Engineering Mathematics II
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Lecture - 27
Polynomial Interpolation (Contd.)

So, welcome back to lectures on Engineering Mathematics 2 and this is lecture number 27, on Polynomial Interpolation, this is lecture, second lecture on the same topic.

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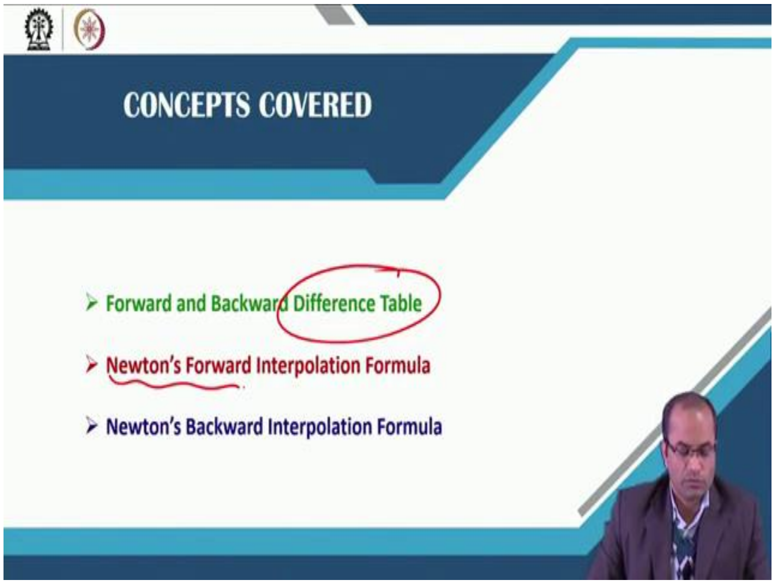

So, in this lecture we will be talking about Forward and Backward Difference Table and these difference tables will be used to derive this Newton's forward interpolation formula as well as the Newton's backward interpolation formula.

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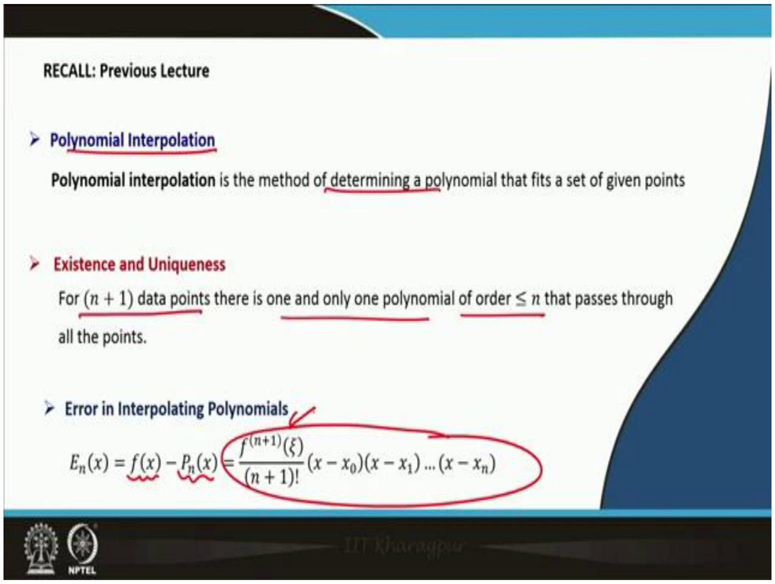

So, just to recall in the previous lecture, what we have done we have discussed what is the polynomial interpolation. So, it is the method of determining the polynomial that fits given data set, and we also talked about the existence and uniqueness. So, basically there is only one polynomial, which passes through a given data point.

So, if there are n plus 1 data points, then there will be only one polynomial of degree less than equal to n that passes through all the points, and we have also derived a formula for obtaining error. So, the error between the function and the polynomial approximation, so that will be computed by this formula which uses the derivative at n plus. So, n plus 1th derivative of the function.

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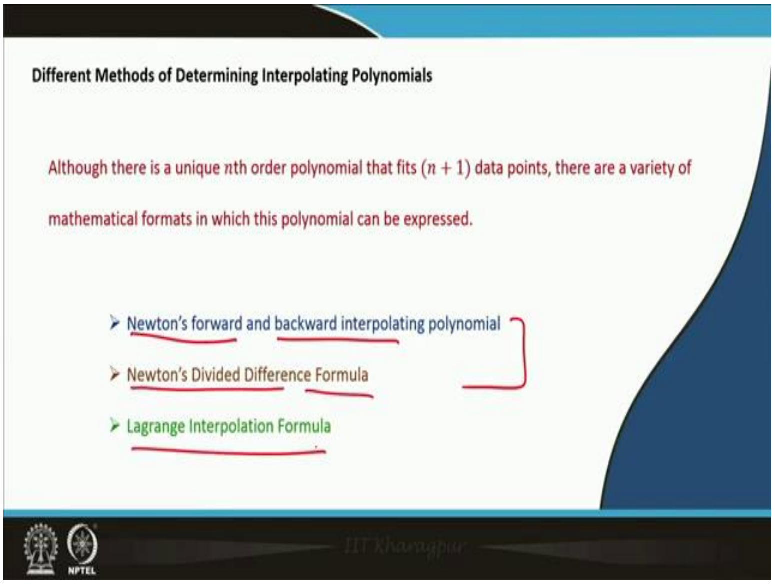

So, having this now we will be talking about different methods for determining interpolation or interpolating polynomials. So, although there is a unique nth order polynomial that fits given n plus 1 data points, but there are a variety of mathematical formats in which this polynomial can be expressed.

So, for instance which we will be talking about in this lecture we have the Newton's forward and backward interpolation polynomial and we have also the Newton's divided difference formula that just slight variation from the previous one, and will be also talking about the Lagrange interpolation formula, next lectures.

