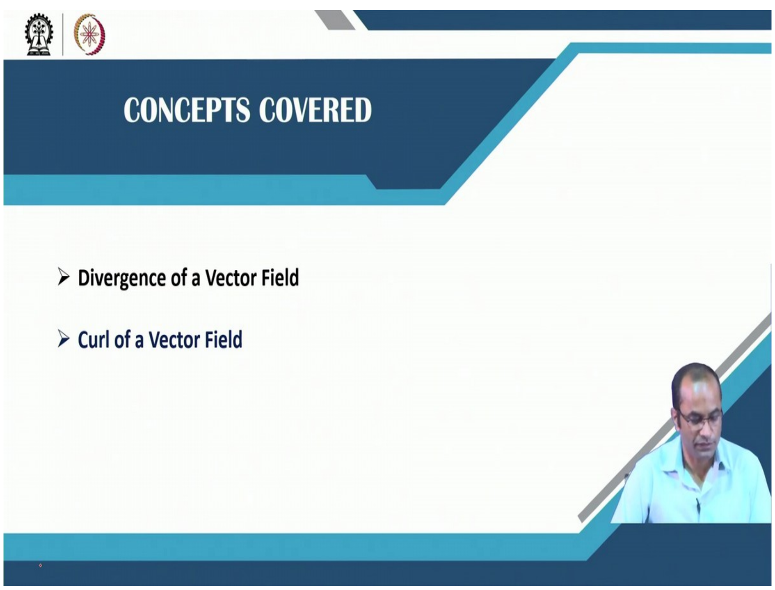
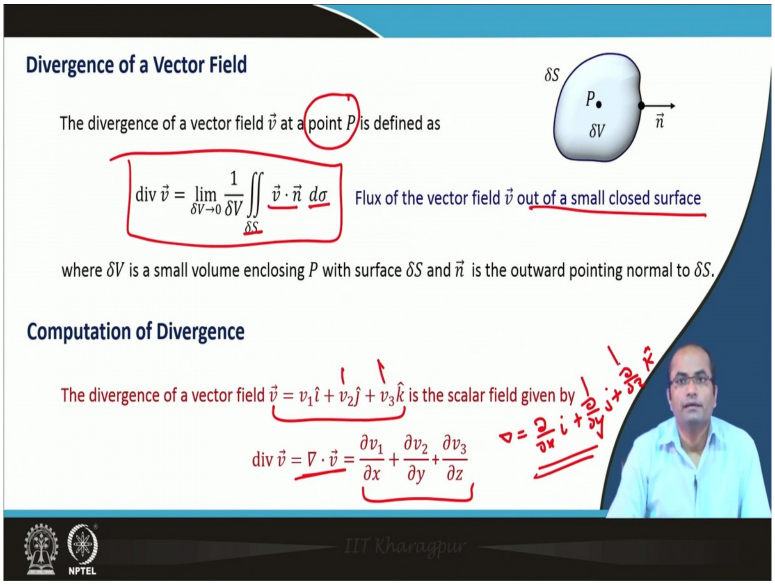
Engineering Mathematics II
Prof. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology Kharagpur
Lecture 03: Divergence and Curl of a Vector Fields

(Refer Slide Time: 00:19)



Welcome back to lectures on Engineering Mathematics 2 and this is lecture number 3 on Divergence and Curl of Vector Field. So, today we will cover the divergence of the vector field and it's geometrical interpretation, also the curl of a vector field and again its physical interpretation.

(Refer Slide Time: 00:40)

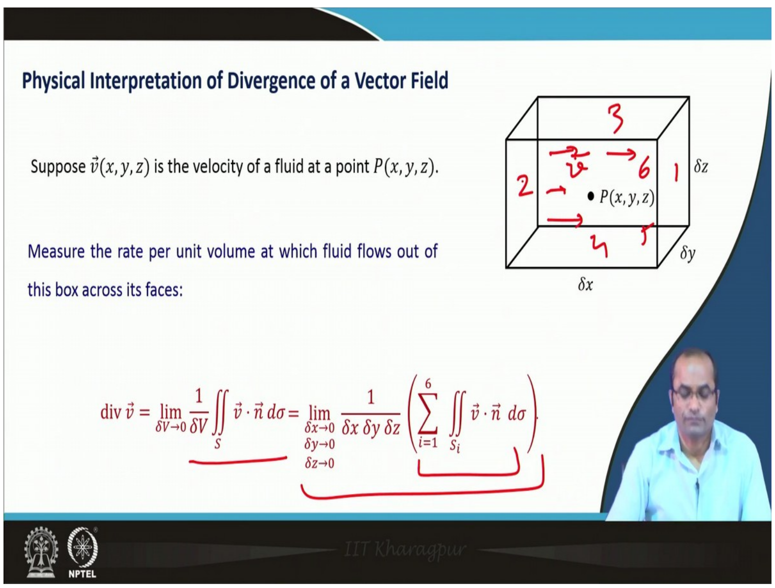


So, coming to the divergence of a vector field, so, the divergence of a vector field v at a point P is defined as by this integral which is actually a surface integral. This is done on the surface here this delta S and this v dot n and this is surface integral which we will learn extensively little later, but today a very special case of this, we will handle with the help of the knowledge which we have gained in integral calculus. So, again here we have this delta v, a volume and that is covered by this surface delta S around this point P and this n is the outward normal pointing, this normal to this delta S. So, this is 1 divided by the volume and this surface integral is done over this v dot n, and here the again the limiting situation we will be considering that what will happen when delta v approaches to 0.

