Engineering Mathematics II
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Lecture - 28
Polynomial Interpolation (Contd.)

Welcome back to lectures on Engineering Mathematics 2, this is lecture number 28 and we will continue with the Polynomial Interpolation.

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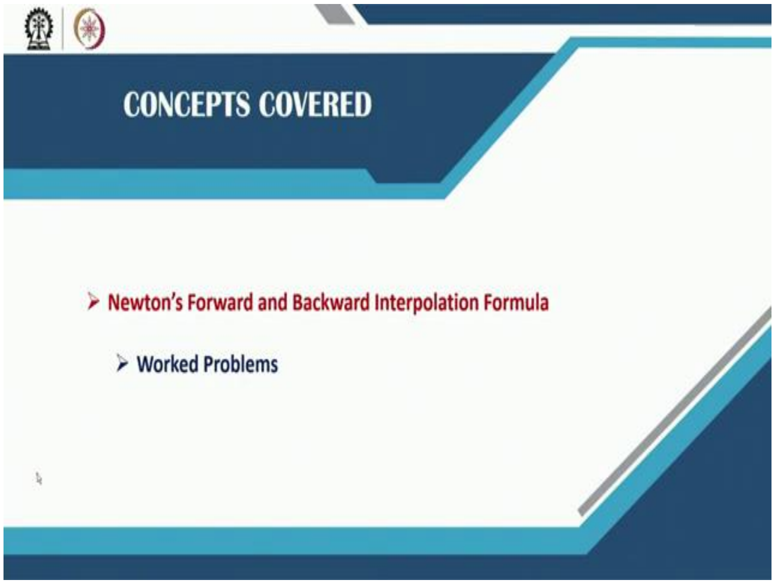

So, in this lecture, we will be basically doing some numerical examples based on the Newton's forward and backward interpolation formula, which were derived already in the previous lecture.

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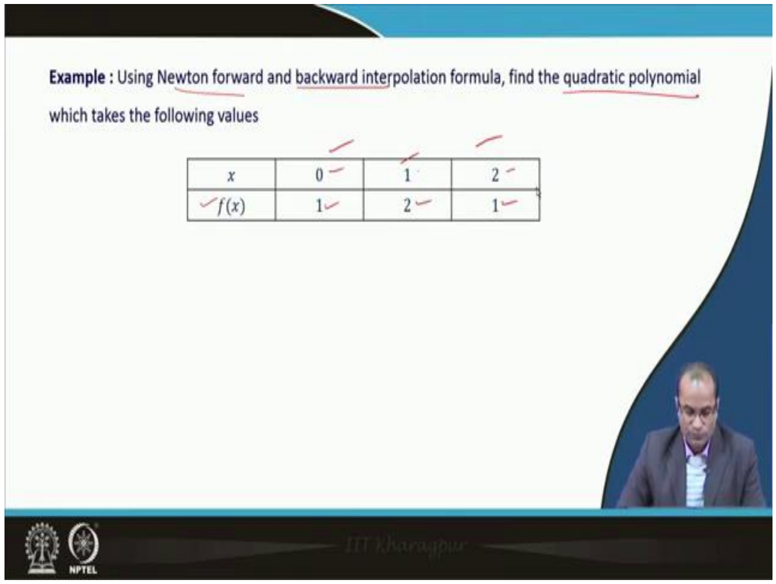

So, let us start with this example a very simple calculation. So, using Newton's forward and as well as the backward interpolation formula, just to recall that whether we use Newton's forward or backward or any other format for the computation of the polynomial, the polynomial will be unique.

So, here we want to find the quadratic polynomial, because there are 3 points given and we can fit polynomial of order 2 or degree 2. So, here the X values are 0, 1, and 2, so they are equidistant, we can use Newton's forward and backward or backward interpolation formula. The function values at these points are given 1, 2, and 1. So, we want to derive the quadratic polynomial, which passes through all these points 0, 1, 1, 2, and 2, 1.

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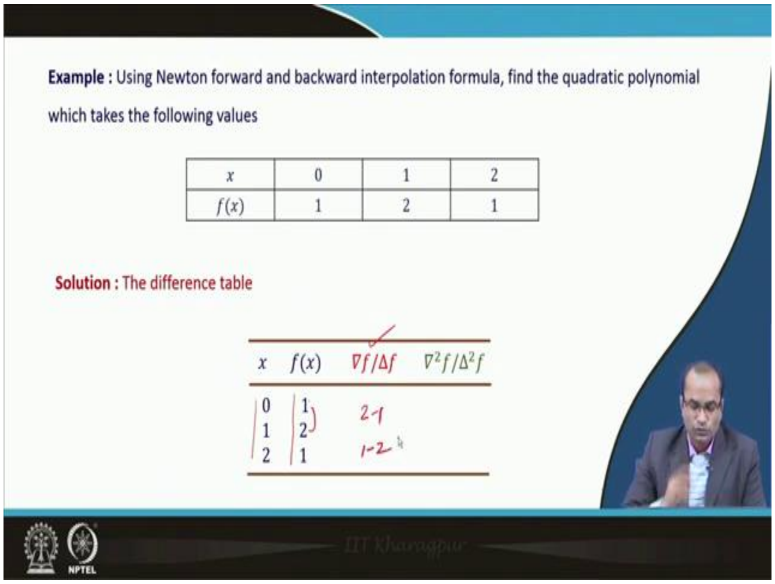

So, we have to construct the difference table, whether we use forward or backward difference formula. To construct this difference table, what we have learnt. So, we will put here, the X value 0, 1, and 2 and corresponding the function value that is 1, and 1, 2, and 1 and then the first order differences will be computed just by subtracting the next value from, the previous value from the next value. So, here 2 minus 1 will come and then 1 minus 2 will appear there.

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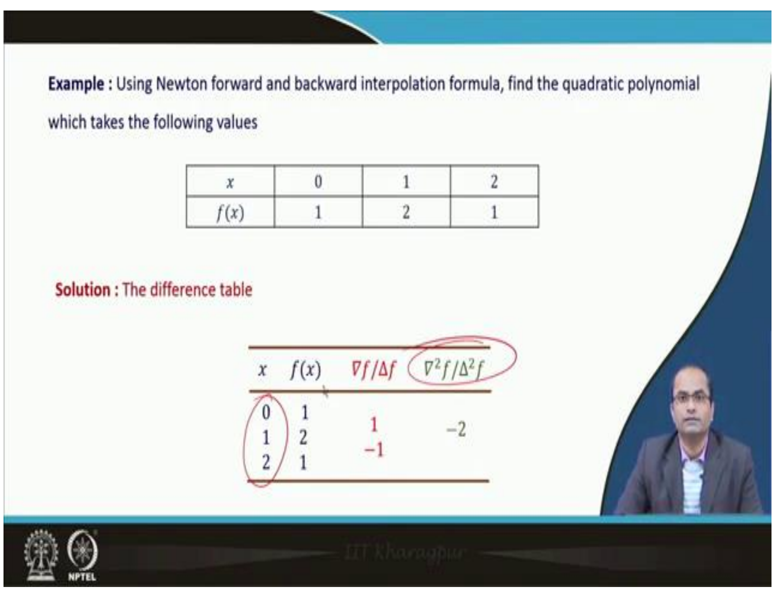

So, basically here, we have then 1 and there it is minus 1 and the higher order differences again we will repeat the process. So, here like minus 1 and then minus 1, so that is minus 2, so we have minus 2 for the second order differences and that is what we can get. So, if there are 3 points we can go up to the second order differences and naturally we will get a second order polynomial.

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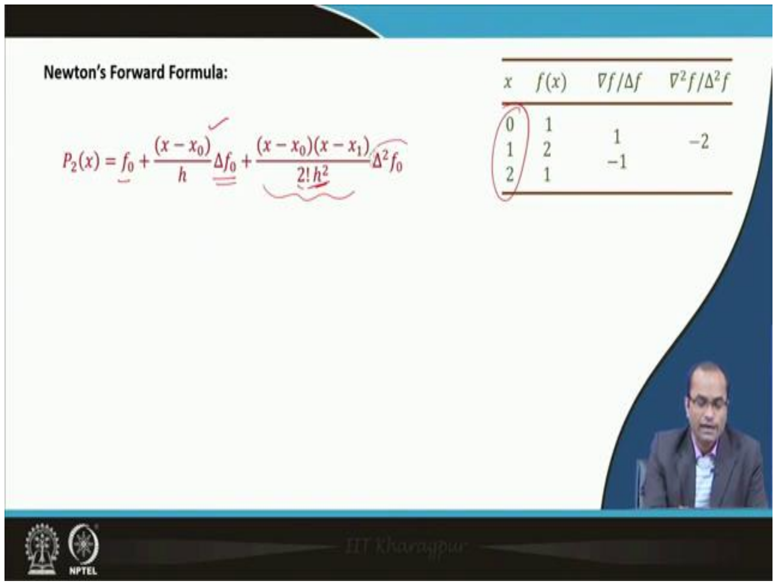

So, having done this difference table, we can apply now the forward difference formula, for instance. So, here we have F naught X minus X naught over h, and the first order difference here, delta F naught, then we have this product X minus X naught X minus X 1 divided by factorial 2 and this h square, h is 1 in our case, h is this distance here, which is equidistant and that distance is 1 and then we have this second order difference there.