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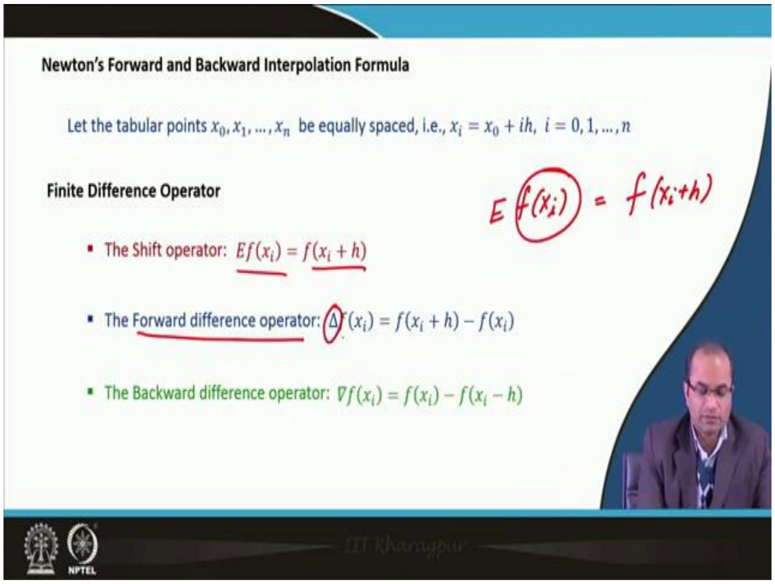

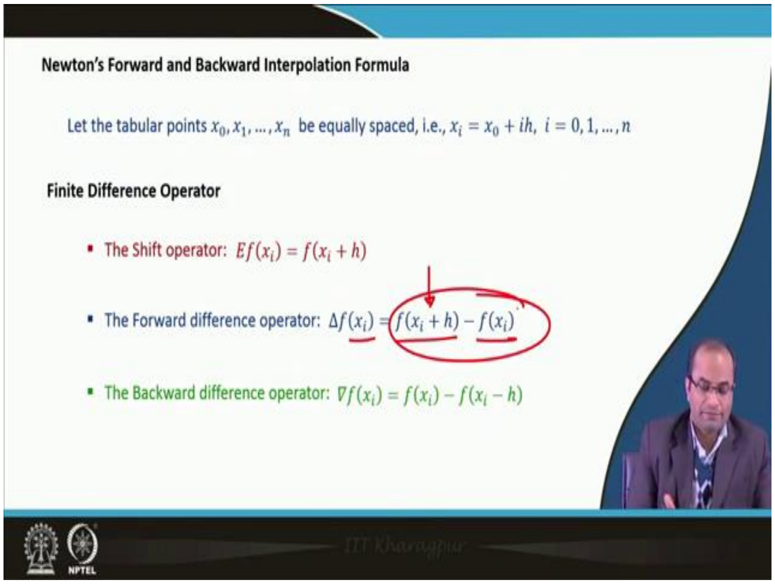

So, coming now to Newton's forward and backward formula we will take these points X0, X1, X2, Xn, and we assume that they are equally spaced, that means X1 is X0 plus h and this X2 will be X1 plus h. So, there will be a difference of just h.

So, we need to talk about these finite difference operator because some operator will be used in these formulas. So, therefore, it is important here, to use this finite difference operator because that is easier to define these formulas, or to derive these formulas, using these shift operators. So, the first shift operator, we are talking about this E here. So, E of fxi will be E of fxi will be f, xi, plus h.

So, when we operate this E on fxi, so if we are operating this E on a point here on a on a data point here fxi, then this is nothing but the value of this function at xi plus h, so at the next data point. So, it will shift the, a functional value, function value here, there is another one forward difference operator, which uses this symbol here delta. So, delta fxi will be just f, xi plus h minus fxi.

So, it is a difference operator this is called the forward difference operator, we are going for in the forward direction, then minus the value of f at the previous point. So, this is called the forward difference operator.

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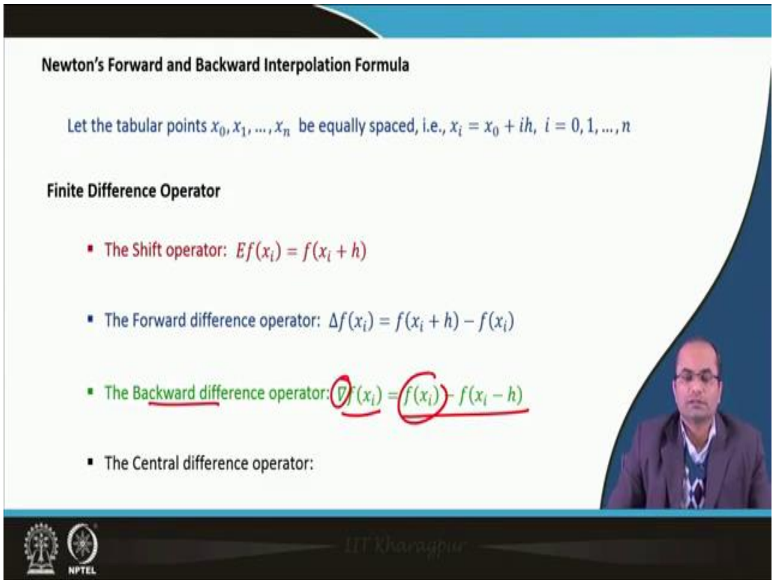