So, that is actually we are talking about the divergence at a point P, well, so, here we can understand this that is the flux of the vector field v out of a small close surface because we are talking about this v dot n. So, v the component of this vector field in the direction of the normal and then we are integrating over the surface and then dividing by the total volume taking the limit. So, it can be considered as the flux of the vector field out of a small close surface around a point P. So, we will go into the detail a bit more about this physical interpretation and the evaluation of such vector field.

So, for instance if we talk about the computation of the divergence, so, this surface integral can be handled in a much more simpler way and the formula which turn up out of this limiting situation, considering a volume around a point P. It is given by this divergent P is equal to this del and the dot product with the v so, that means this del v1 over del x plus del v2 over del y and del v3 over del set because this del here again to remind the del was the operator, del over del x, the ith and del over del y, the j and del over del z in the direction of this k. So this del operator, when we do this product, the dot product with the vector field v then we will get naturally this del v1 over del x, del v2 over del y and del v3 over del z.

(Refer Slide Time: 3:44)

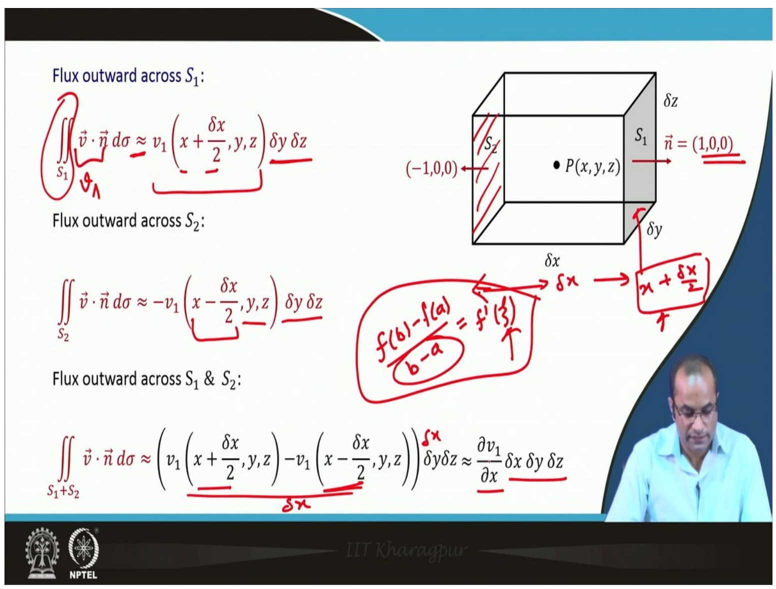


So coming to the physical interpretation of the divergence of a vector field, so we will consider a point is P at which we will compute the divergence and suppose we have this vector field v defined around this point P. So, this is we have the some vector field v there in this domain and around this P or at P we are going to compute that what this divergence signifies and how to compute this divergence. Eventually, we will see that the formula which we have seen earlier that del, the dot product of the v, how this formula is also appearing that we will also get through this interpretation.

So, if you want to measure the rate here per unit volume at which this fluid flows out of this box, so, then we need to compute precisely that surface integral which was defined in the definition and we have considered a simple geometry here the box around this point P. So, the evaluation of this surface integral will be much easier so, what we are going to do now here the surface integral because here with the surfaces are these planes.

So, we have this face here one, two, and then three, four and then there are two more faces so, the five and six. So, there are six faces, which this geometry has. So, we will compute the integral, the surface integral over these six faces, because this is the face, this is the surface around this point P, the surface occupying some volume delta x, delta y, delta z around this point P. So, this integral can be evaluated in a much simpler way by considering the six integrals, one for each face of this geometry.

(Refer Slide Time: 5:49)



So, now, if we compute the flux outward across the S1 so, let us consider this surface S1 whose normal, unit normal vector is 1, 0, 0 because we are talking about so we have let us say direction x here and then we have the y direction and the z, the vertical direction. So, in this direction the unit normal vector is given by this 1, 0, 0. And now if you want to compute the flux across this that means that integral which we have discussed that surface integral, so, in this case, the surface is just the plane here and the integral can reduce to the double integral, the usual double integral which we have already studied. And in that case, this v dot n because here the n is 1, 0, 0 so, only the first component of this v will survive. So you are going to have this v1.