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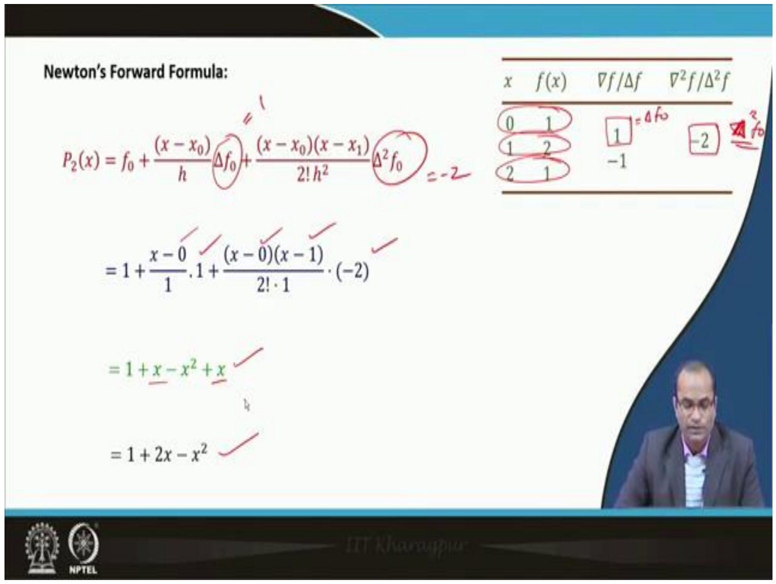

So, in the forward formula I remember so this is what we are going to use in the above values, the values in the table the first values here. So, this is exactly delta F 0 and this is delta sorry the forward one. So, delta square and F 0. So, these two values will be used here, this is minus 2 and here this place this is 1. So, substituting these values 1 and minus 2 and then X naught and X 1 that is 0 and 1.

So, here also 0 we can just simplify this formulation and we will get 1 plus X minus X square plus X and this X and this X we can add. So, finally we have this polynomial 1 plus 2X minus X square that passes through these given points 0, 1, 1, 2, and 2, 1, one can check again and naturally it will satisfy all these points.

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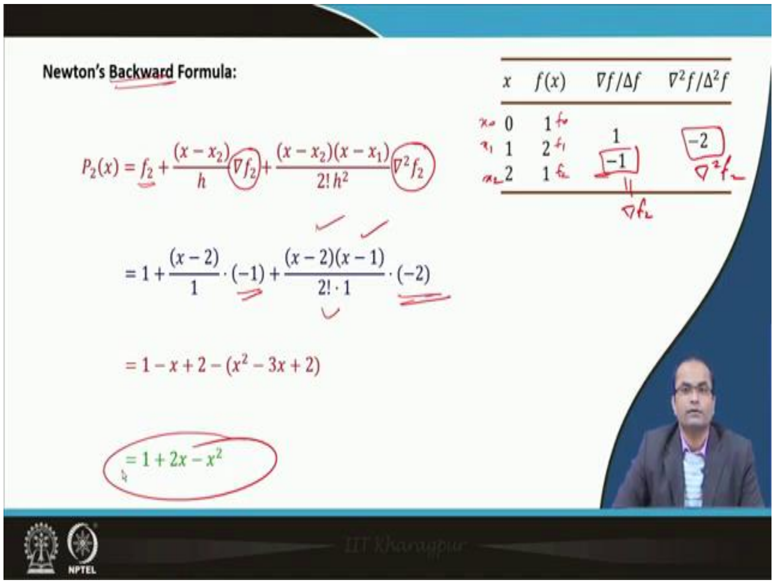

Now, coming to the Newton's backward formula, so the formula looks like we have F 2 now instead of F 0, it will start from F 2 the backward direction. So, that is here F 0 this is F 1 and this is F 2. So, here this is X 0, this is X 1, and this is X 2. So, now these backward difference operator here of order 1 and order 2 and just to recall that now these times, these are the values.

So, this first one is this backward F 2 and this is the second order backward of F 2. So, these values we can substitute now in this formula and we can get the polynomial. So, here X minus this 2, this times and this is 1 here minus 1 and then we have X minus 2, X minus 1, and we have factorial 2 h square, h is again 1.

So, and the second order it is the same as minus 2 and then we can simplify and we will get the same polynomial, same interpolating polynomial, that is 1 plus 2 X and minus X square. So, these Newton's forward and backward formulas, they are applicable when we have equidistant points and this is the simple example where we have seen how to construct the Newton's backward and forward interpolating polynomial.

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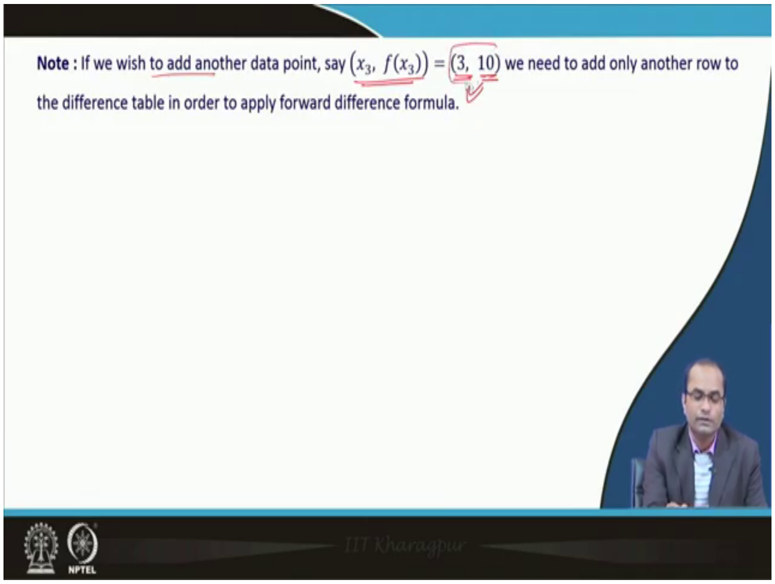

There is a point here, which we should discuss that if suppose now after constructing the second order polynomial, we realize that we have one more data point say X 3, F 3, which is the next point 3 and 10. So, we had given the value 0, 1, and 2 and now this is very often appears for instance in the experiments, that now we have after some time we have next data available that is 3 and 10.

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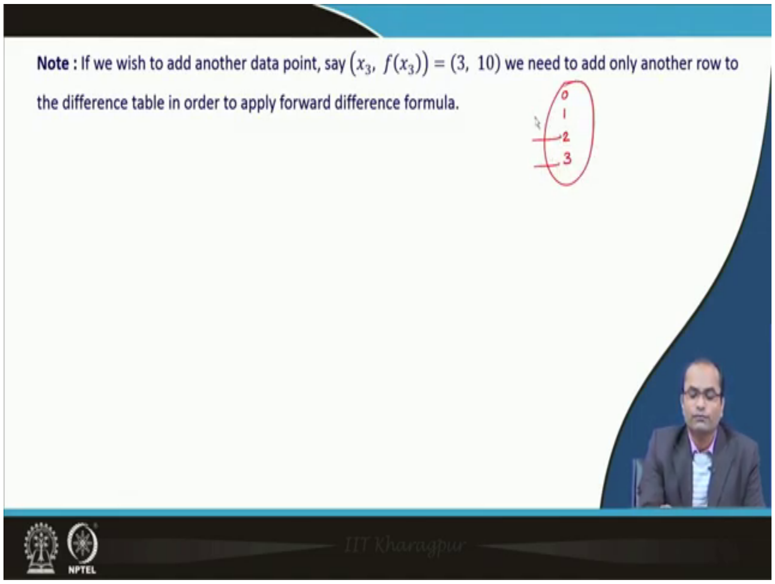

So, we have more data points and now we want to fit a higher degree polynomial, that is the third degree polynomial, we have four points now we have values at 0, we have at 1, we have at 2 and we have a also value at 3, so there are 4 points. So, we can fit a polynomial of degree 3 as well.