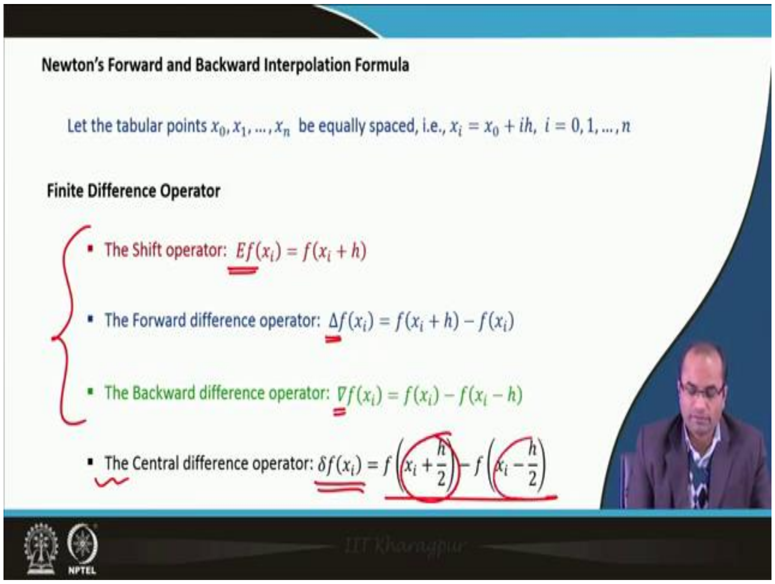

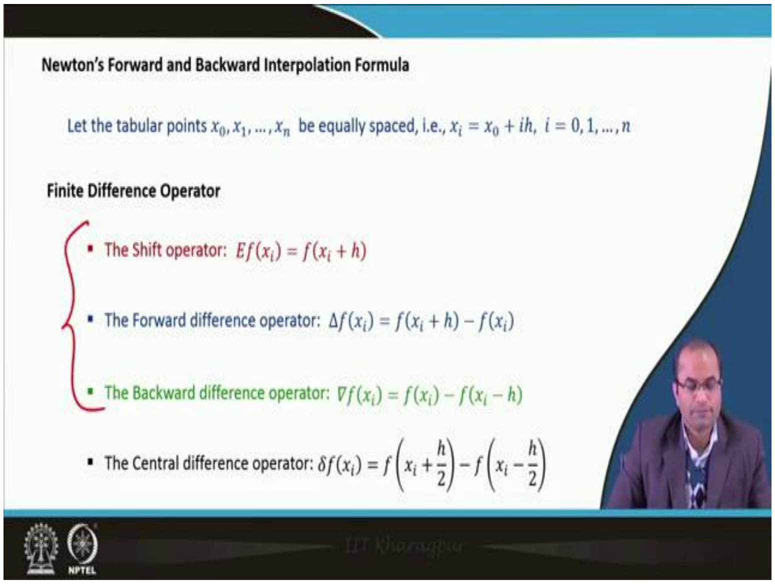

There is a backward difference operator as well which is denoted by this nabla fxi and here it is other way round that fxi minus fxi minus h. So, the value at previous data points, so these are basically the three operators, which we will be using here in this lecture. So, the shift operator E, the forward difference operator delta and another difference operator which is backward difference operator is this nabla.

Then also there is a central difference operator just for completeness, let me explain this also its symbol its delta fxi and this is again a difference operator and these two differences are used f xi plus h by 2. So, it takes the value at xi plus h by 2 and then subtract the value from the xi minus h by 2. So, this is the central difference operator, so having these operators mainly the first three operators shift forward, and the backward we will continue with some relations between them.

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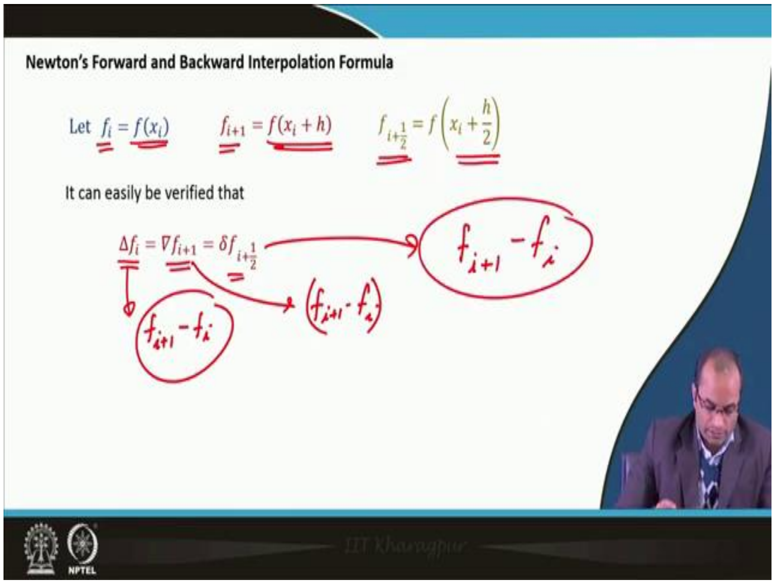

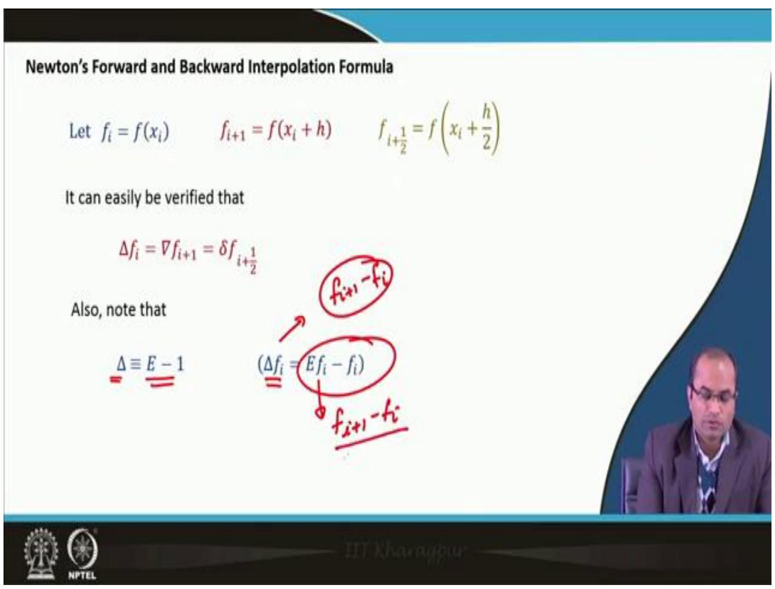

So, let us assume for simplicity that f at xi is denoted by fi, f xi plus h is denoted by fi plus 1 and f xi plus h by 2 is denoted by fi plus half, just for simplicity of writing. So, here it can easily be verified now that this delta fi is backward delta fi plus 1 and also this is equal to this delta when operated on f I plus half, one can easily see all these relation, which is not difficult for instance here.

This delta fi will give as per the definition i fi plus 1 minus fi, when we are here in the backward, so it will give fi plus 1 minus fi, which is same and when we are at this central difference operator, so we will give here fi plus the half, will be added, so we have i plus 1 and then again fi. So, all these three are same.

So, therefore, we are telling here, that the forward operator on fi is equal to backward operator on fi plus 1 and this is equal to central difference operator on fi plus half. So, these 3 are equal here. Also we can easily prove again that this forward delta operator is equal to E minus 1, which again one can easily see when we apply this delta on fi. What we will get? We will get fi plus 1 minus 1 minus fi sorry, and then here also when we apply this fi on E minus 1. So, we have E fi, which is fi plus 1 and then minus fi again. So, again these are same.