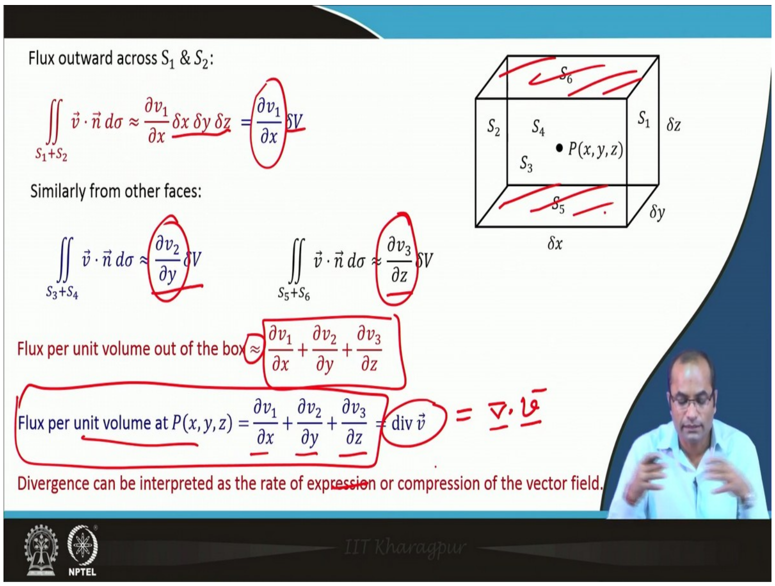
And now at this phase here, if we talk about that, what is the x coordinate, so the x is fixed here, so x and this distance from here to here is delta x, and this is the center so, we have the x plus this delta x by 2 which is fixed over this surface. So, this v1, the x component is fixed by the x plus delta x by 2, this is not wearing over the surface S1 only the y and z will vary. So, we will make an approximation here to this integral so, this is not a very formal mathematical proof, but just to give some idea, because later on we will also consider the limiting situation that delta x, delta y and delta z they go to 0.

So, in that case this approximation will become equality so, in this case we will just take so, the v1 is evaluated at x plus this delta x by 2 and then we have y, z though we have the integral over the y, z over this surface S1, but we have taken just the approximation taking as this constant value over this surface, but again as I said that in a limiting situation, this will become actually the equality. So, then this is the area of the surface which is coming out of this integral and similarly, we can do this for the S2. So, the difference in S2 is that we have this minus 1 there. So, this component will become minus v1 and the x component will be x minus delta x by 2 because this is left to this point x, y, z. So, in this case, we have now v1 and x is replaced by the x minus delta x by 2 and we have this y, z and again with the same argument we have written the surface integral over this surface which is S2.

Now, if we add these two that means the flux out toward across these two faces S1 and S2. So, if we add the two so we have v1 and then we have minus v1 again at the x minus delta x by 2. So, these two, the first argument here of v1 varies, this is x minus x plus delta x by 2 and we have x minus delta x by 2. And now, we will apply the mean value theorem. So, just to recall that we have for instance here f b and minus f at a and divided by this b minus a.

So, the mean value theorem says that, there will be a point between a and b where this quotient will be equal to the derivative of f. So, here also you will apply for the first argument, so, this derivative will become the partial derivative with respect to x and then there will be some point between these two x minus delta x by 2 and x plus delta x by 2 this is I there, but again we will just write the approximately equal so, at the point x, y, z because in the limiting situation again, this will approach to only this point P. So, we have the partial derivative del v over, del v1 over del x and this delta x because we have to divide by this delta x also to apply this mean value theorem term. So, you multiplied by delta x and divide by delta x. So, this is delta x, delta y, delta z multiplied by this partial derivative with respect to x.

(Refer Slide Time: 10:26)

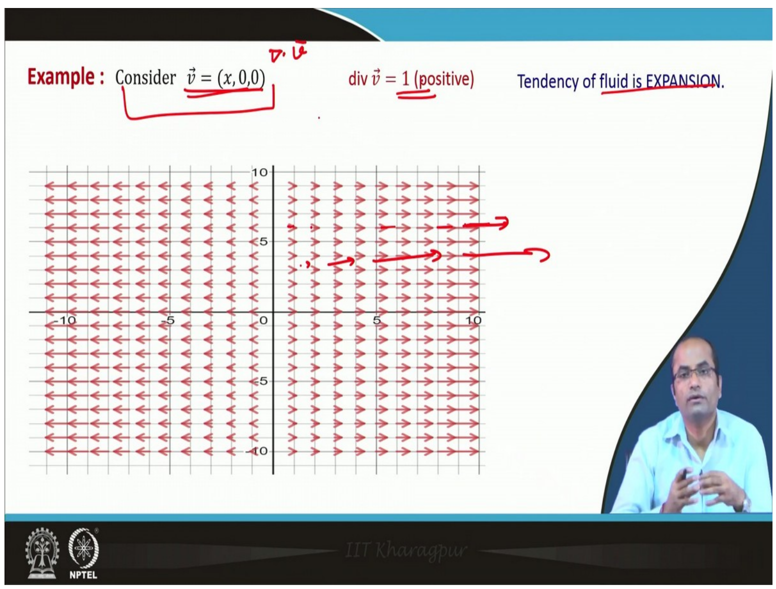


Now, moving next, so, we have the flux across the S1 and S2 and this is what we have computed. So, this delta x, delta y, and delta z that is we can also denote by the delta v, the volume of this box and then similarly from other faces also we can compute this easily so, for example, for S3 and S4 which is in the direction of this y. So, we have similarly del v2 over del y and then the delta v and again for S5 and S6, which is here on the bottom and the top one, so in the direction of z, so, we will get this del v3 over del z and delta v. So, the flux per unit volume out of the box here, we can add all these and divide by this delta y to get this per unit volume.

