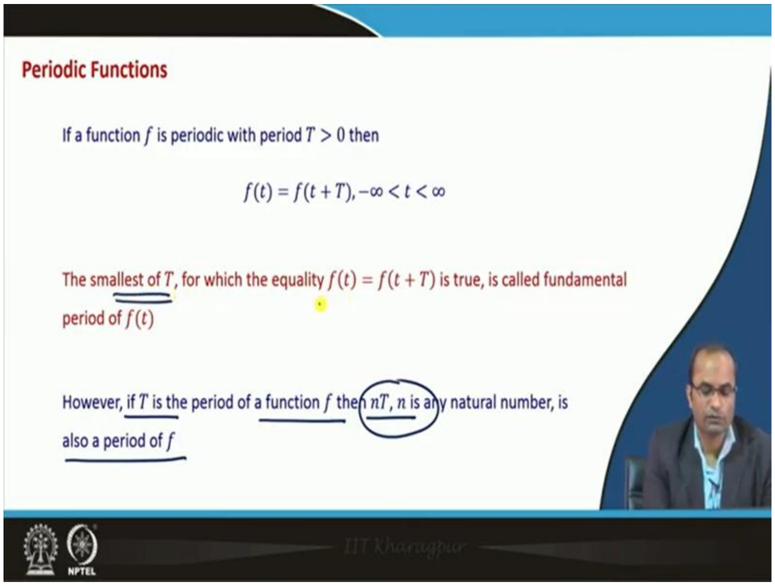

So, let me start with this periodic functions, so if a function f is periodic with period this T then the ft the function value at t must be equal to the function value at t plus this T. And then, we call that this T is the period of the function and the function is periodic if this property holds for all t. And the smallest value of t this capital T which we are calling period for which this equality holds.

Because once I mean we will observe also later that once we find some this capital T such that this property holds here than the 2t or the 3t any integer multiple of this t will also work and this property will be true. So, if ft is equal to ft p plus this capital T than also ft is equal to ft plus 2 times capital T or 2 times 3t etcetera.

So, the smallest value for which this equality is true is called the fundamental period of the function ft, so I if this t is the period of the function f than nt which I have just discussed for any a natural number is also a period of t. And as a whole we are talking about the smallest of such t for which this equality is true and this is called the fundamental period of t.

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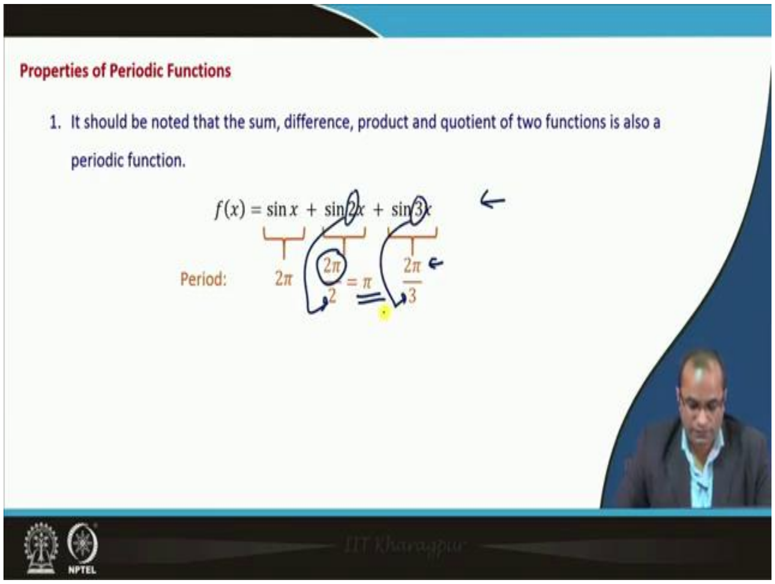

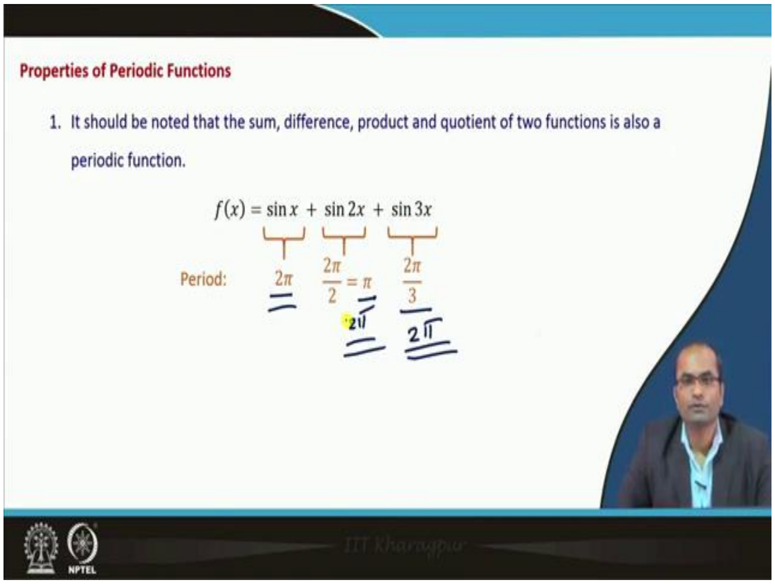

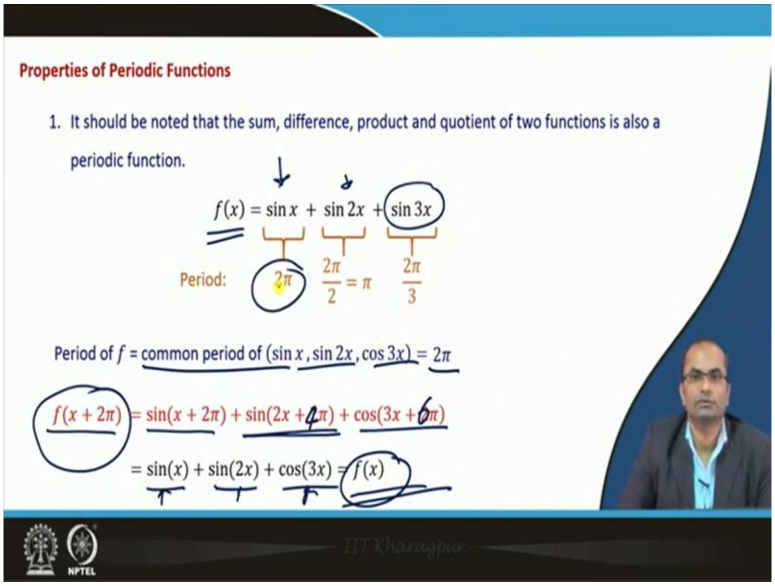

So, there are some interesting properties of the period function some of them I will list here, so the first one that the some difference product or quotient of two functions is also periodic. So, if we have two functions there sum will be also periodic their product will also periodic their difference will also periodic their quotient will also periodic.

This is a nice property we have for this periodic function. So, for instance if we take this fx is equal to sin x plus sin 2x plus sin3x so we have added 3 periodic function so it is well known that the sin x is a periodic function or sin2x or sin3x all these are periodic functions with different period different fundamental period.

So, what is the period here for this function which is sum of the 3 different periodic functions, so the period we will look individually first. So, what is the period of this first function that is 2 pie its known and for 2x when are talking about sin2x the period will be half because we are talking about 2x there. So, usually we compute 2 pie and divided by this number will give the period of this sin or cosine function. So, here we have the period pie and similarly for the sin3x we have 2 pie over this number 3 here. So, here the period is 2 pie by 3, so 2 pie period pie period and then 2 pie by 3period.

So, to get the period of this sum what we have to do we have to just find the common period of all this functions. So, for example sin2x has 2 pie period than this sin2x has pie period but we also in the previous slide we have already discussed that if pie is the period so if it is repeat repeating its value after pie, definitely it will also repeat its value after 2 pie.

So, and as a result, we are calling that if pie is the period than n pie is also a period, so here the 2 pie is also period for this function. And for this function if I multiply by 3 so again the 2 pie is also period of that function. So, the common period of all these 3 functions what we observe is actually a 2 pie and as a result we can tell something about the period of this fx because this is repeating its value after 2 pie period. This is repeating after pie and naturally after 2 pie also and this is also repeating after 2 pie.