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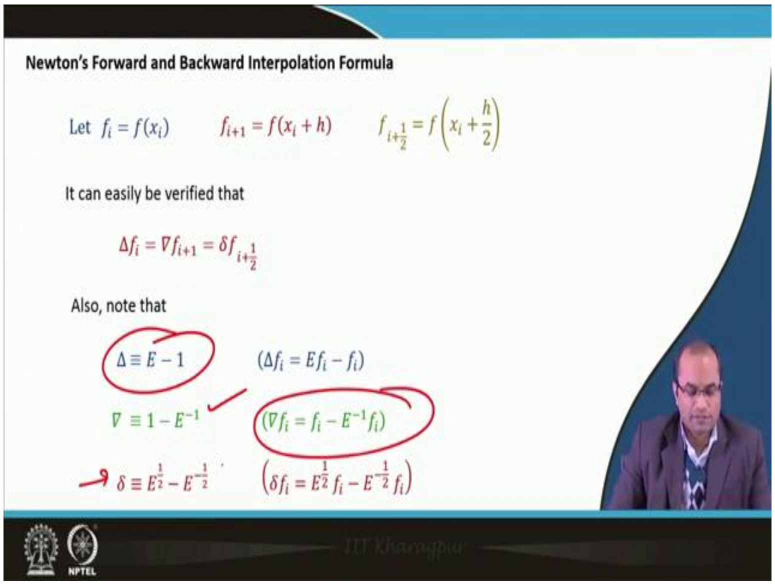

So, we can say that this delta is nothing but this E minus 1, similarly the other relations, the backward delta here, it is 1 minus E minus 1 and similarly we can prove this by just operating two operators on fi and the result will be the same. So, also we have the relation that delta is nothing but the E power half and minus E power minus half. So, again one can see, when we operate these two operators on fi and we will receive the same value at the end. So, having these relations we can now move further.

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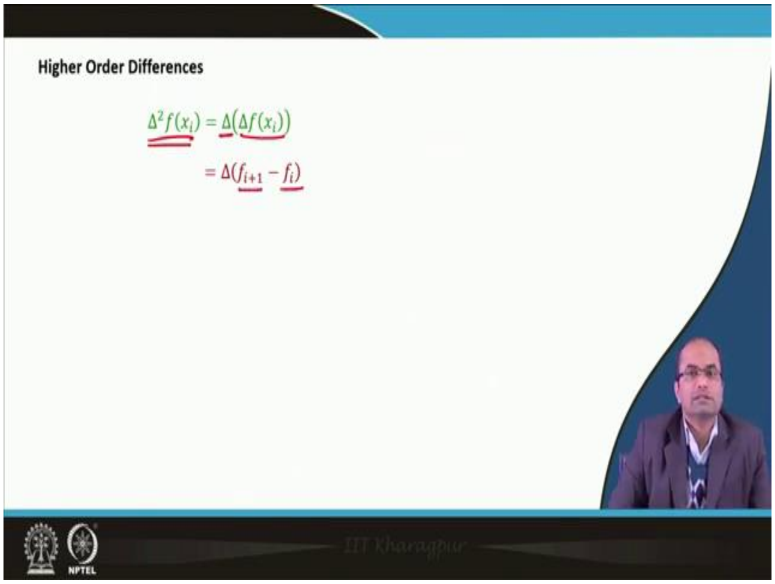

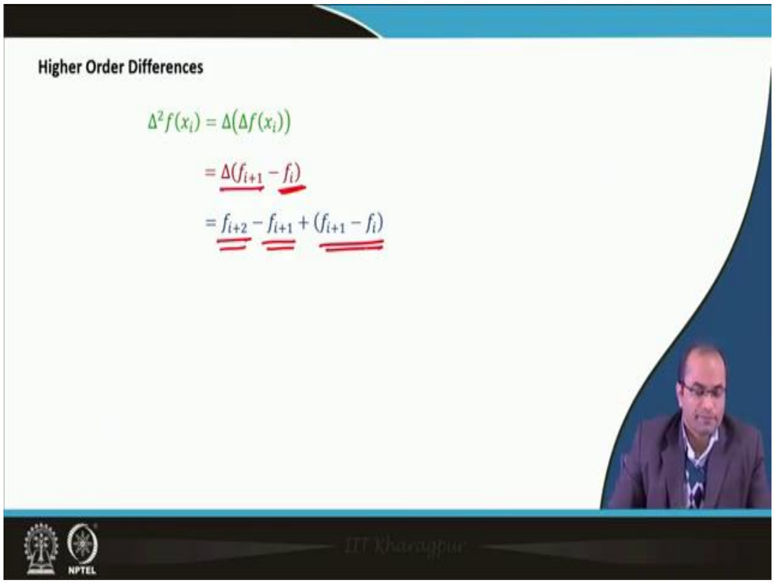

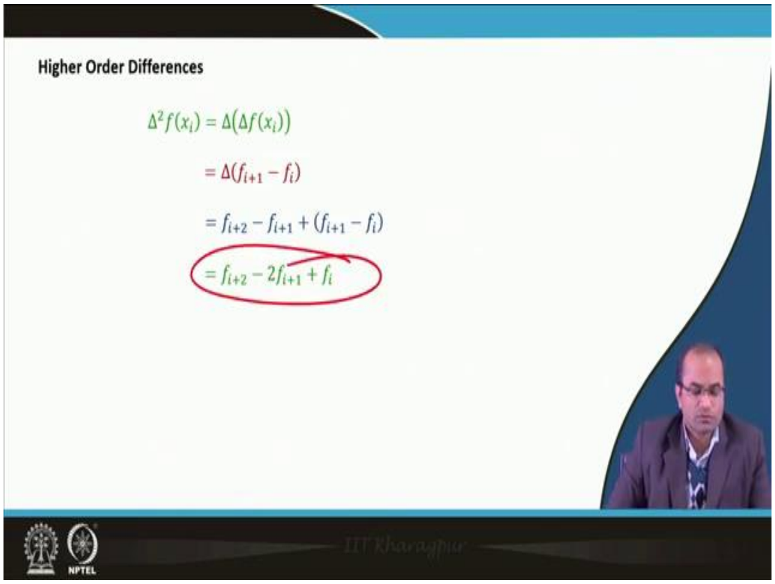

So, we have the higher order differences also, this is, this is what we have discussed, they are first order differences, we can talk about the higher order differences as well for example delta 2, operated on this f xi, that means delta operated on this delta f xi and this delta. So, delta fxi is fi plus 1 minus this fi.