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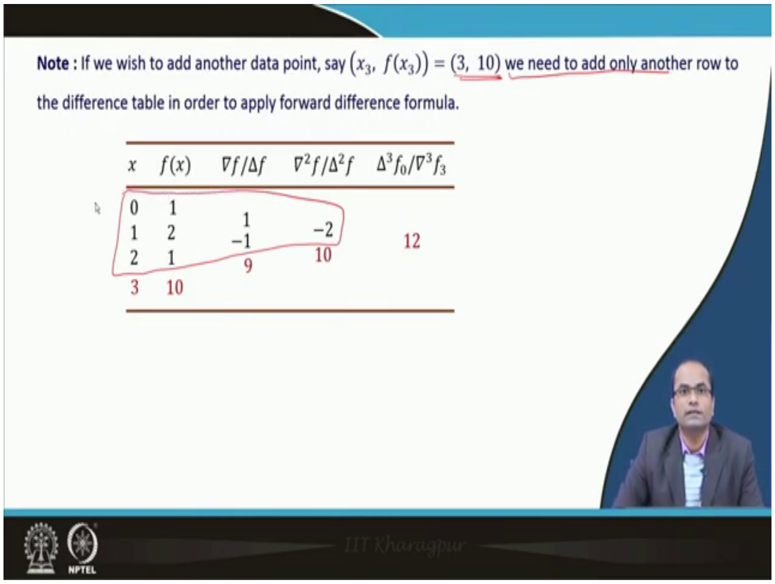

And the beauty of this Newton's forward and the backward formulation is that we do not have to do all the calculations, we do not have to repeat the whole table again, only we will add the last row in each column. So, this was already the table the difference table, which we have used for constructing the second order polynomial.

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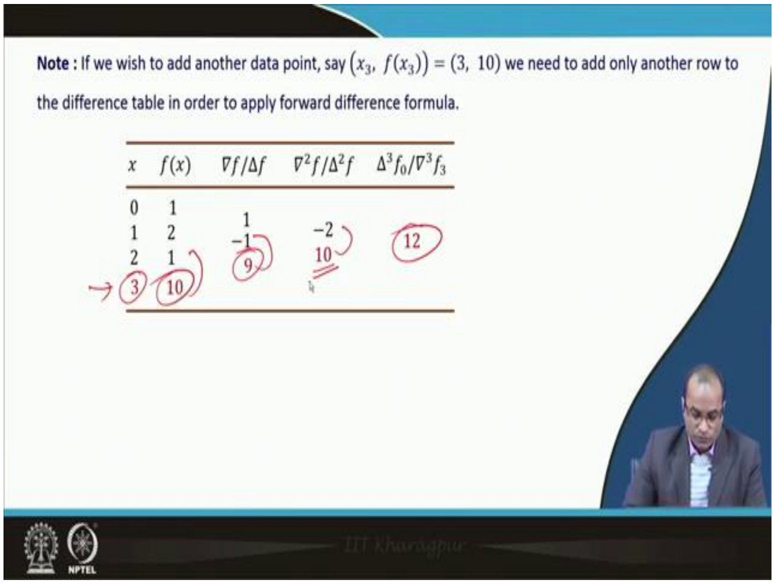

And now we have this additional data that 3, at 3 the value of F is 10. So, we will simply add this in this table and accordingly now from here one more difference will come that is 10 minus 1, 9 and then here also 9 minus 1, minus 1 that is 10 and then here also 12.

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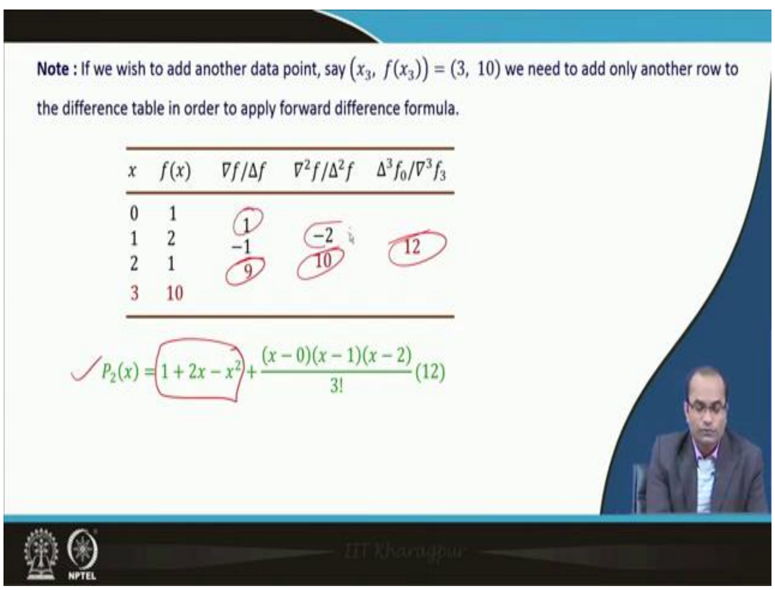

So, these are the new values now which have come because of this additional data but we do not have to repeat the whole calculation to construct the table, we have only done some calculations, which were extra now because of this extra data and we can apply whether forward or backward. So, for instance here we are using the forward again and just remember that this portion, which was already evaluated because that will not be changed those values were used there.

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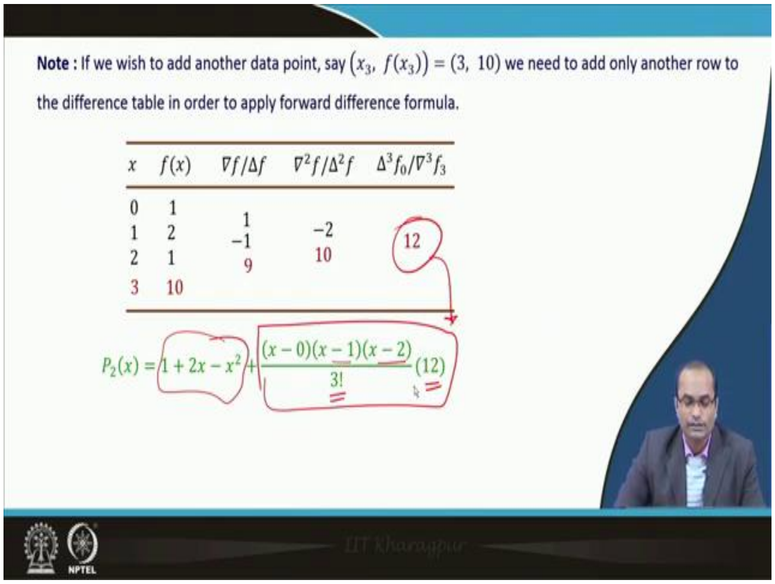

And we have just used here that polynomial which we already got that was the second order polynomial only one extra term will come because of this additional data that is X minus X 0, X minus X 1, X minus X 2 over factorial 3 and this third order difference will come here as 12.

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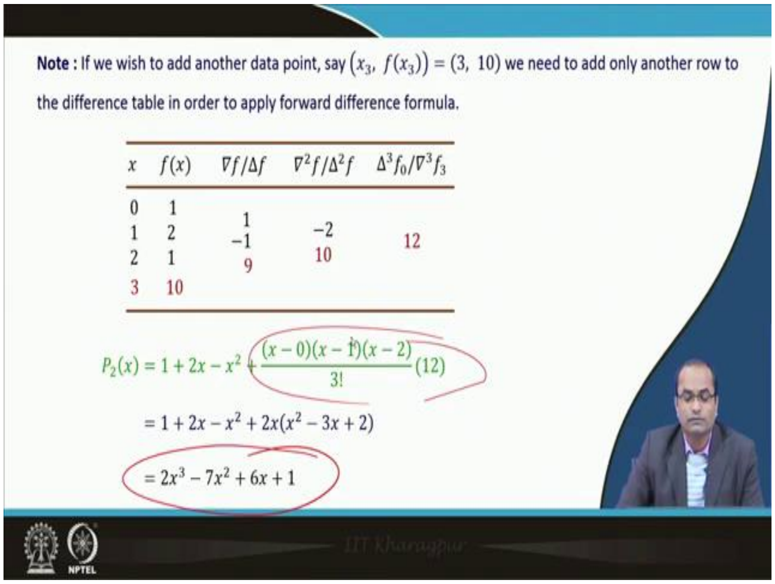

So, now we can simplify this and we will get a third degree polynomial, we will realize later on when we will be talking about other formats other than the Newton's forward and backward formula, that this is not always possible when you have the additional data you have to repeat all the calculations.