So, the sum will also as the sum function will also repeat its value after this 2 pie period which is the common period of all these functions sin x sin2x and cos3x. So its a 2 pie here, we can observe we can clearly see that fx plus 2 pie is equal to sin x plus 2 pie sin 2x plus 2 pie and cos 3x plus 2 pie we have added this 4x we have listed here x plus 2 pie.

So, what will happen so here f x is replace by x plus 2 pie here also x plus 2 pie so this will become as 4 pie this will become as 6 pie whatever. So, here this sin x plus 2 pie is sin x here also sin x plus this 4 pie is also sin x and this is also cos x because all these functions have the period pie, 2 pie or 2 pie by 3.

So, this is sin x, sin 2x cos 3x again the same function fx and therefore this a periodic function with period 2 pie because fx plus 2 pie we are getting this fx and then this is indeed the fundamental period because the common period is 2 pie and not less than 2 pie. So, that is going to be also the fundamental period.

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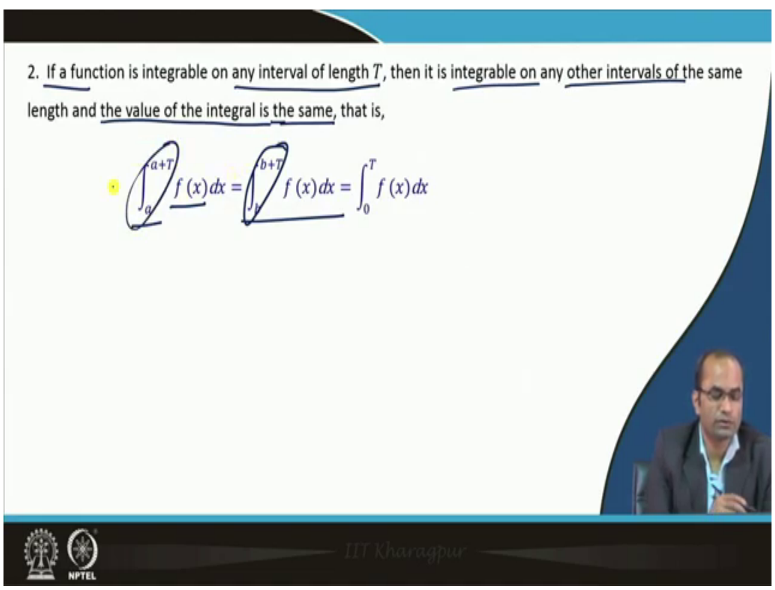

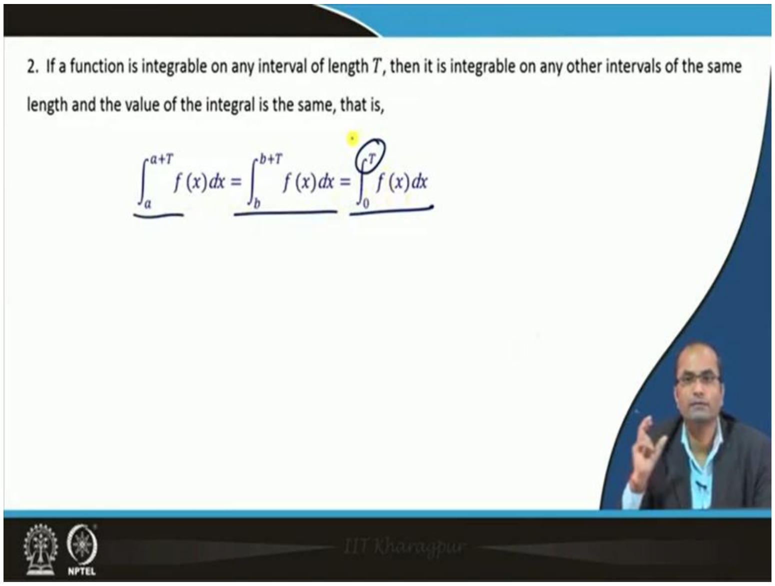

So, another interesting property which we will discuss here so if a function is integrable on any interval of length t than it is integrable on any other interval of the same length. And the value the interesting point is that the value of the integral is same over as long as the length of the interval is same and we are talking about the periodic function as its integrant than the value of all the integrals will be equal only we have to see that the length of the of the interval were we are integrating this.

So, the then the value of the integral is going to be same so let me just elaborate more, so here if fx is periodic with period t then if we integrate this a to a plus t the value will be same if we integrate from other point b to b plus t b plus t. So, here the length is t here also the length of this interval is t and that value will be same if we integrate from 0 to T so in the whole this period t so this all these values will be same as long as we have the we are covering the whole period T.

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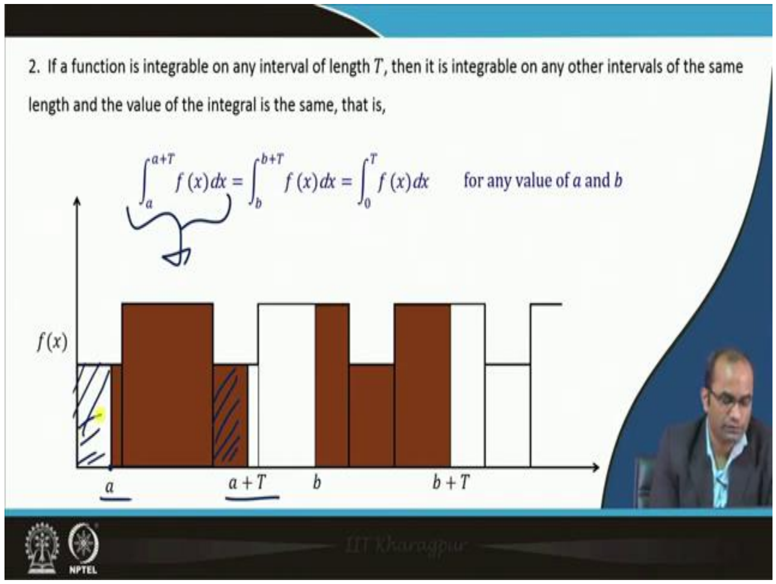