And so we have the delta on fi plus 1 that is fi plus 2 minus this fi plus 1 that is when we operate delta on fi plus 1, and when we operate delta on fi, we will get fi plus 1 minus fi. So, forward difference operator, we will go in the forward direction and then subtract the value fi. So, which can be written in this form, so fi plus 1 minus 2 fi and plus fi.

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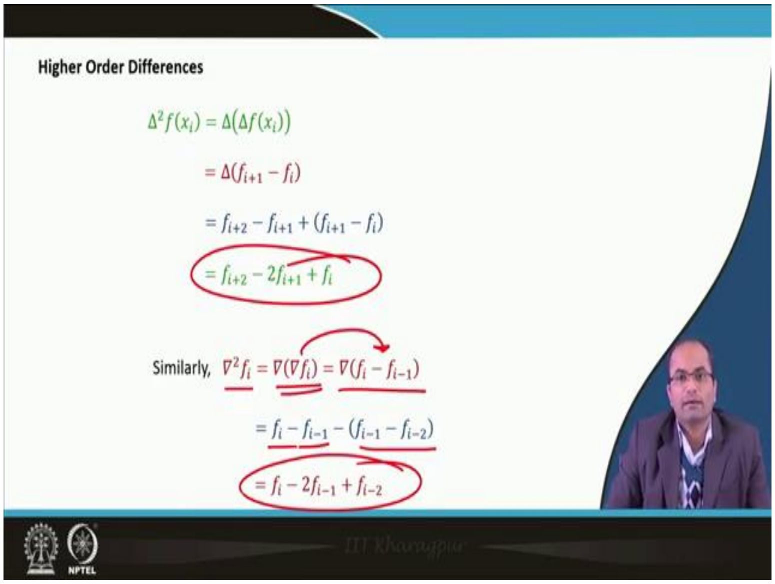

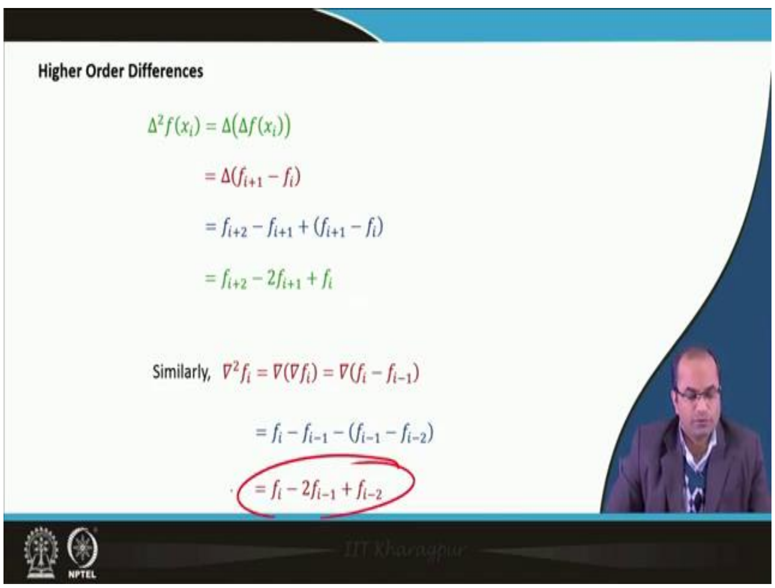

Similarly, we can talk about the second order backward difference operator, which is delta, del 2 here fi and here we have again the two times, we operate and so the first time when we operate is fi minus fi minus 1 and we will operate again this backward operator. So, fi minus fi minus 1 and here again fi minus 1 minus fi minus 2 and when we combine, so we are getting fi minus 2 fi minus 1 and plus fi minus 2.

So, these are the second order difference operators, we can talk about the third order, fourth order or nth order in general, but we do not require at present. So, we will move further just, but the idea can be extended of course for the third order and the fourth order, it is very clear now from here.

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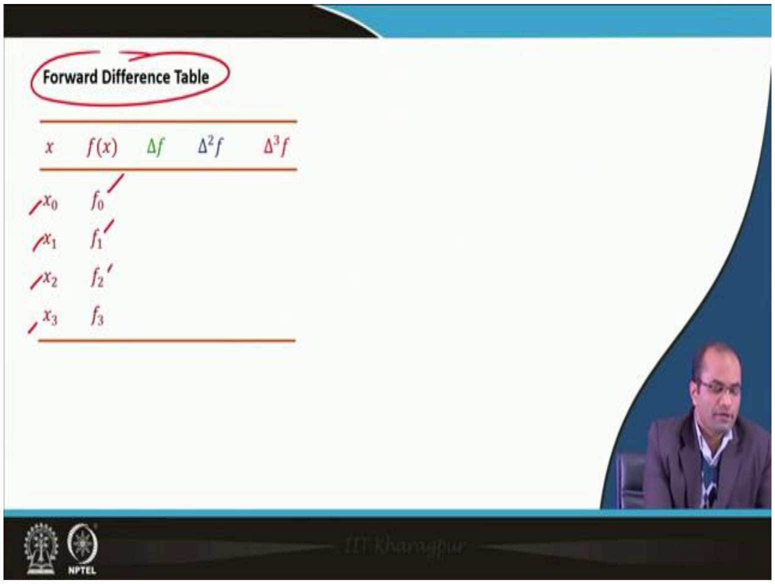

So, the Forward Difference Table usually when we compute these differences, so we can use naturally the formula, which we have derived but they are bit difficult to again to remember and then to apply on the given data set it is much easier to form this forward difference table and from the table we can easily pick all these values for the first order operator, or second order operators, or third order, fourth order operators.

So, therefore, these tables are very, very useful so given this X, so for instance, this is just an example X naught, X1, X2, and X3, the four data points are given and the value of the functions were F0, F1, F2, F3, is also given.