So, that means, you will be adding del v1 over del x, del v2 over del y and del v3 over del z. So, this is what we have this flux, and the flux per unit volume at P. So, exactly this quantity but it is exactly equal because in the limiting situation these approximations what we have taken for computing all these integrals will be actually equality. So, finally, we have this formula for this flux per unit volume given by this del v1 over del x del v2 over del y and del v3 del z, which is the divergence which we denote buy this, divergence v or we can also write down this as del dot v, the dot product of these two vector the del operator and the vector v.

So, what we conclude now that the divergence can be interpreted as the rate of expansion or the compression of the vector field because this is the flux here, we have computed per unit volume at P. So, if it is coming to be like positive that means, at that point the tendency of the vector field is for the expansion. So, here its expansion and on the other hand if this comes to be negative number, there is a tendency of compression at this point for a given vector field.

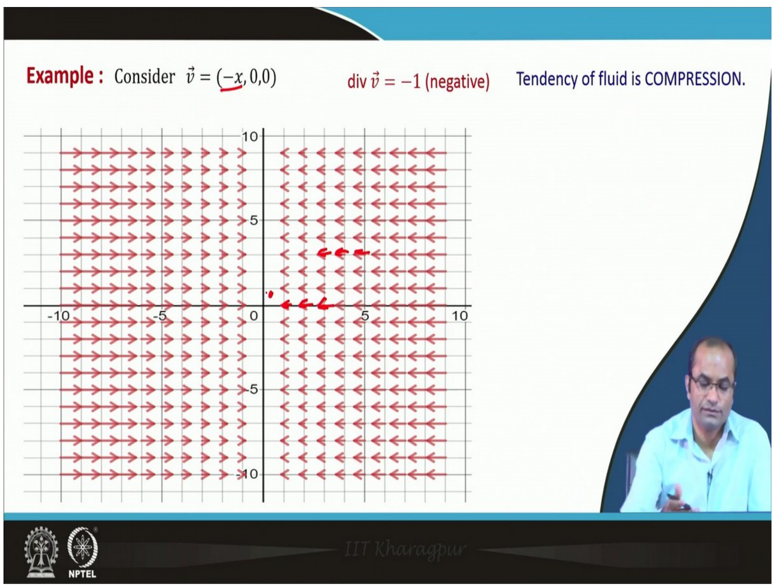
(Refer Slide Time: 13:01)



So, this situation will be much more clear when we consider these examples. So, for instance, if we take the vector field here, v is equal to x and 0, 0. So, in that case if you compute the divergence, so, the divergence as discussed this is just the product with this del operator of this v. So, since the two component are 0, 0 so, we will have only the first component whose partial derivative is 1. So, we have the divergence for this given vector field is 1, which is positive that means, the tendency of this vector field is for the expansion. So, the tendency of the fluid if we consider that this is the velocity field for the fluid than this signifies that the tendency of the fluid at that point is the expansion.

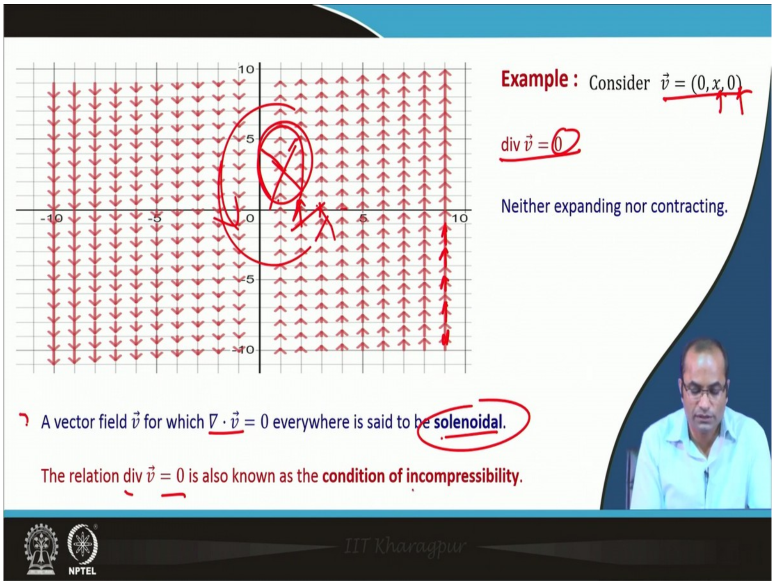
So, if you plot this, if you visualize this vector field whether how to visualize that we have seen already yesterday, then you can see here for instance, this is a very small, then it is increasing, increasing, increasing. So, as we are going away in this direction this vector field, the length or the magnitude is increasing and that exactly tells that the tendency of the fluid is for the expansions, so we have small vector field big and then big and then the bigger one. So, the tendency clearly also visible from this picture that the fluid at any point indeed in this case, because this is one at any point so, the tendency at any point of this domain here is for the expansion of the fluid.