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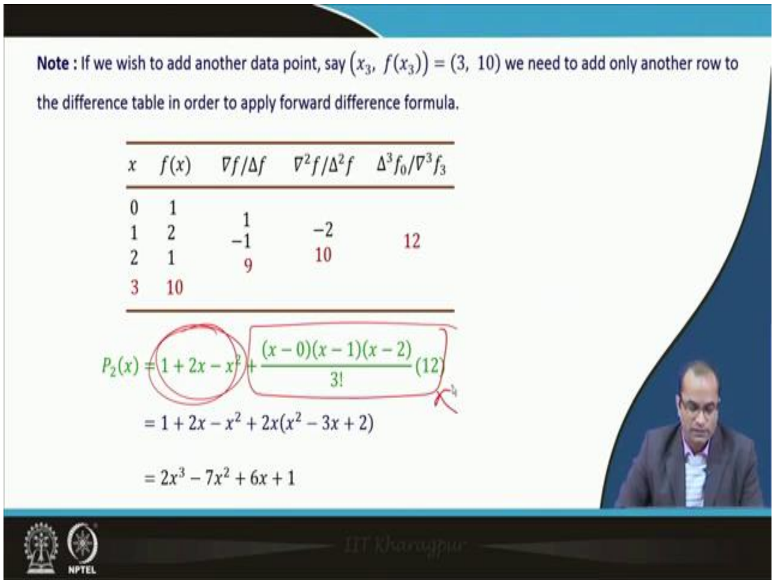

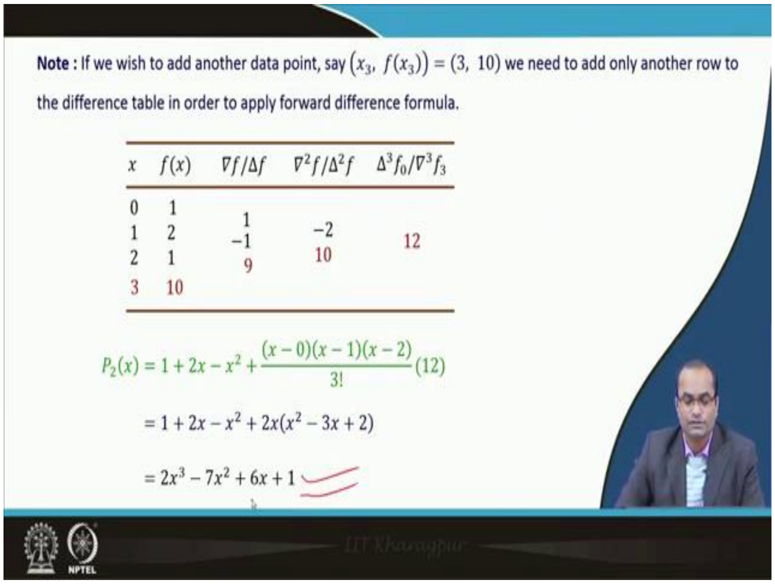

But here in this forward and backward formula we have to just add one extra term indeed we do not have to do this calculation, which we have already received as a second order polynomial, only this third degree polynomial term will be added there and then accordingly we have to simplify this and we got the new polynomial, which is of degree 3.

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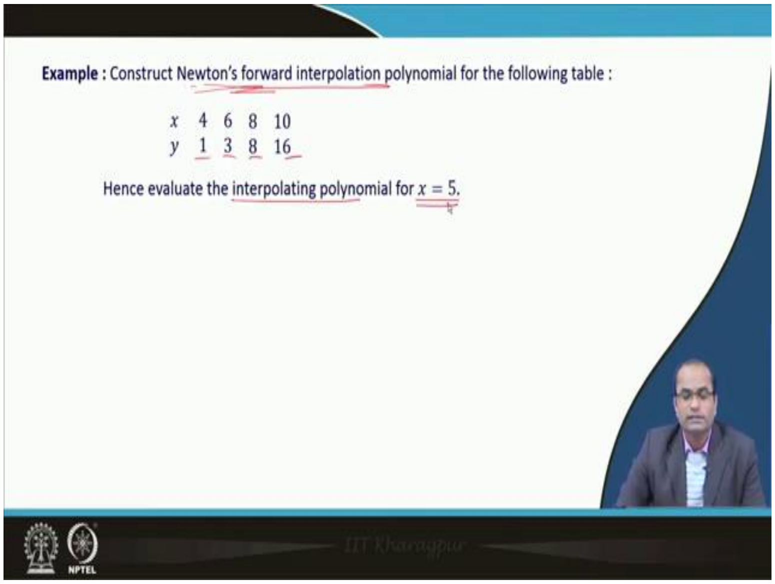

Next example, we have the data given and we want to construct Newton's forward interpolation polynomial. I mean using Newton's forward formula the interpolating polynomial will be the same, whether we use forward or backward. So, here the data are given 4, 6, 8, 2, there is a difference of 2. So, h will be 2 in this case corresponding values are given 1, 3, 8 and 16. So, we want to evaluate interpolating polynomial for X is equal to 5.

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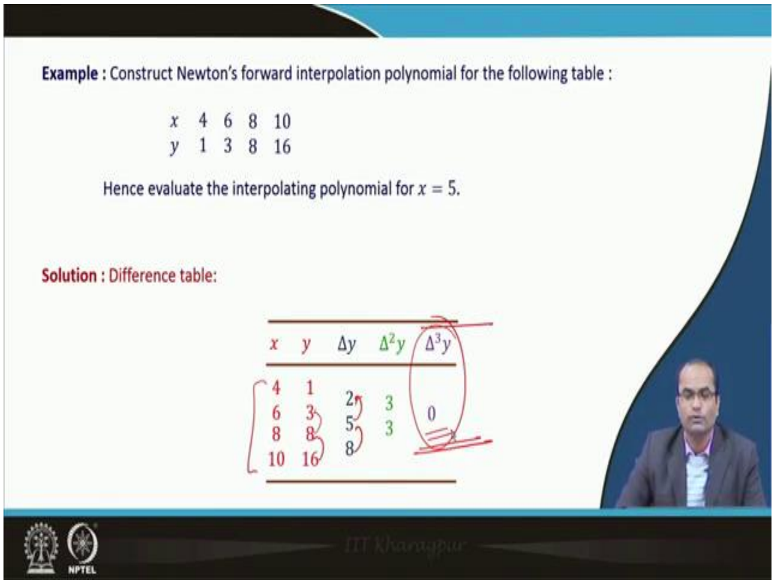

And to construct the difference table. So, we have 4, 6, 8, 12 and then corresponding values are 1, 3, 8 and 16. So, we will construct this table, so here 3 minus 1, there will be 2, then 8 minus 3 will be 5, and 16 minus 8 will be 8, so we have 2, 5, and 8 and again for the second order derivative will be do this, we will be doing the same thing.

So, 5 minus 2 is 3 and 8 minus 5 is also 3 and then finally we have 3 minus 3, 0. So, there is no actually third order term and it is, it will reflect now in the formula also because our calculation will be simplified, there will be no third order term is though there are 4 data points here, so we are expecting a polynomial of degree 4, but because of this here the third order differences become 0 and therefore we will not get the third degree term in our polynomial, the polynomial will be a second degree polynomial.

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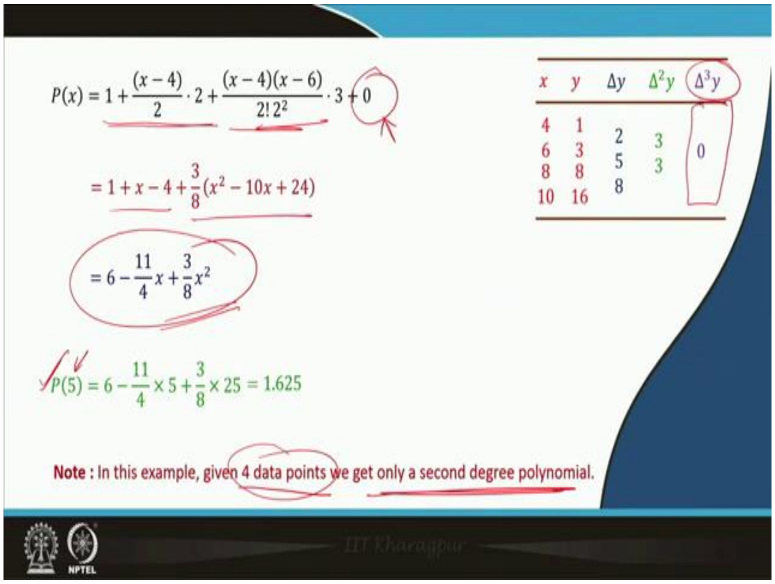

Because this term here where it uses the third order difference that becomes 0 and we have only these terms. So, we have 1 plus this X minus 4 because of this and then this can be simplified to this one and finally we can combine everything we have this second degree polynomial and now we can compute value at whatever point. So, for instance X is equal to 5, if we put we got 1.625.

So, what is interesting here, that given 4 data points we get only a second the polynomial, which is which is nothing surprising because this was a result that given n plus 1 data points, there will be a unique polynomial of degree n or less than equal to n, which passes through all these points. So, here we have seen that though there were 4 data points and but we got only second degree polynomial, which was also clear from the table because the second or third order differences were 0 in this case.