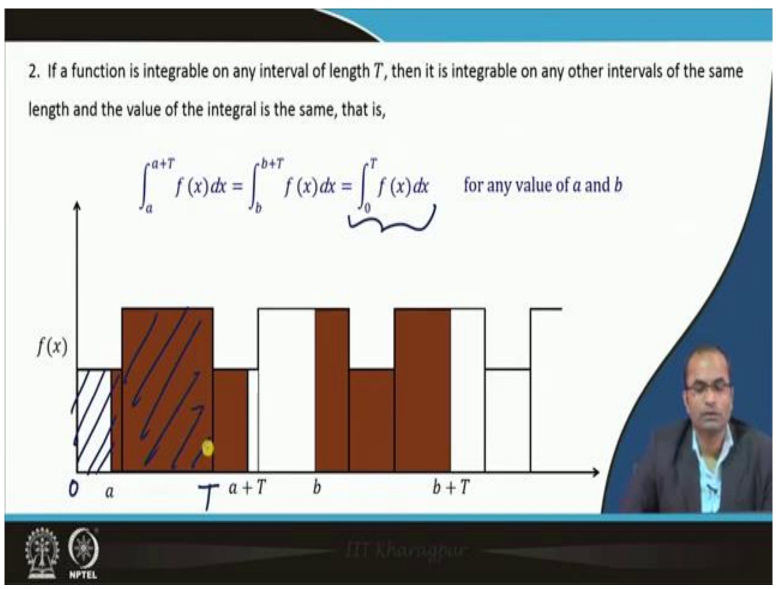

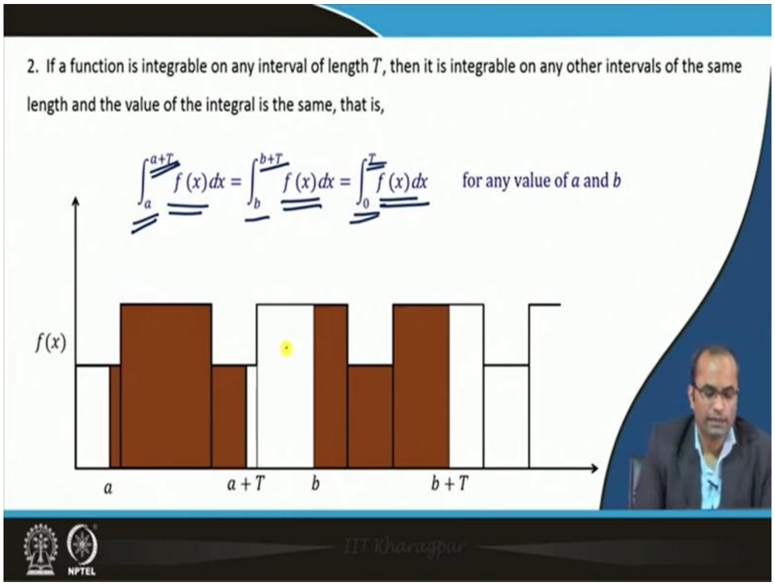

This is this can be visualize from this figure so if we are integrating suppose this is a periodic function here from here to this then we have this here and then again this starts from here and so on. So, it is a periodic function so from here to here that period ends this point to this point there is a period.

So, if we integrate this a to a plus t so what is interesting here if we started at this point a and then covering this period t. So, whatever portion was left here we have covered actually here and this a plus this t so the area under this curve fx is going to be the area of these 2 rectangles. So, one this one and the other one is this one, so and either we start from other point this b the same situation whatever we have left here for instance its covered here.

So, ultimately in any case whether we are integrating form a to a plus t or b to b plus t the area of these 2 rectangles the smaller one and this bigger 1 is covered. Which is same as if we integrate from 0 to for instance t again this the area of this 2 rectangles will be covered. So, that is the interesting property that whenever we have this periodic function we can integrate from any point to that point plus the period t b to b plus t or from 0 to 0 plus t. So, from any point we can start and then cover the whole period that will bet the all those integrals will be equal.

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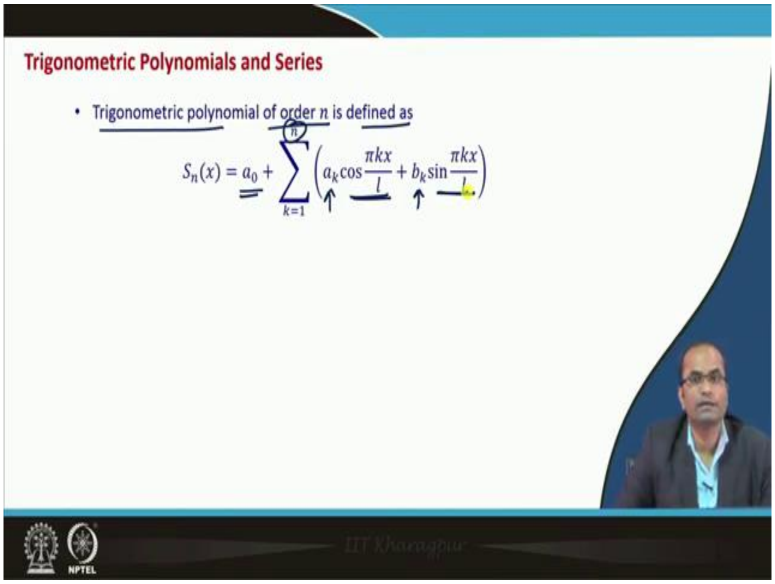

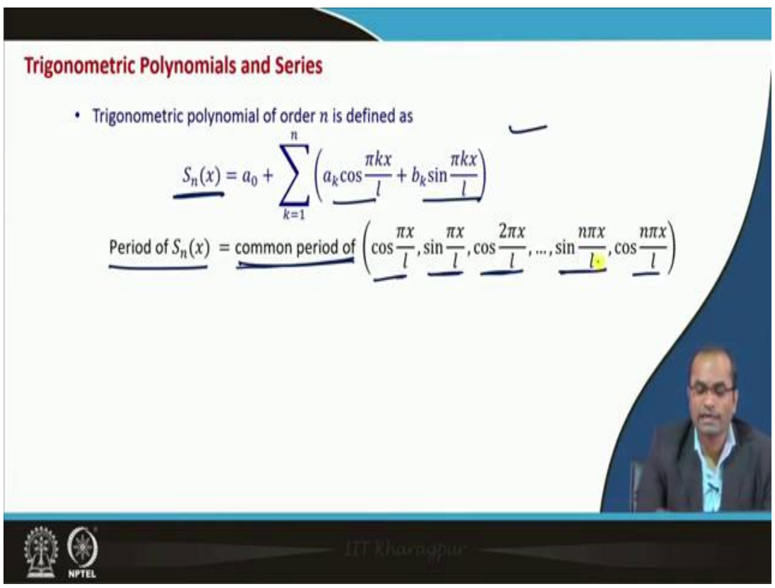

So, what are the trigonometric polynomials and then trigonometric series will be considering now, so the trigonometric polynomial of order n is defined as this some constant and then sum of the cosine and the sin functions with again some coefficient here sitting in front of cos and sin. So, and since there are n such trumps are added for having this cos and sin trumps, so we call this trigonometric polynomial of order n. And the period of this polynomial which we are denoting as Sn x will be the common period again of all these sin and cosine.

Because here what we are doing? We are adding the cosine sin cosine sine functions, and the common period if you want to see for this finite sum in this case, will be the common period of all these functions which are being added here to construct this such a so called the trigonometric polynomial. So, the first here when the k is 1 it is a cos pi x over l sin pi x over l then we have cos 2 pi x over l sin 2 pi x over l and so on. This is continue until we got this sine n pi x over l and cos n pi x over l.

So, if you want to get the common period of this remember from the particular simple example the common period was coming with this the first function itself because here we have two times here and three times n times, so the period of those functions will be smaller, the largest will be from the coming from the first function and others are just the multiple. The other period we can multiply by natural number to get this period.

So, that is going to be the common period, so common period will be 2 pi and then we have to divide by this pi by l number this number pi by l so the 2l.

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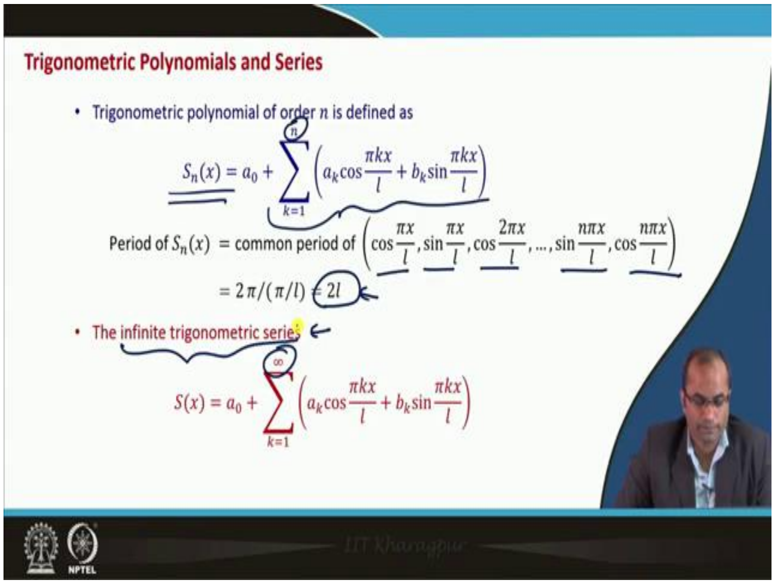