(Refer Slide Time: 14:46)



For instance, if we consider the v is equal to minus x instead of this plus x so, that is the only difference now, so, naturally the divergence will be minus 1 and in this case it is the other way around, that the tendency of the fluid is for the compression and if we look at the visualization of this vector field, then we can realize that something is happening in the other direction now, that towards this line here the y axis, so, we do see that the magnitude is becoming smaller and smaller. So, if we take any point anywhere, and then we see that the around this point, the fluid, the tendency is for the compression, so because this divergence negative itself signifies that or we can visualize that and conclude the similar effect.

(Refer Slide Time: 15:45)



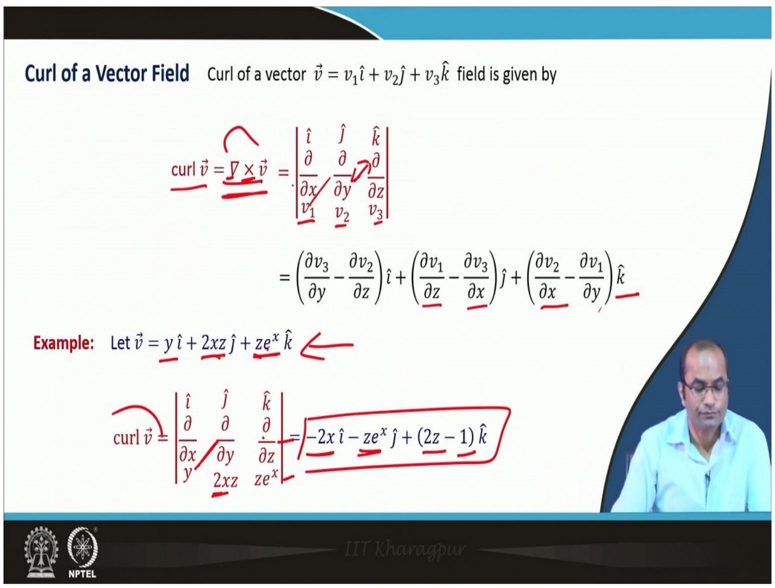
Another example if we take for instance 0, x, 0 and the vector field is given by the 0, x, 0 and the divergence v, in this case will be 0 because del over del x it is 0, del over del y for this x will be also 0 and then component is also 0, so the divergence v is 0 in this case. So now what this tells that the, the vector field is neither expanding and not contracting at any point, actually, because this does not depend on a point so it is a 0 everywhere. So this is what we should also see from the visualization.

So if we visualize this vector field, so this is what happening at for instance, you consider at some x here so, the vector field is of the magnitude is the same. So there is not like, the magnitude is increasing so, any value x here we fix then in this direction, there is a flow but the magnitude is same. So there is no tendency here that the fluid is expanding or contracting. Indeed, so that is what also clear from the visualization as compared to the earlier figure, it is clearly visible here that the divergence is 0.

What we will see later on there is another term which we will use to also define such fluid that is curl, which signifies the rotation of the, or the tendency of the rotation of the fluid and in this case it is clearly visible that because here the magnitude is increasing, here the magnitude is decreasing and so on. So, there is a clear tendency of the fluid that there is a rotation happening, if you place some object in this fluid here, so, because the magnitude, this side is more than this side, so, there will be a rotation of this object so, that is what we will see now next.

So, there is a one more point here that the vector field v for which this divergence is 0, everywhere is called Solenoidal. So, for instance, for this vector field the divergence is 0 everywhere. So, this vector field is called Solenoidal or this relation divergence v is equal to 0 is also known the current condition of incompressibility because there is no compression or contraction happening here. So, this is the condition for incompressibility.

(Refer Slide Time: 18:25)