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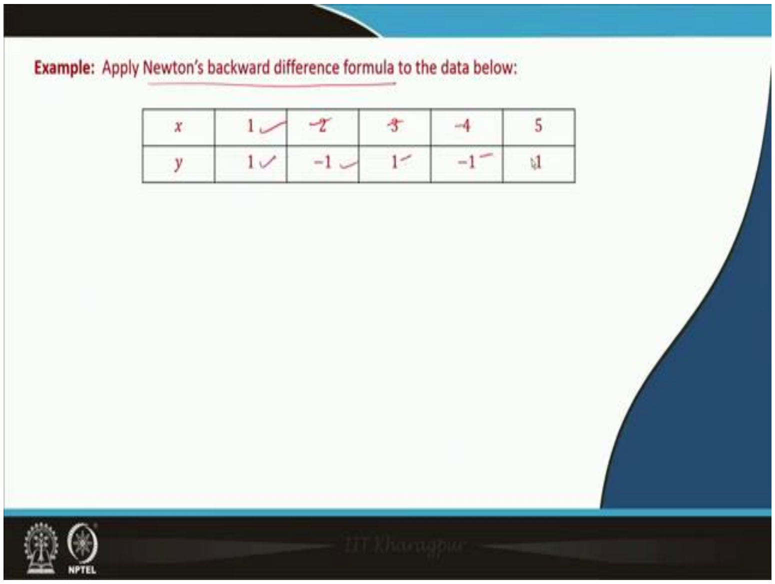

In this example, we will be talking about again Newton's backward difference formula for this table, where X values are 1, 2, 3, 4, and 5. So, there is a difference of 1 and then the corresponding value of Y are 1, minus 1, 1, minus 1, and 1.

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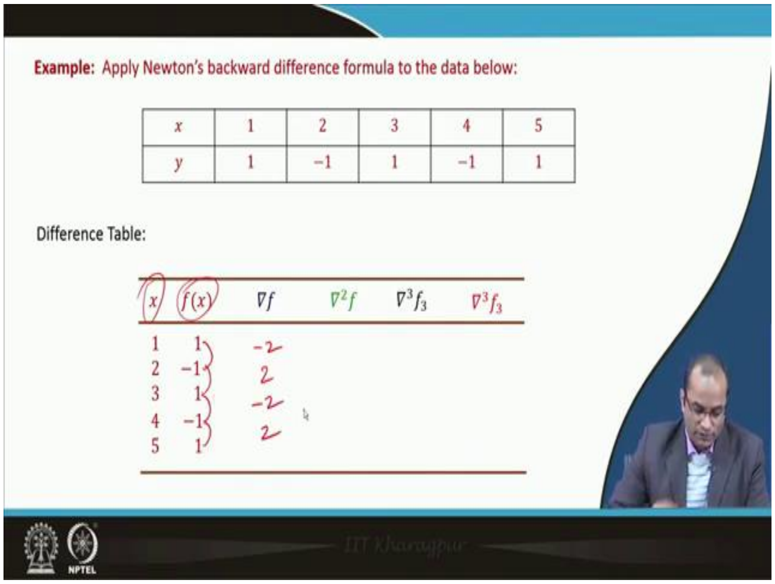

So, having these data points, we can construct the difference table, where we will list all these X values in this first column, then the corresponding value of function in the second column and then the differences will come. So, minus 1 and minus 1 so will be minus 2 then here 1 minus 1 will be 2, then here minus 1 minus 2 and then 1 minus 2.

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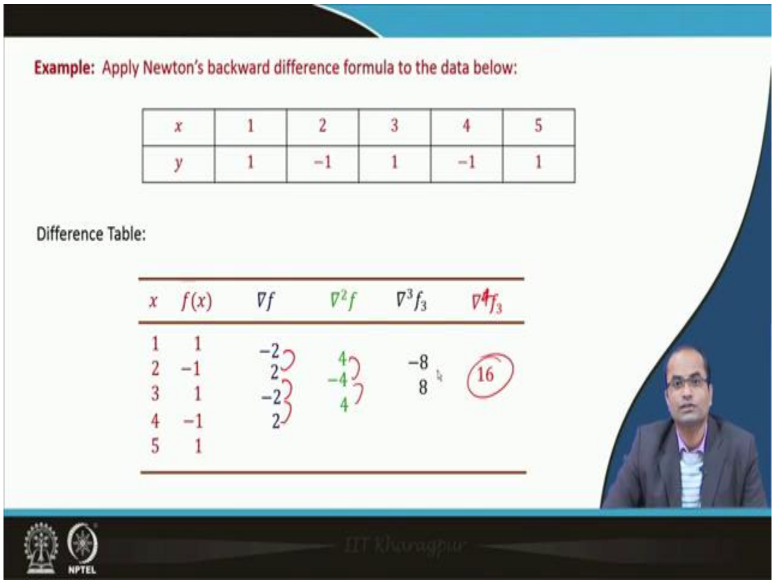

So, all these minus 2, 2 and minus 2, 2 will come and again similarly, we can have this differences there and we will get the second order differences and similarly, we will get the third order differences finally third order and then we have this fourth order differences that is minus 16, plus 16.

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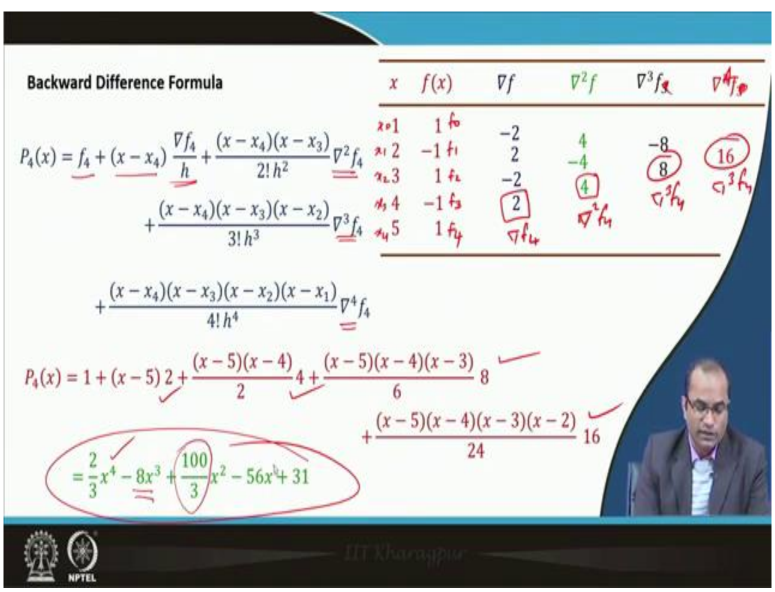

So, having this difference table ready we can now talk about the, or you can we can use this backward difference formula, for the calculations. So, the backward difference formula we will start for this F 4, we have F 4 X minus X 4 and this forth delta F 4 then we have delta 2 F 4 delta 3 and delta 4 F 4. So, what we are getting now here, we can substitute all these X 0.

So, this is X 0 here, this is X 1, this is X 2, this is X 3, and this is X 4, correspondingly here, we have F 0, F 1, F 2, and F 3 and F 4 and these are the other differences and this in the backward table, we know that this is actually delta F 4 and here we have this delta 2 F 4, then delta 3 F 4, this value and then delta 3 F 4.

So, we can use all these the 2, 4, 8, 16, so here 2, 4, 8 and 16 all these values are used and then X 0, X 1, X 2, X 3, and X 4 are substituted here in this formula and after the simplification, we can get this fourth order polynomial 2 by 3 X 4 minus 8X 3, 100 by 3, X square minus 56 X and this plus 31.

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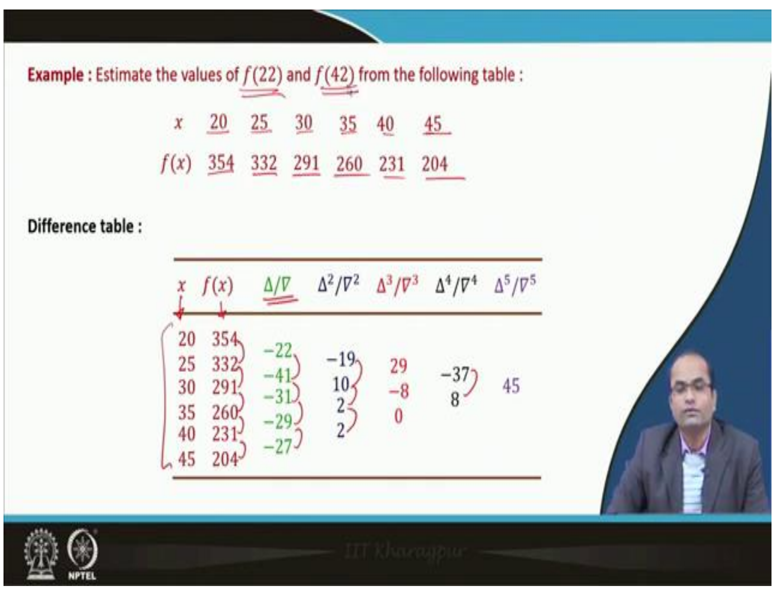

Estimate the values of this F 22 and value F 42 from the following table. So, here we have the data from 20 to 45, 20, 25, 30, 35, 40, and 45. So, there is a difference of this 5 and corresponding values are given here for the function. And again we have to construct the difference table as usual. So, the first column will contain the values of X here, the second one its corresponding values of the function.

So, then we can compute the first order differences which just by subtracting these we can get here these values which appears to be all negative and then again here of minus 41 plus this 22 then minus 31 plus 41 and so on, we can again get this second order polynomials for second order differences, then accordingly we will get here the third order differences and fourth order and finally the fifth order which is 45. So, here we want to estimate the F 22 and F 42, so we can use the forward or we can use the backward difference formula.