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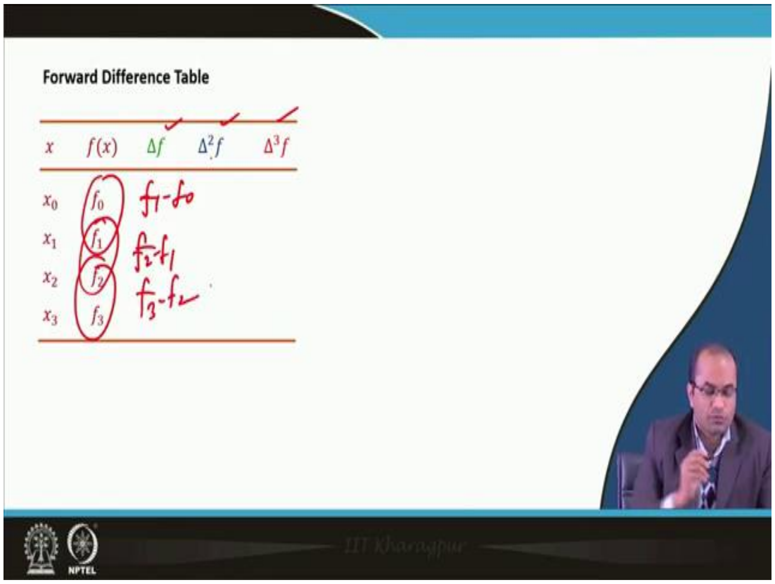

And then we want to compute the forward difference operators, first order, second order and the third order. So, what we do usually and we can verify with the formula, which we have derived, this first order differences here will just come when we subtract F1 minus F0. So, from here F1 minus F0, from here F2 minus F0, so here F2 minus F1 will come, here F1 minus F0 will come and here third place F3 minus F2 will come.

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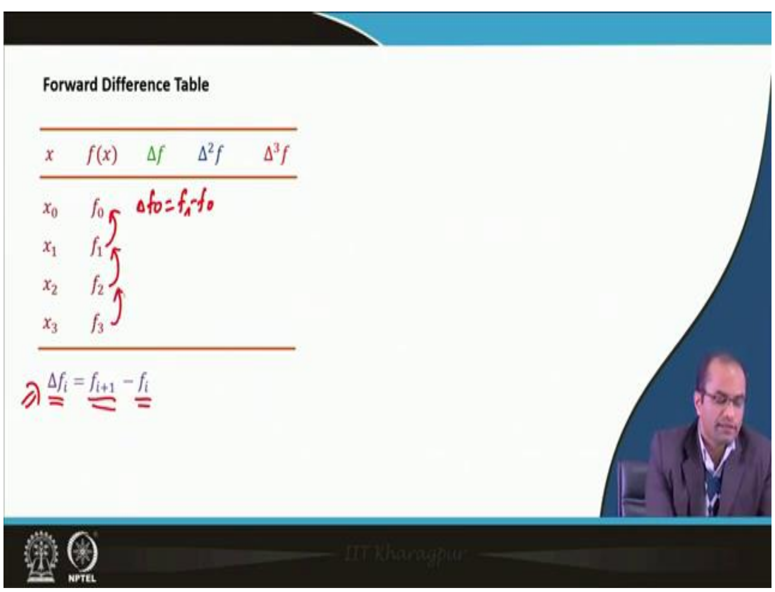

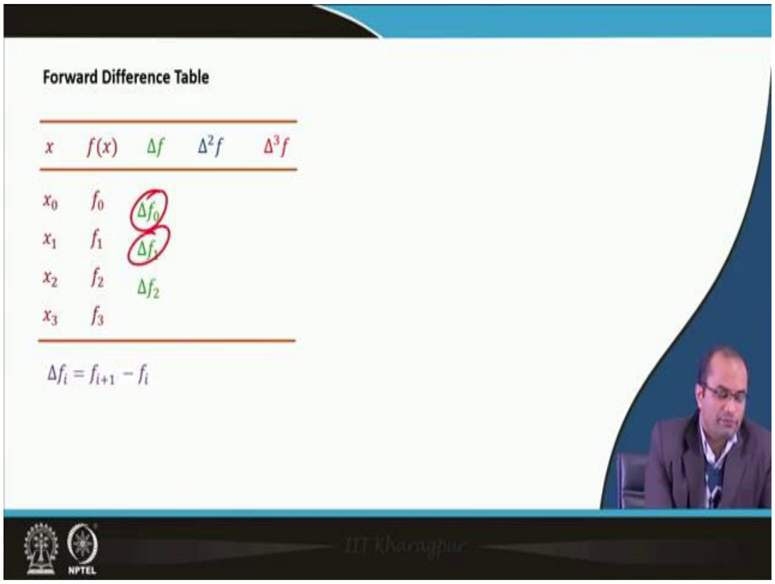

So, with just a simple differences here, we can get the first order difference operator, which can be noted here, that this first order difference operator is nothing but we go in the forward direction and subtract the same. So, here the delta F0, which will appear at this place delta F0 which will be nothing but F1 minus F0. So, we go with the difference here F1 minus F0, then F2 minus F0 and then F3 minus F2 and we will get these delta F0, delta F1, and delta F2.

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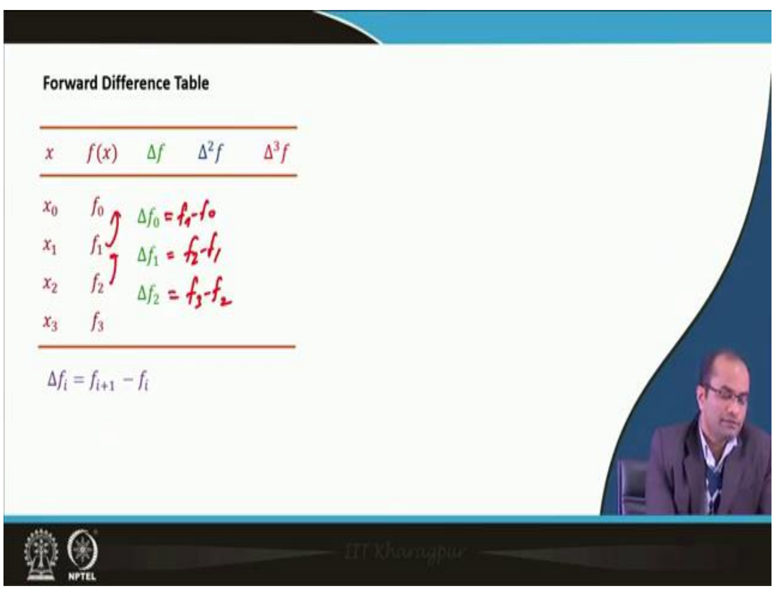

So, delta F1 is nothing but F1 minus F0, delta F2 is nothing but the F2 minus F1 and delta F2 is this F3 minus F2. So, just by doing these differences, we can just get these first order differences delta F0, delta F1, and delta F2.