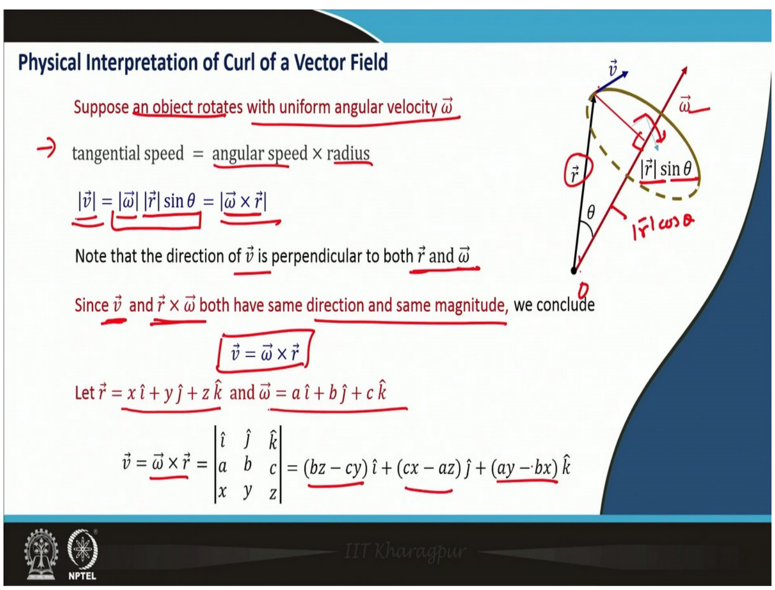


Now, coming to the vector, this curl of a vector field, we will define now that the curl of a vector given by this v1 v2 v3 this is given by the curl v is equal to the del and the cross product now, instead of the dot product, now we are talking about the cross product with this vector field v. And naturally this cross product we can define so, i, j, k and these are the component of the del here del over del x del over del y del over z, and then the component of v so v1 v2 v3.

So, if we simplify this, so, with this component i will have del over del y v3 and then here del v2 over del z similarly, for j we have this del v1 over del z and del v3 over del x for k we have del v2 over del x and then we have del v1 over del y. So, this is the expression for evaluation of the curl and now, we will see in the next slide that what this signifies physically.

Now, let us just take an example to evaluate this. So, if our vector field is given by this y 2xz and this ze power x, in that case we can evaluate this curl v, i, j, k again these are the components of the del operator, these are the components of this v operator, so, we can compute this. So, we have del v3 over del y so v1 v2 v3. So, this is v3 with respect to y that is going to be 0, v2 with respect to z so, v2 with respect to z we will get this minus 2 x there, when we compute for the component j so, we have del over del z of y that is 0 and minus with del over del x of ze power x so, ze power x will come and similarly, we can compute for the case so, del over del x here, so, we have two z and del over del y for this y so, that is 1 there. So, we have this expression for the curl v of this given vector field so, by simple product here, the cross product we can compute curl of a vector field.

(Refer Slide Time: 21:01)



So coming to the physical interpretation of the curl of a vector field we suppose, an object here, it rotates in a clockwise rotation, anti-clockwise rotation here in this situation with the uniform angular velocity, this omega, so we have the angular velocity, given here we have the tangential velocity, and this is the angle theta with the axis of rotation to this circular path here, that is given by the r, vector and this is theta and then we have this radius of that circular path. And this is theta, then this distance here or the radius we call it so this is given by the length of this vector or the magnitude of this vector. And the sin theta so, this is the 90 angles and so this will be r the magnitude of r cos theta so, magnitude of r cos theta and the vertical one this will be the absolute value of r or the magnitude of r with this sin theta.

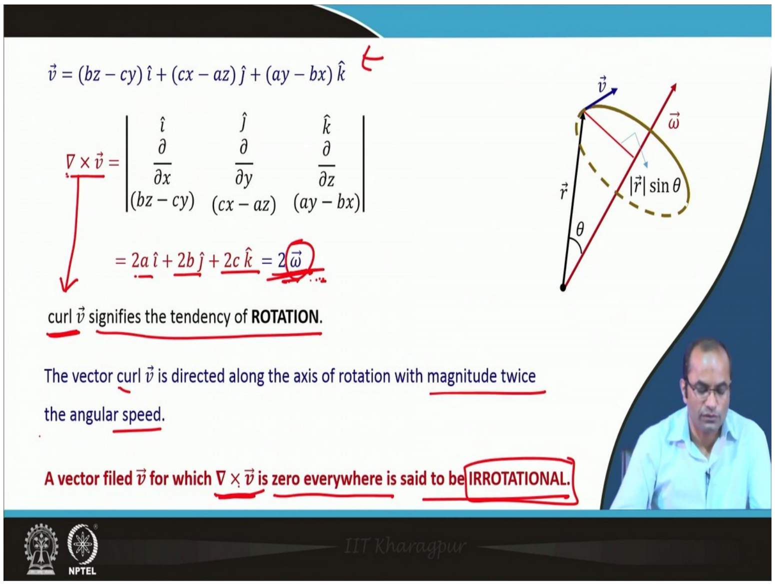
So, this is what we have here, the absolute value or magnitude of r and sin theta this distance. And we know the relation that the tangential velocity that is or the speed rather to say speed this v is angular speed into this radius, which we have just seen that is absolute value of this r, the magnitude r sin theta.

So, we can say that the tangential speed that is a magnitude of v is equal to the angular speed that is the magnitude of this omega and the radius which is given by the magnitude r sin theta. So, this we can write we know that this is precisely the cross product, the magnitude of the cross product of omega and r. And now what we also note down that the direction of this v, so, you can just visualize this motion. So, there is this r vector which from the origin, it connects to this perimeter of the circular path, and then we have the tangential component there.