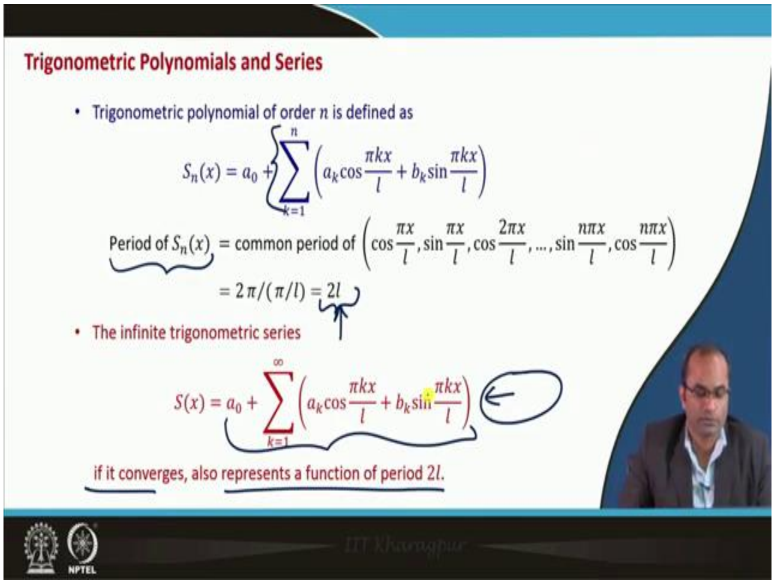

So, for this trigonometric polynomial which is just the edition of all these functions whose common period is 2l. So, the period for this Sn x for this function here for this polynomial will be 2 times l. And the trigonometric series we call the infinite so infinite trigonometric series when the k goes from 1 to infinity, so instead of this n if we are going for infinitely many terms we call this infinite trigonometric series. And in this case if the series converges and this is another interesting point that if this series converges, because now we are talking about infinite series and the series may not converge.

But if this converges in that case if it converges this also represents a function of period 2l. Because for this when n is finite we have seen that the period of this 2n or Sn x is 2l. And now here we are keep on adding these terms but that common period will not change that will remain 2l itself. So, if this series converges then that function to whom this is converging that will be a 2l periodic function.

So, that is another information which we need here for explaining the Fourier series, in the next lecture indeed. So, here we are we have now introduce in this trigonometric series or infinite trigonometric series and the trigonometric polynomial in terms of the cosine and sine functions.

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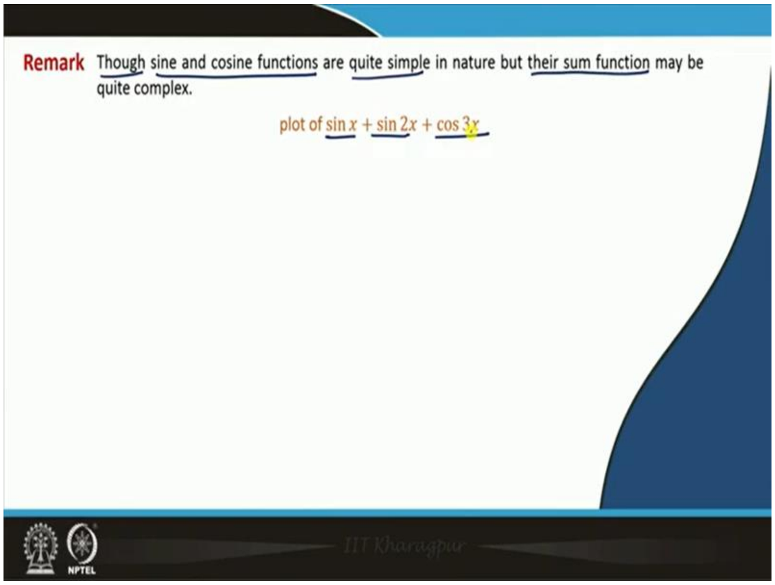

So, just a remark so though this sin and the cosine functions are just very simple functions and we know the graph of each of them. But what is interesting that they are some may be quite complex and I just want to clear this point because later on we will see that many many complicated functions can be represented in terms of this sine and cosine functions.

Just a simple example if we take which we have just seen before sin x sin2x n cos3 x for instance if we take this function whose period is going to be 2 pi, if the common period of all these functions here is 2 pi, so the period of this function the sum function is 2 pi.

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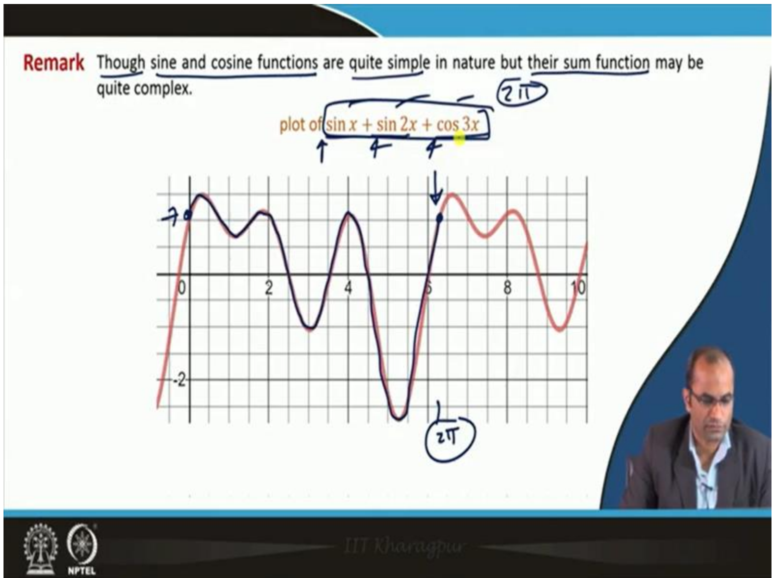

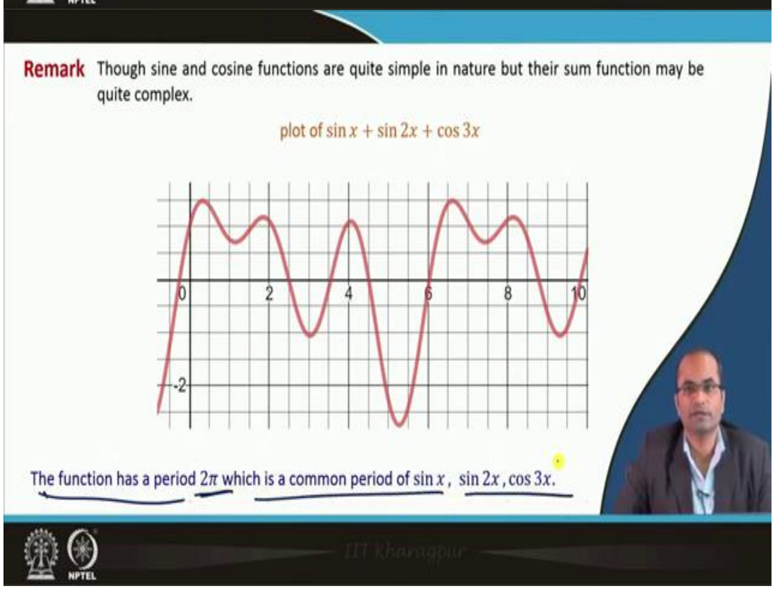

So, if we just plot this, what will happen? We can see this is not as simple as we perhaps expected from the sine and the cosine functions. So, here for instance if we say this is starts from 0 then it is having this curvature here and then goes up to this up to this 2 pi and then again this repeats its value. So, this is in one period the function starts from here and continue up to this point, and then it repeats its value, so from here to this point it is one period covering one period it is a 2 pi. So, at this point here we are somewhere 2 pi and then it will repeat its value.

So, this sum of these three just three simple functions represents this complicated behavior then we can imagine that we have if we have possibility of having many more terms or infinitely many terms that too with some coefficients here setting with this cos and sine. So, we have a full flexibility of getting a desired function in terms of the sine and the cosine functions and that is what we will exactly do in the Fourier series, in the next lecture we will observe that. So, the function has this period 2 pi which is a common period of all these three functions which I have already discussed.