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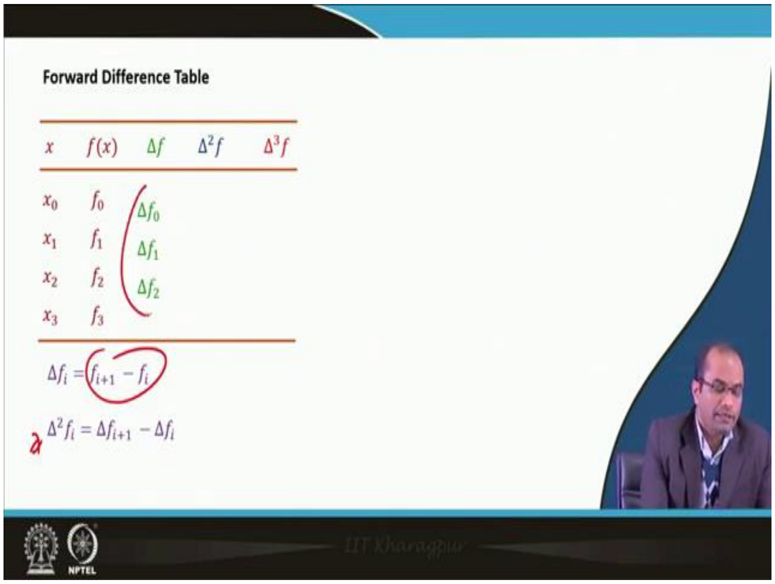

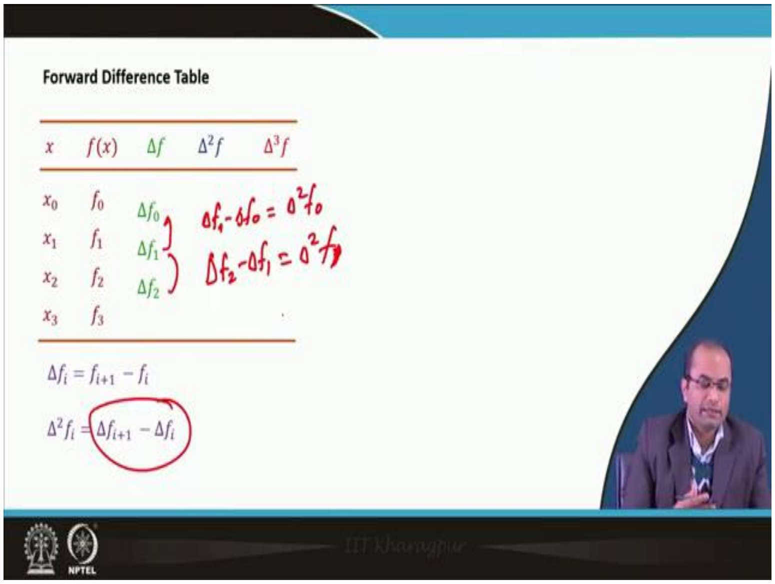

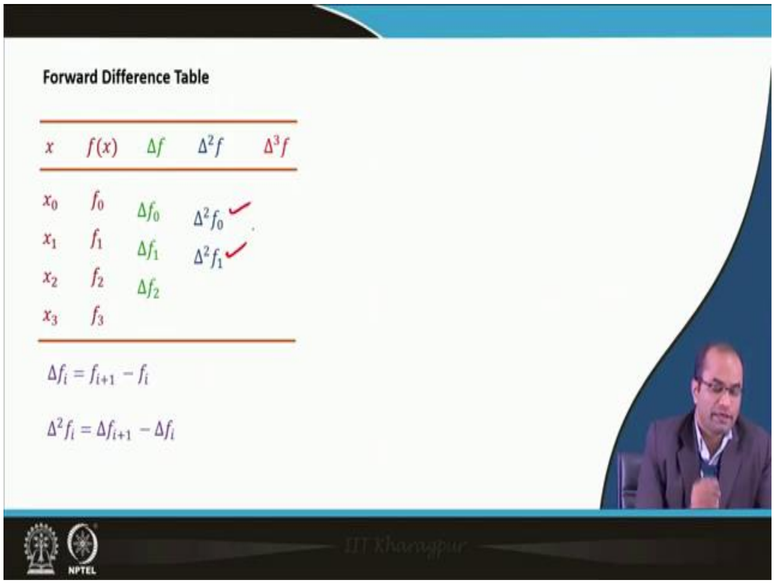

Again if we note here that the second order derivative, second order difference operator is nothing but the when we apply this operator here. So, that means delta fi plus 1 minus delta fi, so again we have obtained already here the first order differences that means these are the deltas here and again if we do the same exercise.

So, we compute here delta F1 minus this delta F0 and as per the definition here, this is nothing but delta 2 F0 and similarly here when we compute this delta F2 minus delta F1 this difference we will get delta 2 F1. So, by repeating the same exercise you will get the second order differences, that means delta 2 F0 and delta 2 F1.

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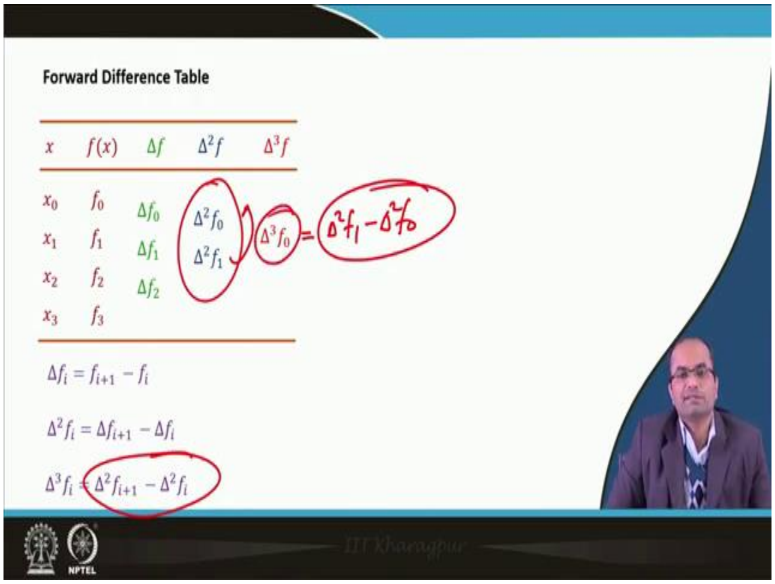