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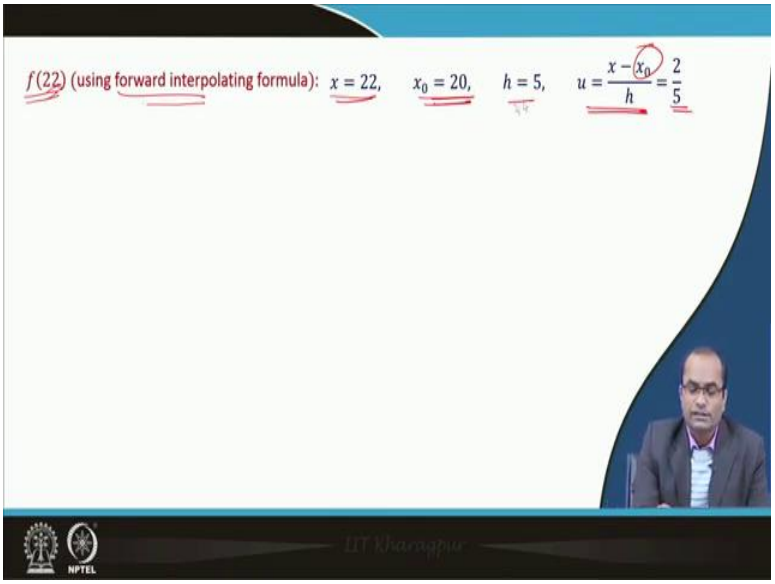

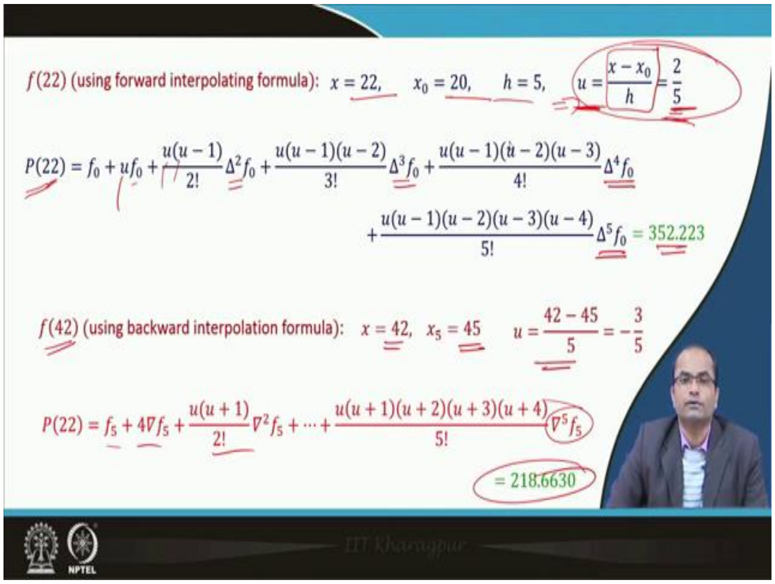

So, here for computing this 22, which is at the beginning because our data starts from 20. So, we will use forward difference formula because it is at the beginning and we it is better to use the forward because let us do this change of variable which this formula also we studied in previous lecture. So, X here is 22 and X naught so for the forward difference this X naught will come here for the backward the X and the last point will come here.

So, since this is close to the first point we will take the forward one because the difference will be just 2 and we will get a small number for u. So, here h is 5 and if we substitute here X minus X 0 over h we will get 2 over 5.

And then we have this formula where this polynomial in terms of u is written and to the value here at 22 we want to get this done. So, the u is 2 by 5 we will substitute here for u 2 by 5 and we can use from the difference table all these differences, which we have computed and after calculation, we will get this number which is exactly at the value of this interpolating polynomial at 22.

So, sometimes when the values are large we can work with this shift here because u becomes smaller and the calculations might become simpler and similarly when we want to compute this 42, 42 is close to the 45, which is the last point there we can use the backward difference formula where again we have to compute u and use this backward difference formula and again after substituting all these differences and this u here minus 3 by 5 we will get 218.6630.

So, this is also where, though we do not have to use this shifting of the variable we can use directly but these numbers will be a little big one to compute and therefore we have used here this shift of this variable.

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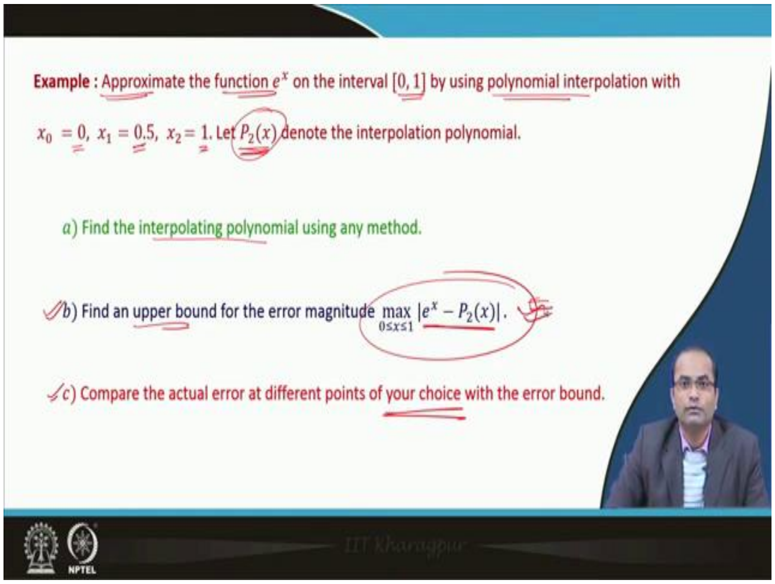

Here we want to compute, we want to approximate this function the exponential function on this interval 0 to 1, it is given by the polynomial interpolation. So, this function, the exponential function we want to approximate by a polynomial interpolation with these points, the X naught, 0.5 and 1. So, we can get this exponential function at 0, 0.5 and 1. So, only these 3 values we will use and naturally if we have 3 values, we can construct a polynomial of degree 2 or less.

So, in this case we will get a polynomial of degree 2, so if we denote this P 2 by this interpolating polynomial then we want to compute now what is this interpolating polynomial first the usual what we have already done several examples based on this. This is new here we want to find the upper bound for the error e power X minus P 2 X without a computing the direct error we want to have the bound for this error and we in the first lecture on this interpolation, we have already discussed that what is the interpolating error, what is the error between the actual function and the interpolating polynomial.

So, that formula we can use to estimate this error here and then we will also compare the actual error at different points, we can choose different, different points where we can compare that error whether this error bound really gives some meaningful bound for the error.

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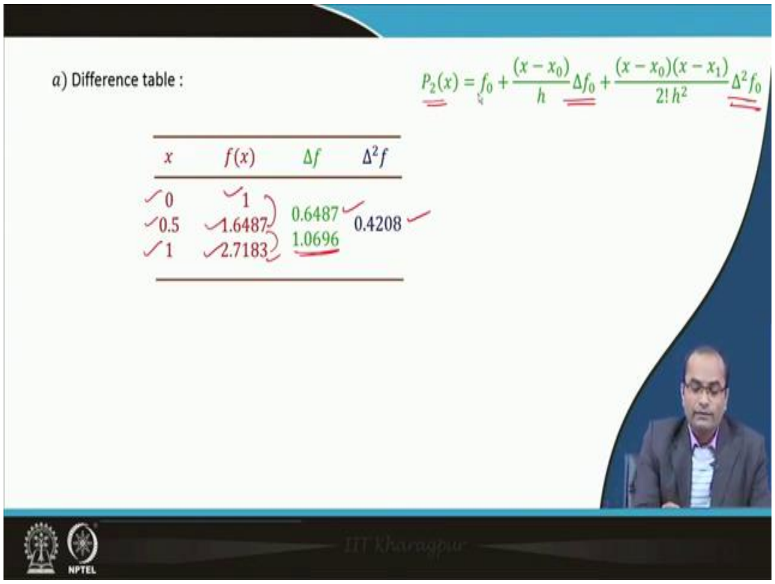

So, again we have to construct the difference table, we have the second order polynomial, that forward difference formula we can use and we have to compute this table. So, X values were 0, 0.5 and 1, we have computed here the exponential 0, 1, exponential power 0.5, exponential power 1. So, the value of E and then the delta F. So, first order differences we will compute by subtracting this, so it will be 0.6487 and again we will do the shift here or the difference here 1.0696 and we will get here 0.4208, as the second order difference.