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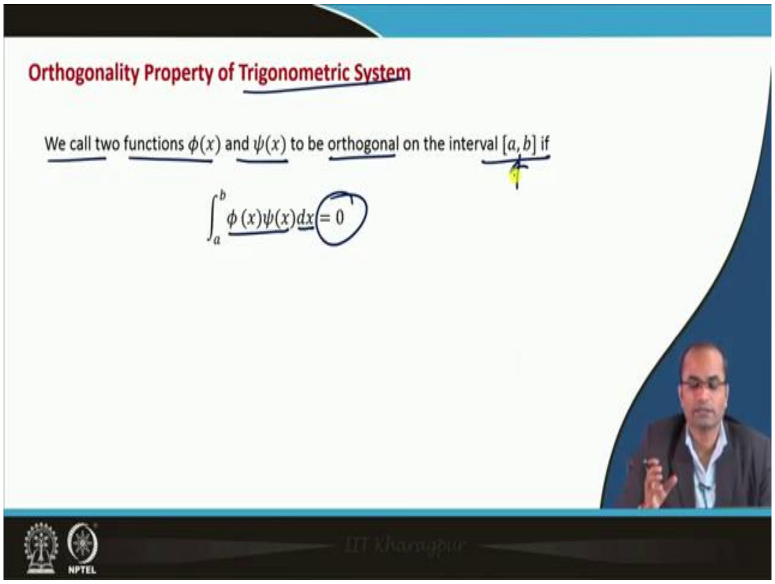

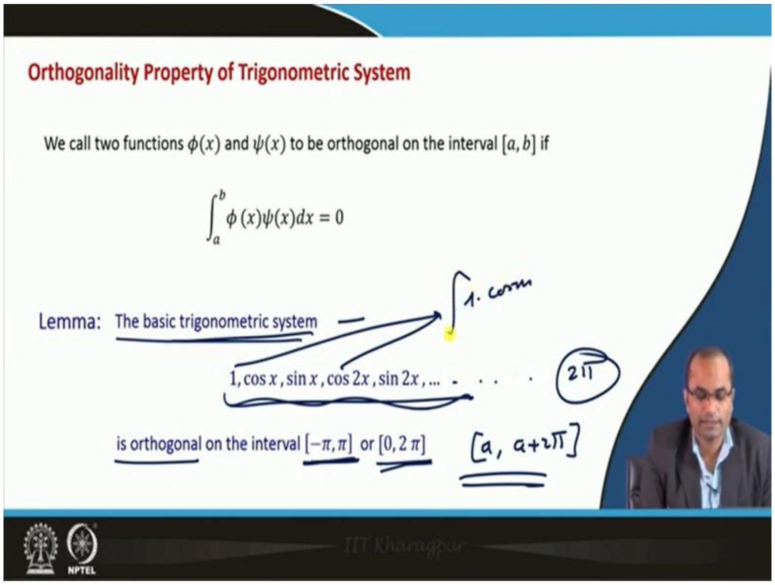

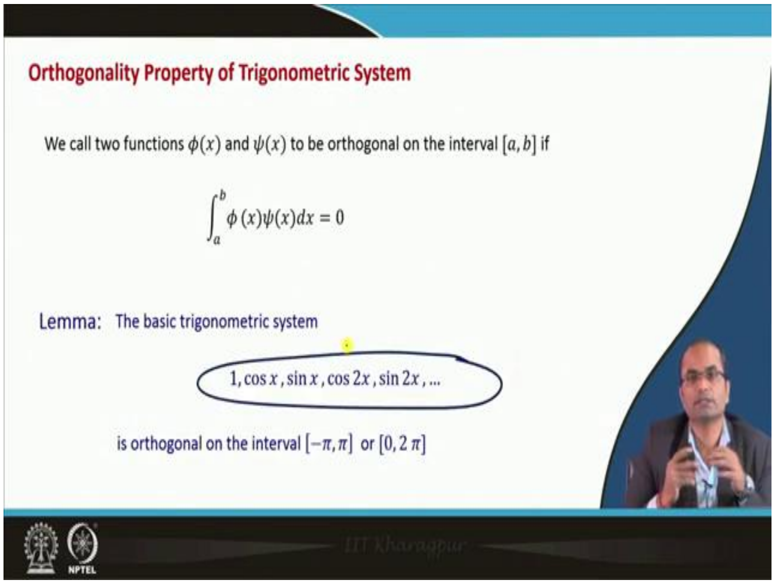

And very interesting property now we will look into for trigonometric system. So, what is the trigonometric system? We are calling this sine cosine and so on sin 2x cos 3x, that is system with having cosine sine functions and that is called trigonometric system. So, we call that two functions phi x and psi x to be orthogonal on this interval a, b, if we integrate from a to b the product of these two functions and if it is 0, if the product of the two functions integrated over the certain interval here a, b is 0 then we call that these two functions are orthogonal in this interval a, b or on the interval a, b if this integral is 0.

So, now we have the result for our trigonometric system, so the basic trigonometric system we call the basic trigonometric system and we have the simple functions 1 cos x sin x we have cos 2x sin 2x etcetera and this can continue to whatever number we want. So, this basic trigonometric function what is interesting here and we will prove this fact that this is orthogonal on this interval minus pi to pi or we can take any other interval for instance 0 to 2 pi or a to a plus 2 pi as long as we are covering the length this 2 pi than for because this is the common period for this is 2 pi for this system here.

So, this is orthogonal on the interval minus pi to pi or 0 to 2 pi or any number a we can take and a plus 2 pi this is also fine. So, this is we will observe that this basic trigonometric system this is orthogonal, what do we mean? If we take any two functions from this system, for instance, we take 1, we take cos2x and 1 into cos 2x and then integrate from this minus pi to pi dx. This must be 0. This is clearly visible here too but we will formally prove in the next slide.

So, if we take any two members from this of this family any two members form this system and their product when integrated over this interval minus pi to pi, the value of this integral is going to be 0. This is what we will observe now.

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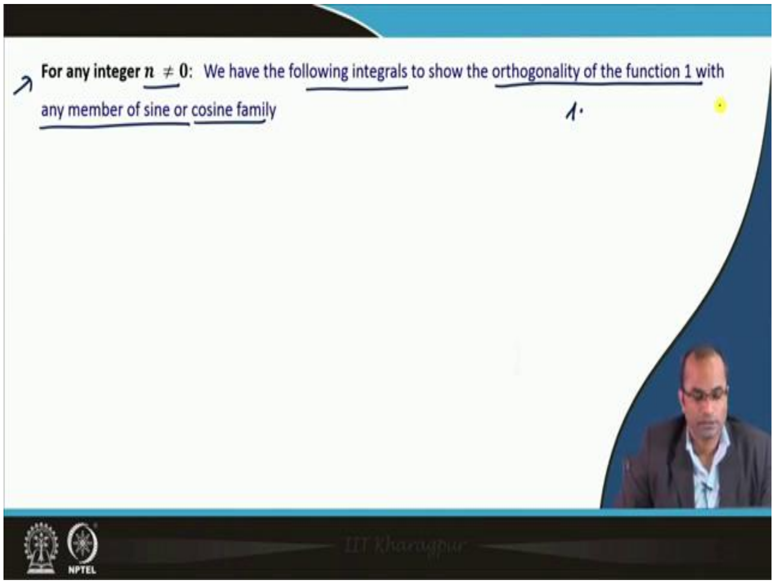