So, the tangential component which is given by this velocity v, so, these two are perpendicular because this tangent on the circle will be perpendicular to the line which is meeting to this arc here. So, what we observed that this v is perpendicular to r and also perpendicular to the omega. So, we have this axis of rotation and then there is a rotation there. So, this vector v is also perpendicular to this axis of rotation and this is all also perpendicular to the line which is meeting to the origin there.

So, this v is perpendicular to r and omega since, this v the velocity, the linear velocity or the tangential velocity, and this r cross omega, which we have just observed there that though the magnitudes are same and both have the same directions, because the direction of the v is perpendicular to both the r and the omega. So, that means, the v and this r omega they are the same actually because they have the same direction and they have the same magnitude. So, we can conclude that this tangential velocity not indeed the speed tangential velocity is equal to this omega and the cross border product with this r so, if we assume that r is xi yj zk and omega is ai bj and ck, in that case we can compute this v by this cross product of omega and this r which is given by this determinant and which we can simplify here. So, we get this bz minus cy cx minus az and ay minus bx as the third component.

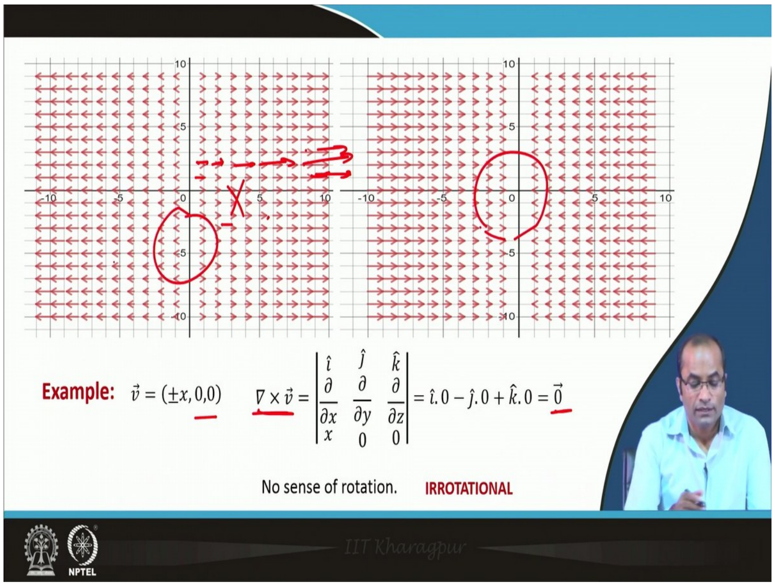
(Refer Slide Time: 25:30)



So, we have this vector field v which is bz minus cy and cx minus az second component, the third component ay minus bx. So, if we now consider this curl of v that means, the cross product of del and v so, v is given by this expression there. So, we again compute this determinant and we will see that we get 2ai and 2bj and 2ck out of this determinant. So, what does that mean? That this is two time's the omega so, here we have the curl v is equal to two times this omega, which is the angular velocity.

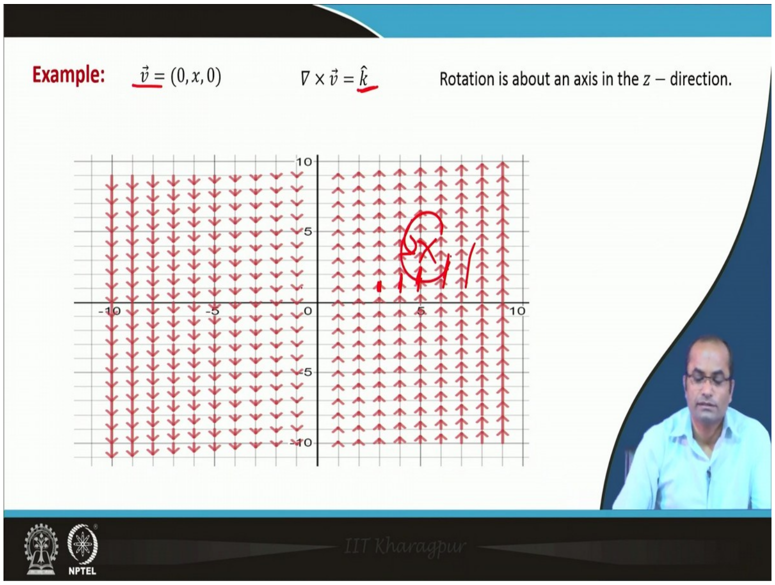
So, what its signifies the curl v which is given here, the curl v is two times omega, it signifies a tendency of rotation, because this is nothing but the two times the rotational velocity, the angular velocity of that particle or that body there. So, the curl v is directed along so, it is the same the curl v, the direction of this omega, the angular velocity that is the axis of this rotation and the magnitude is twice the angular speed. So, that is the physical interpretation which is clearly visible with this mathematical formulation that the curl v is equal two the omega, the angular velocity. So, a vector field v for which the curl is 0 everywhere is said to be irrotational so, that is the term we use normally, that this is a irrotational if this is 0, if this is not 0, then the fluid has tendency of rotation, whether clockwise, anti clockwise that depends on the situation there.