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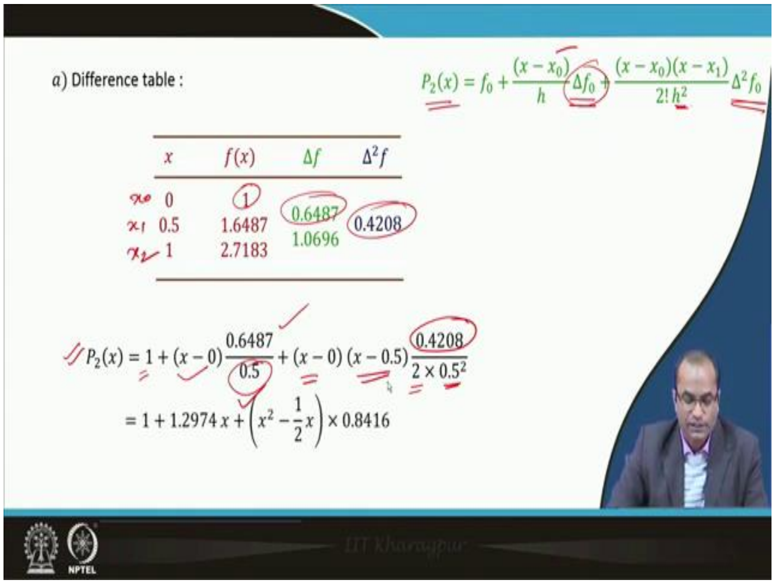

And then in this formula we can use this F 0, so F 0 is already here 1, then we have delta F 0, which is here and then we have delta 2, F 0, which is given here and this X naught is this one, here we have X 1, we have X 2. So, all these values are known now we can substitute there this formula and we can get the polynomial.

So, here we have the second order polynomial as 1 plus this X minus X 0 is 0 and h is 0.5 in this case. So, this 0.5 is taken and then here the difference for this delta F 0, which is use here then we have X minus X 0, X minus 0.5 and this second order difference and then factorial 2 and we have this h square.

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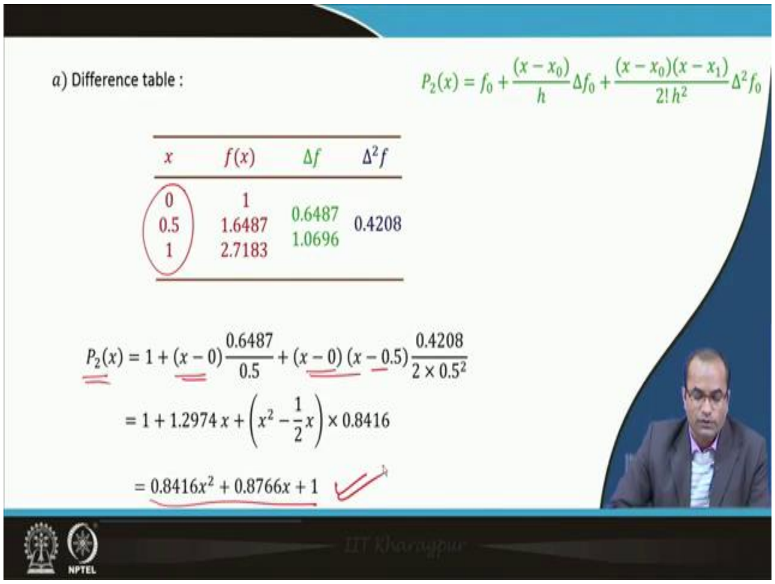

So, having these all values we can now substitute here and simplify this which will turn up to be a second order polynomial. So, this is the second order polynomial, which approximate the exponential function, where we have used these 3 values of from the exponential function and we got this polynomial of degree 2.

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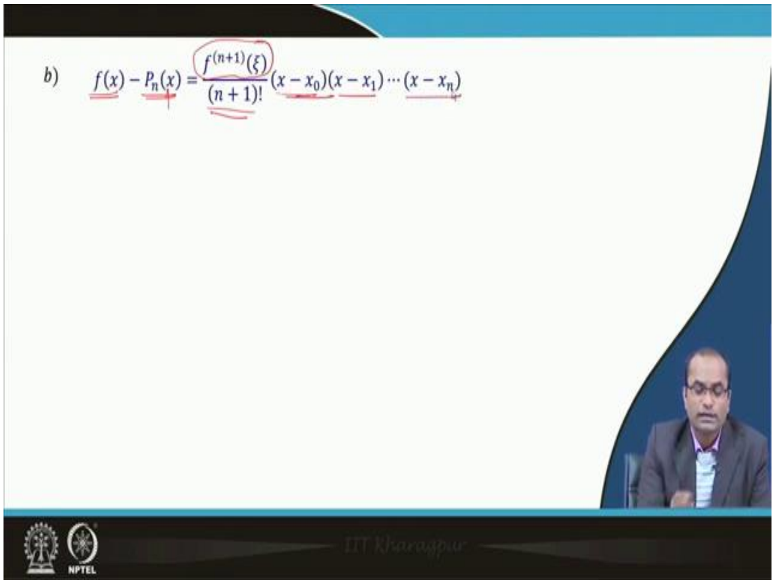

Now, coming to the error part so just to recall there was a formula which we have derived that if FX is the actual function and the corresponding polynomial is Pn X taking n plus 1 values of this FX and this was the formula, which uses this n plus 1th derivative at some points Xi in this range, which contains X 0, X 1, X n and this X point. So, and then this n plus 1, we have this product X minus X 0 X minus X 1 up to X minus X n. So, this formula will help us to get the upper bound of the error, which occurs in this polynomial.

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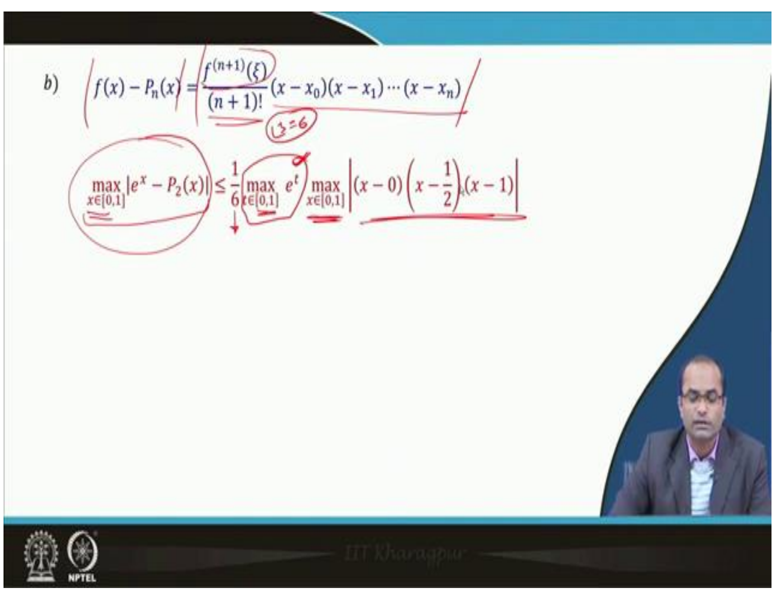

And to get this error the maximum error here, so what we do we will take the absolute value both the sides here and then the maximum because X can vary from 0 to 1 our points are 0 to 1, we are talking about the what is the maximum error between 0 and 1.

So, X this maximum value we are interested in for X and here also now we will get the n plus 1th derivative. So, we have the exponential function, so that exponential will remain as it is the derivative and the because of this upper bound to get this upper bound we will take the maximum value here over this exponential function, when this T varies from 0 to 1 this 6 here is because n is 2 now, so 3 the factorial 3, factorial 3 will give 6, so which is 6 here and again here also we will maximize this. So, the maximum again X over 0 to 1 and then we have this X minus 0, X minus half, and X minus 1.

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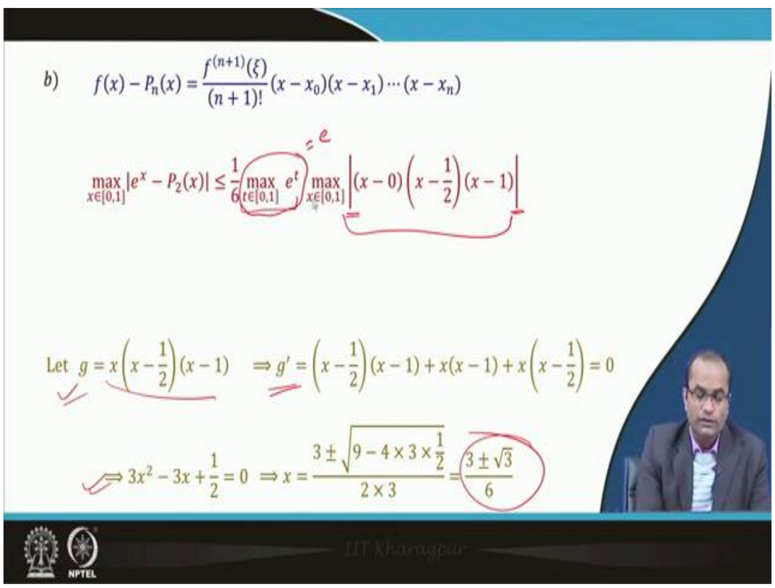

So, we need to have this maximum first of the exponential function it is an increasing function and at 1 when T is 1 then will be the maximum. So, here the maximum of this will be nothing but e and here this maximum of this function of order 3, we have to compute then.