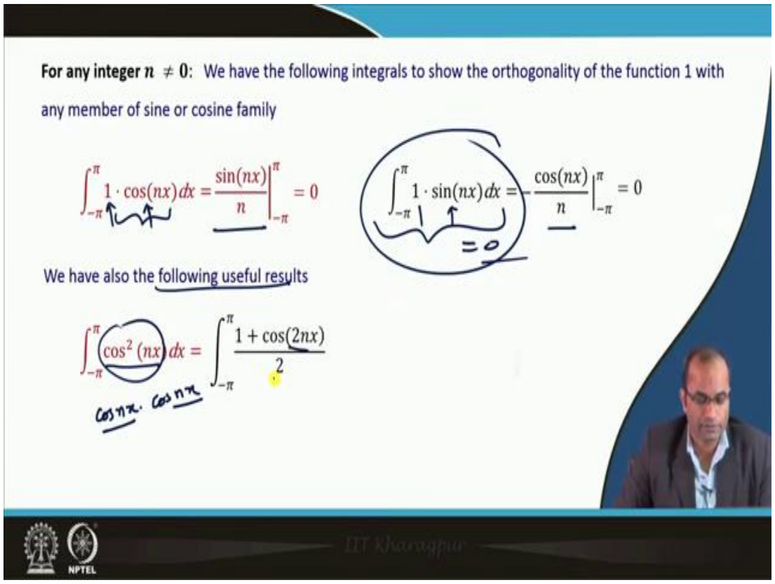

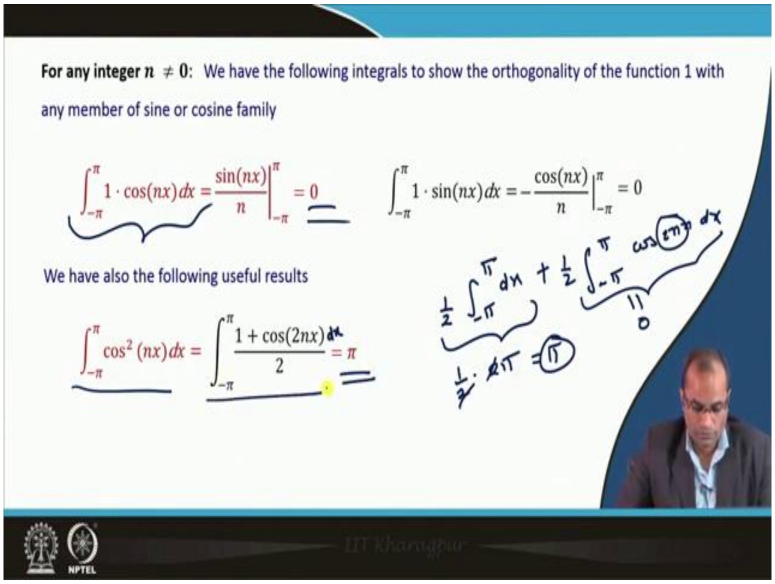

So, we will consider different cases, first we are considering here that if n is not equal to 0 its n integer we are talking about, but which is not equal to 0. And then we have the following integrals to show the orthogonality of the function 1 with any other member of the sine or cosine family. So, at first what we are looking for.

So, 1 is also a member of this trigonometric system, so this 1 and we will take with any other sine or cosine function, either sin x, sin2x, cosx, cos2x etcetera. And we will show here that these two are orthogonal 1 with any other trigonometric any other member of this trigonometric system from the sine or cosine family. So, here if we take 1 and for instance, is cos nx, n could be anything other than 0 because if n is 0 this is going to be 1, so which that the same member we will be making the product so that we are avoiding here the two different members.

So, this is one member and this is the another member of the of this trigonometric system and if we integrate from minus pi to pi, so it is just the product is cos nx so the integral will be sin nx over n and then minus pi to pi. Sin n pi is 0, sin minus n pi is also 0 so the result of this is 0. So, here we have seen that with 1 if we take any member of the cosine family that means the cos x, cos 2x, cos 3x, cos 5x etcetera, these two are orthogonal.

Similarly we can also prove that with 1 if we take any other member of this sine family from our trigonometric system, then again we can see that this is cos nx over n and here or indeed from here directly we can so sin nx is a odd function and the value is 0. So, the value of this integral is also 0.

So, this at least we have seen that with 1 we can take any member of the cosine family or any member of the sine family and these are orthogonal. We can also check this following useful results that if we take the same member that means the cos nx, cos nx mean cos x, cos x or cos 2x with cos 2x, so they are not different members, so they are the same members.

What we have said in the earlier result that, the system is orthogonal that means any two different members if we take they are orthogonal but not the same member. So, here we are taking the same member we have taken the cos nx and with cos nx for same n of course. So, if we have taken the same member in that case this is not 0 and we will observe now that what is the value coming. So, this cos square nx we can convert into this two angle double angle 2 nx.

So, 1 plus cos 2nx by 2 with the trigonometric identity we have this result. And then this value is coming to be pi, how pi? So, the first integral is this half is integral minus pi to pi and the dx, so here is dx missing dx. So, and the second one so the plus again half we have minus pi to pi and we have cos or 2nx and dx. So, this cos 2nx dx we have just seen above that this is going to be 0 with any number nx there, so this integral is 0, whereas the first one gives half and this is 2 pi. So, we get the phi as a value pi there.

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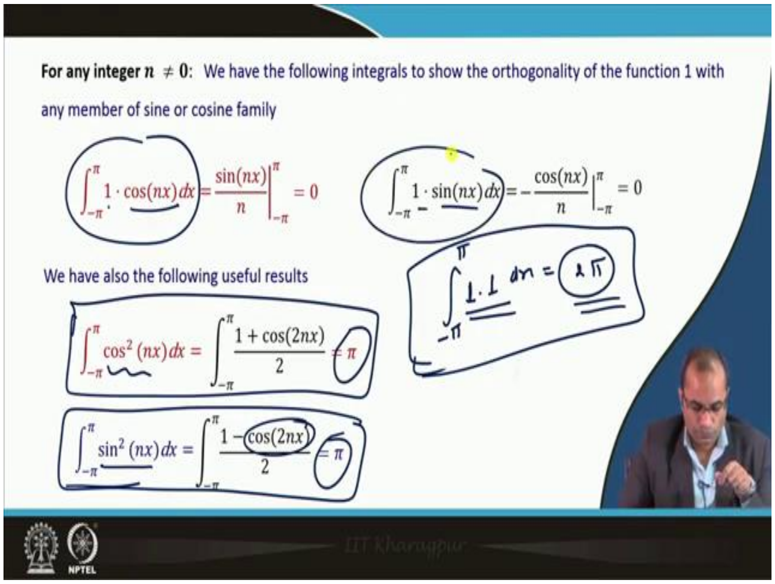

So, here we have the value pi so cos square nx dx is equal to this one and is equal to pi. So, when we take the same member of the family than the value is pi. Similarly, if we take the sin from the sine family the same member sin nx sin nx. In this case also we can observe that this is 1 minus cos nx by 2 and again the second integral will lead to 0 and this half integral minus pi to pi will give us pi.

So, here the value is pi in either case the only member left is 1, if we take 1 with 1 and then integrate from minus pi to pi dx, in this case the value will be 2 pi naturally. So, these are three situations we have covered, one when we have taken the same member from the cos family value is pi, the same member from the sine family again the value is pi, and the same member of this 1 itself, because 1 is also a member of the trigonometric system in that case this is 2 pi.

And 1 with any other member of the cos is 0, 1 with any other member of the sin it is 0. And there is a now something left here to be proved that any two members we can take of the different two members of the family it is a 0. So, here we have only proved that with the 1 if we take any other member from the system it is 0.

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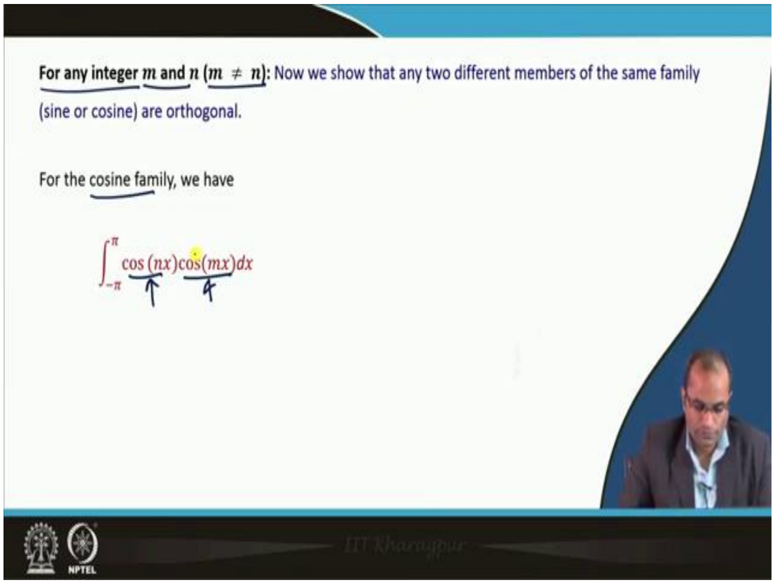