(Refer Slide Time: 27:25)



Let us just consider this example, that v is given by either plus or minus this x, 0, 0. So, if we compute, if we compute this cross product or the curl of v, then we realize that this is 0 in this case, irrespective of the plus or minus sign. So, that means there is no sense of rotation, the vector field is irrotational. And if we plot these two vector field with plus one and then the minus one, then we do see that indeed there is no rotation, there is no tendency of rotation. So, if this is the flow just the magnitude is increasing in this direction and it is the same for the next as well. So, there is no tendency, if you place some object there, it will not rotate. And similarly, here also it will not rotate and we have seen that there was a tendency of the fluid to compress in this case or for the expansion in this case, but there is no rotation.

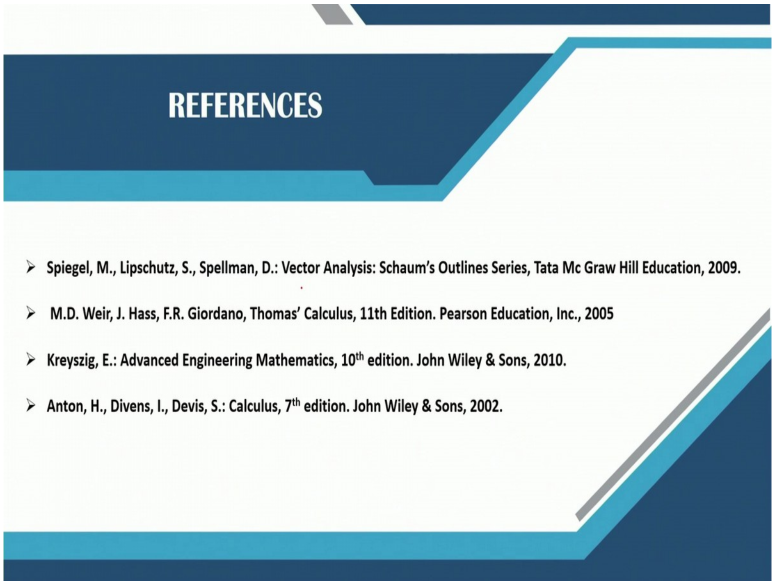
(Refer Slide Time: 28:37)

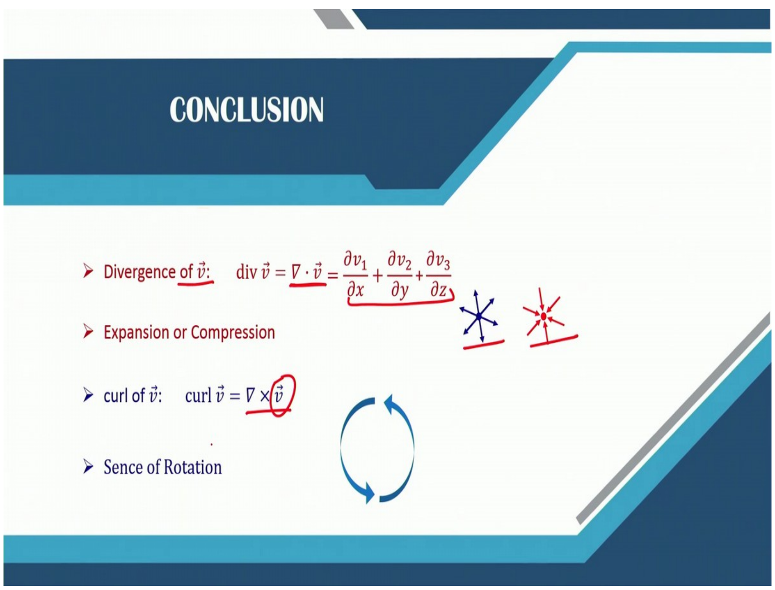


So, we take the next example where 0, x, 0 is the vector field and in this case if we compute the curl it is coming to be k. So, the rotation is about the axis in the z direction so, that is just the k.

And if you visualize this vector field, then we can realize that, so, for instance this and then increasing, increasing, increasing so, in this direction increasing so, if you place this object this will move, this will rotate because here you have the larger magnitude of the vector field as compared to the left. So, this object will start rotating and so, there is a tendency of rotation and again this thumb rule will be applicable. So, if you have anti clockwise rotation, then this is the axis of rotation in the direction of this thumb. So, here it is exactly the k, the axis of the rotation is in the z direction.

(Refer Slide Time: 29:32)





So, here we have the references, these are used for preparing this lecture. And to conclude that we have learned that the divergence of a vector field v is given by this dot product or finally, this is just the sum of these partial derivatives. And we have also seen that the physical interpretation says that this tells the tendency of the vector field to expand or to compress. So, which is visualized here again the second thing we have studied the curl of v, which was defined by this cross product of this del and the vector field v. And we have also realized that the value of this curl tell something about the rotation, so, the tendency of the rotation of the vector field. So, that is all and thank you very much for your attention.