So, let us assume that G is this one and then we get this critical points by taking its derivative and finally after solving this we will get that these 3 plus minus square root 3 by 6 are the points where it attains the maximum and minimum but here we have the absolute value. So, actually at these two points it achieves the maximum value over this range 0 to 1.

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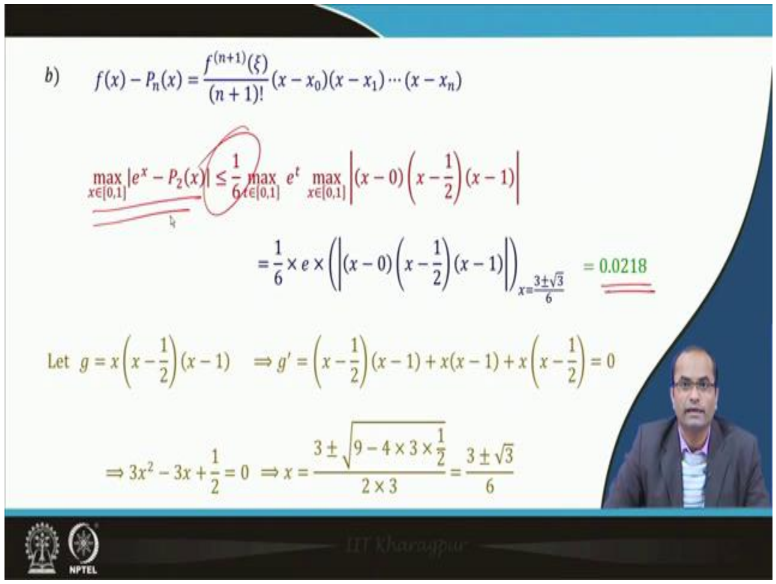

So, having this we will get the value of this at X is equal to 3 plus minus 3 by 6 and this comes out to be when we multiply by this e and minus and this 1 over 6 we will get 0.0218. So, that is the, that is the upper bound we are getting for this error when X varies from 0 to 1 and now we can verify whether the actual error actually is below this error bound because this is the upper bound, this is the upper bound. So, the error should be less than 0.0218 at any point in this range 0 to 1.

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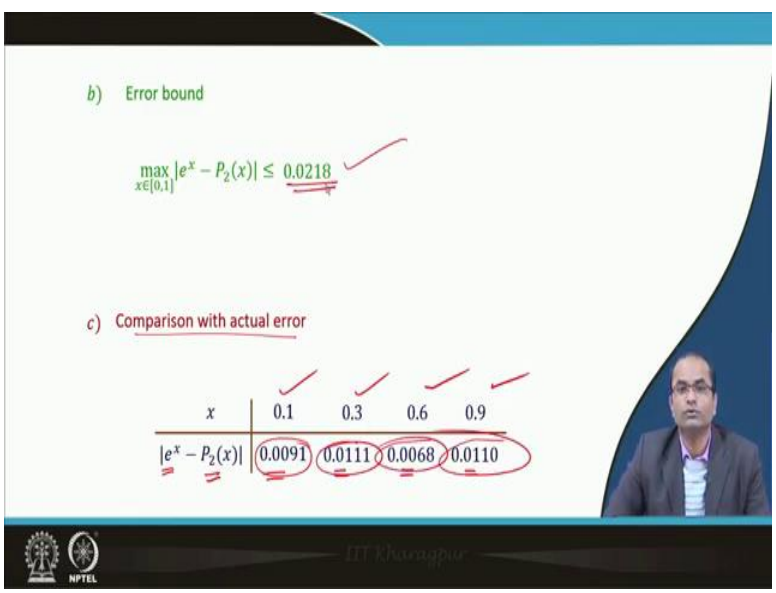

So, we have this error bound and now if you compare with the actual error, so we have taken for instance 0.1, 0.3, 0.6, 0.9, some points we have taken from 0 to 1 and this difference of the actual value and the polynomial value is computed here, and what we can see that these actual errors are always less than this 0.0218.

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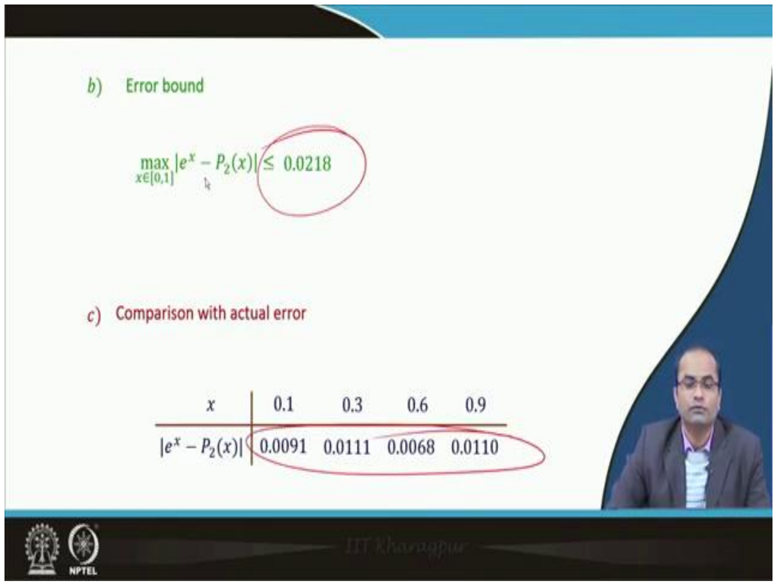

Well, so in this particular example we have also seen that this error bound without calculating the actual error, we can just estimate that what could be the, what would be the upper bound of the error.

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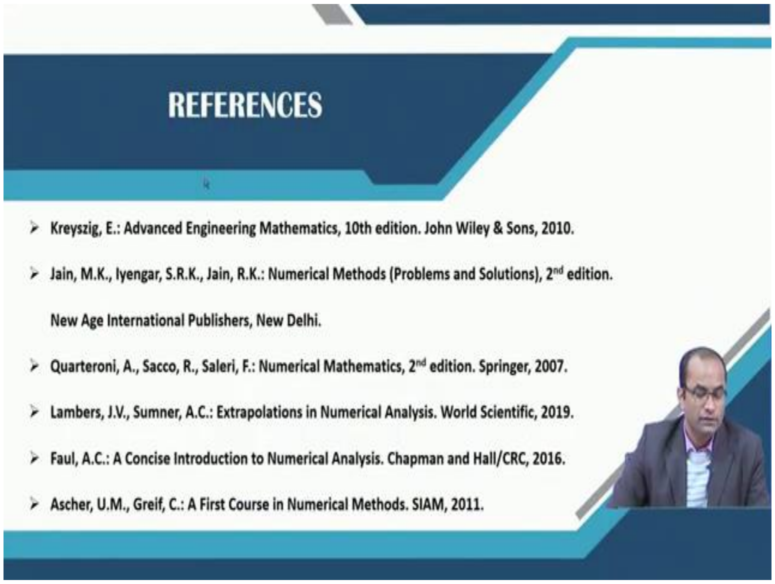

So, these are the references, we have used for preparing this lecture.

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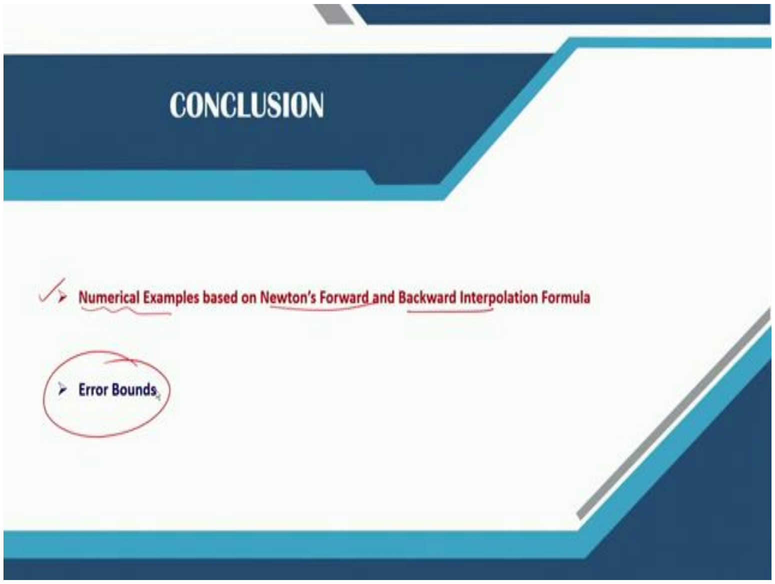

And the what we have done, we have we have demonstrated through several examples, how to construct the interpolating polynomial using Newton's forward and Newton's backward interpolation formula and in this last example we have also make use of the error bounds which was, which were derived earlier in the first lecture on this interpolation. So, that is all for this lecture and I thank you very much for your attention.