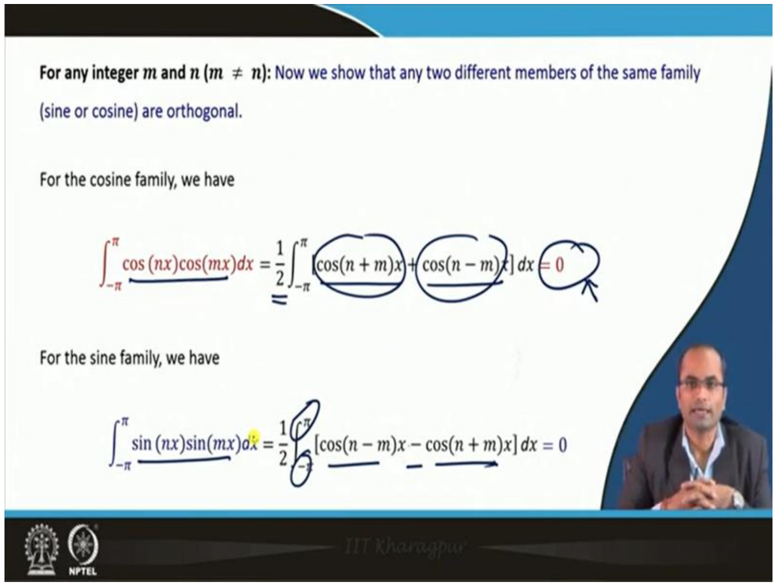

So, now we will continue this proof. So, we will take now for any integer m and n, when these two are not equal that means first for the cosine family we will take. So, cos nx and cos mx, so when m and n are equal we have seen the value is pi. So, but now we have two different members, one is cos nx another one is cos mx, when n and m are not equal. So they are different member of the family.

So, in this case we again use the trigonometric identity 2 cos a cos b with half here, so cos this n nx plus my and then nx minus my and integral is from minus pi to pi. So, here the sin will come and we integrate here also sin will come and then with pi and minus pi, in either case the value is going to be 0.

So, can when we take the two different member of the cosine family the value is 0, similarly if we take from the sine family also the same situation will happen so again this trigonometric identity will lead to the cos only with the negative sign there and when we integrate this again sin n minus m sin n plus m will come and x and when we have to substitute the value pi n minus pi so sin integer times pi is 0 and sin n integer times pi is 0.

So, this will also lead to 0, so we have also seen that if we take two different members from the same family it is going to be 0, if we take fix 1 as a member and take any other member from the cosine or from the sine that is 0, so what is left now? That for any m, n if we take 1 member from sine family other member from the cosine family then what will happen?

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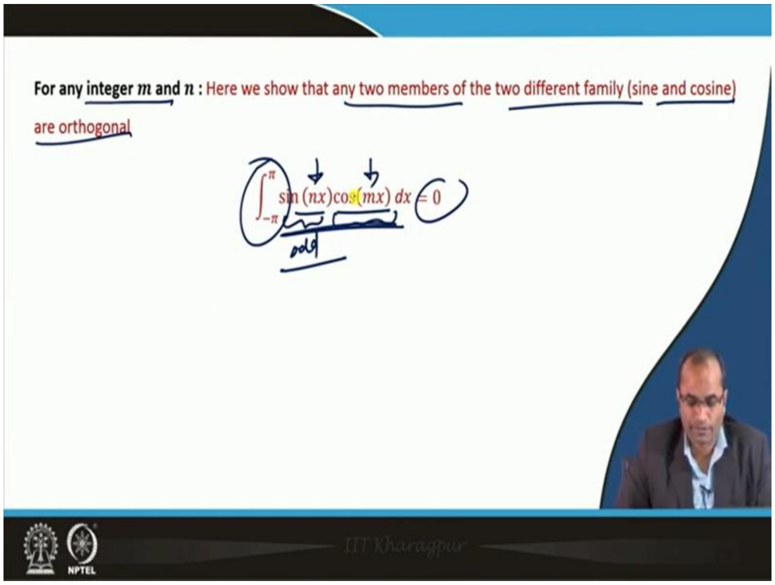

So, for any integer m and n we will show now that any two members of the two different family means sine and cosine they are also orthogonal, so we take sin nx and then the cos mx, the two here the two members the two different members, so one from sine and other from cosine. So, here m and n may be the same because the family is different, one is coming from sine another one is coming from cosine.

So, we can talk about sin x cos x so that value will be also 0, and here the reasoning is very clear he cos is the even function here sin is the odd function and when we integrate minus pi to pi this integrant is odd, so the value is 0 irrespective of whatever n or m is.

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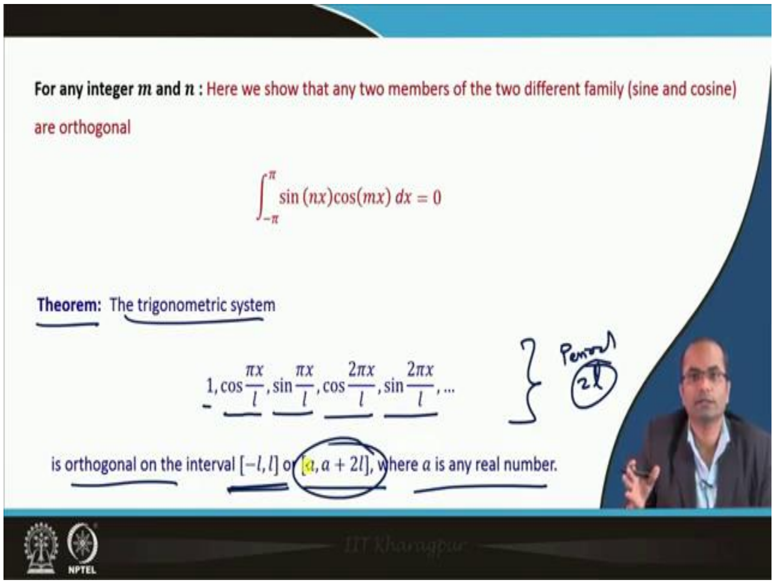

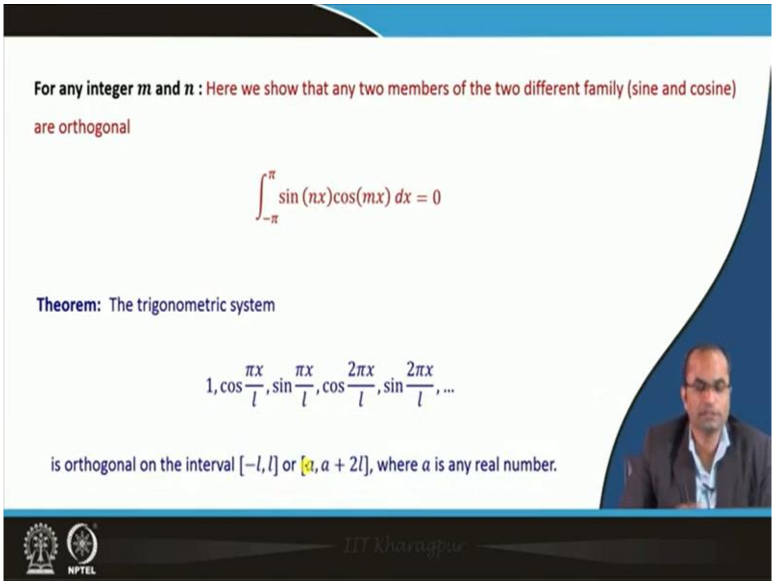

So, the theorem, what is we have a more general result than what we have just proved where we have taken the basic the fundamental trigonometric system where the period was 2 pi. But for a rather general trigonometric system where the period is not 2 pi but it is 2l, we can also prove the same similar result that this trigonometric system and now we have generalize the system. So, we have taken 1 and cos pi x over l, sin pi x over l, cos 2 pi x over l, sin 2 pi x over l and then we will continue with cos 3 pi x over l, sin 3 pi x over l, etcetera. So, here this trigonometric system the period is period is 2l now instead of 2 pi.

So, in this case, this is orthogonal in on the interval minus l to l, if we go from minus l to l we are covering the period 2l or a to a pus 2l we can go, where a is any real number. So, this trigonometric system is also orthogonal in these intervals minus l to l or a to a plus 2l. So, this is a more general result than the earlier one which we have proved, so this we are not proving but exactly the proof follows the similar argument what we have just done there.

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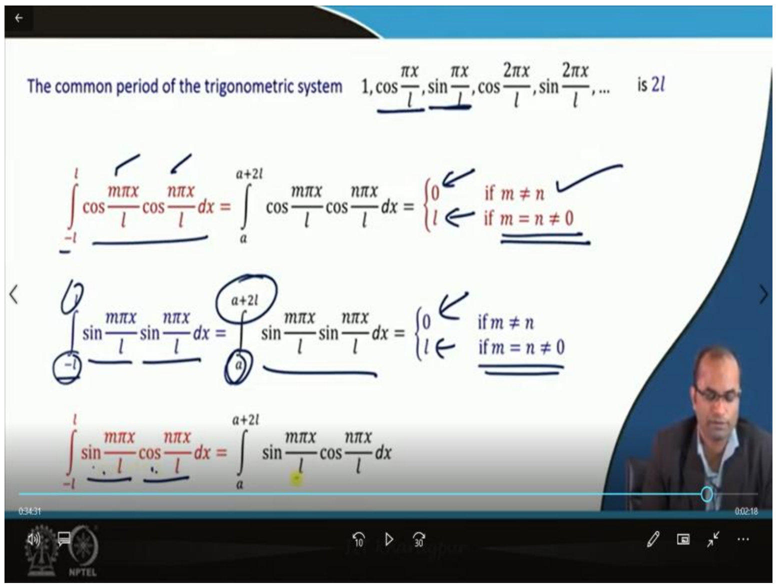

So, the common period of this trigonometric system is the 2l, this can be easily obtained from the period of this first two functions which is 2l. So, and what results we have now it is if minus l to l we integrate the cos mx and the cos nx so for different m and n than the value will be 0. And if have same m and n then the value will be l earlier it was pi. So, if m is not equal to n the value is 0 so they are orthogonal if they are two different members but if they are the same members then the value is l, this is the result can be proved.

Or minus l to l sin for the sine family also we have the same result that it is 0 when they are two different members of the family and if they are the same then the value is coming as l. And we can we know already for periodic function where are you integrate from minus l to l or from a to a plus 2l, the value will be the same. Another result that we can have two different members now sine and cosine, one from sine family another from cosine family. And then we will observe that this is also 0. So, this is a very important result which will be used in the next lecture.

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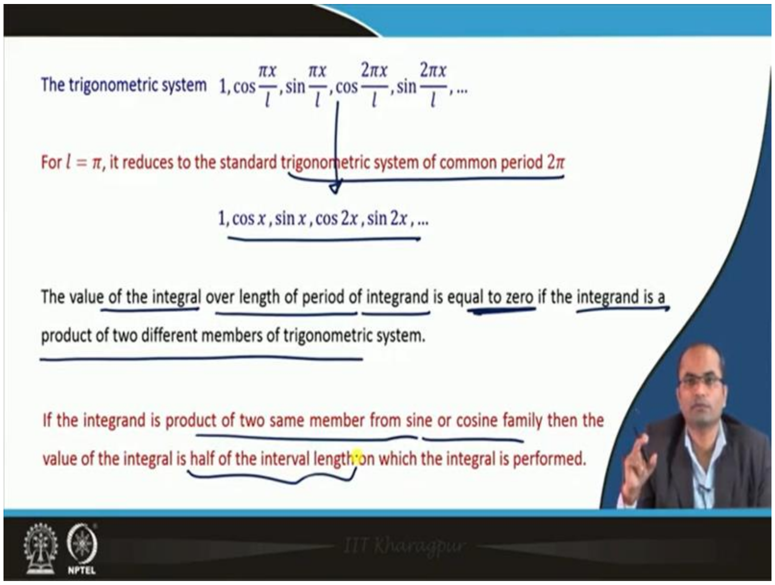

So, what we have now this remark, so if l is equal to pi this system reduces to so if we put l is equal to pi, so it is simply that system which we have discuss before 1 cos x, sin x, sin 2x, etcetera. So, this is the same system the standard trigonometric system of pj 2 pi, this one which can be just obtained from this one. And the value of the integral so which was the earlier result which we have just discussed then, over the length of the period of integrant is equal to 0, if the integrant is a product of two different members of the trigonometric systems.

And if the product is from the same member from the sine or the cosine family. So we are taking sin x sin x or sin 2x sin 2x, in that case it is the value is the half of the interval length. So, if we are talking about minus l to l so the value will be l or we are talking about minus pi to pi the value will be pi, this is what we have seen.

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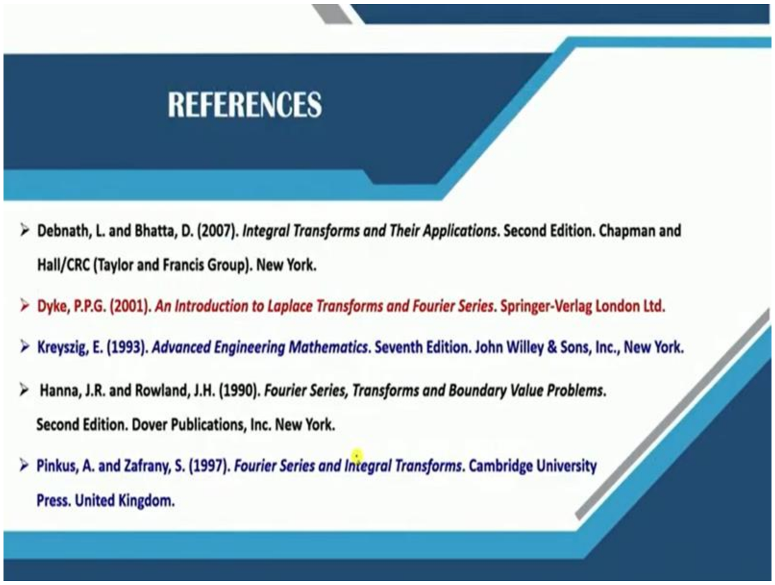

So these are the references which we have use for preparing this lecture.

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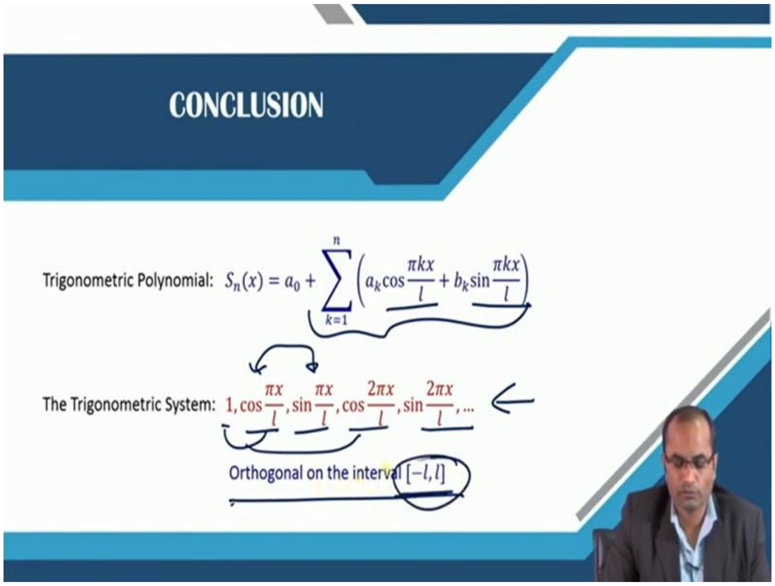

And just to conclude, so we have basically discussed that trigonometric polynomial today which is this finite sum of the cos and the sine function, and we have also discussed that trigonometric system which is 1 cos pi x, sin pi x over l, and so on. And the interesting property of this trigonometric system which we have also proved it is that.

The system is orthogonal on this interval minus l to l, that means any two members we take and the product if we integrate over minus l to l the value will be 0. So, you can take any two different members here and the product if integrated over from minus l to l the value will be 0. So, that is all for this lecture and I thank you for your attention.