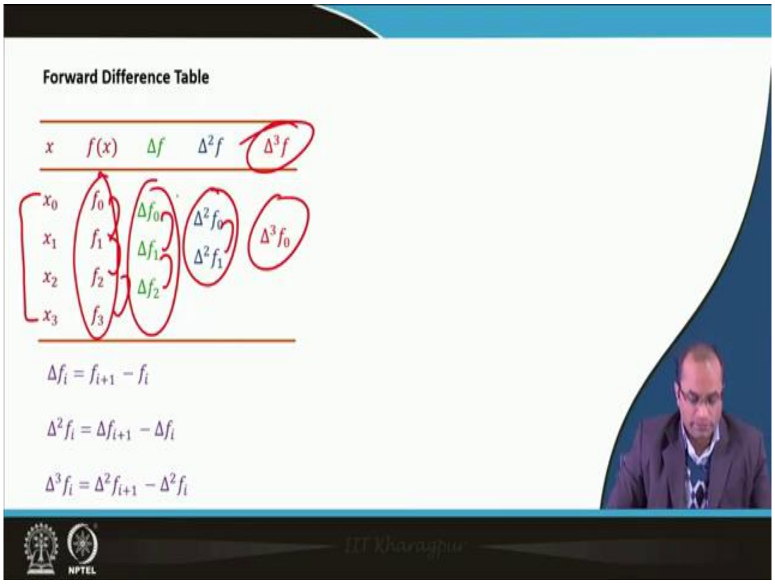

And similarly when we go for the third order again we have to use the same thing, so we have already the values for these delta twos and then we can go for the differences again and we can get this delta 3 F 3 that means this is delta 2 F1 minus this delta 2 F0. So, this difference here will give this delta 3 F0.

So, it is very easy to construct this difference table given the values here of this F we will go with the differences here and then we will get these first order differences, again we will go with the differences we will go second order and again the differences we will get the third order differences and so on there are more data points so we can go with fourth order, fifth order etc. So, given these four data points we are going up to the third order differences.

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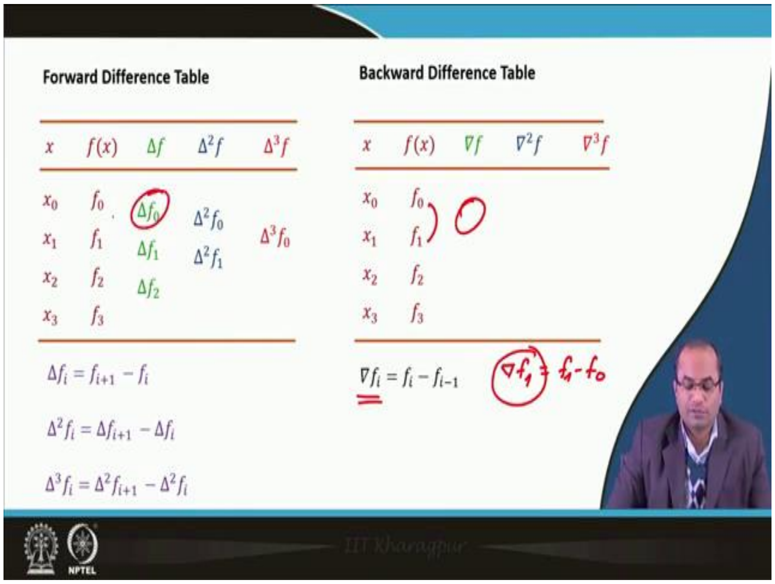

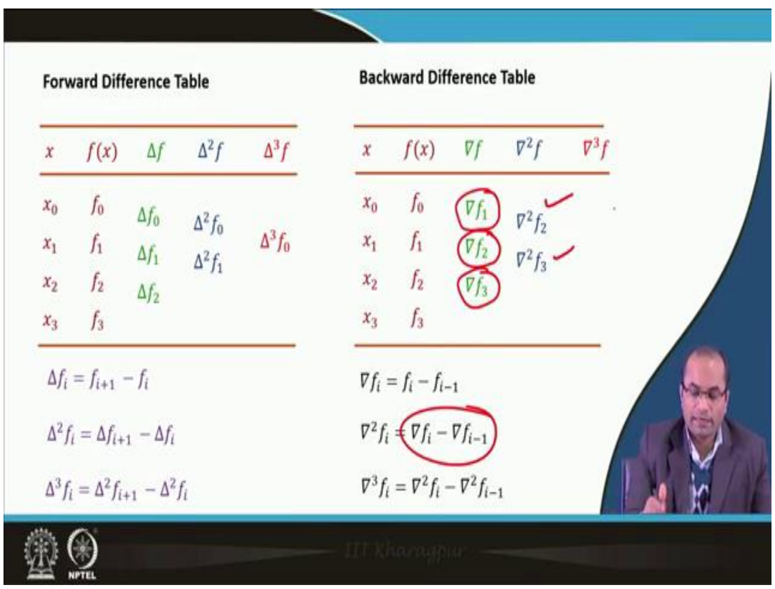

Similarly, the backward difference table, we can construct and here also the idea is same. So, we will go, we will construct exactly in the same way. So, there is no difference here in the calculations, the same table can be used for forward difference operators and the backward difference operator only as per the definition which we will see backward delta for instance now here I am talking about delta F1.

So, what will be the delta F1, F1 minus F0 that means the value which we will get here by subtracting this F0 from F1, we will get now backward delta F1, though here it was forward delta F0. So, that is the only difference.

So, constructing table will be the same but now we will get instead of this as delta F0 in the backward notation we will say it is a backward F1, this is backward delta F2, backward delta F3. So, that is the only difference we have and again we go further with these differences and we will get the second order delta 2, this F2, delta 2 this F3 and then again we go with the difference. So, we will get a third order.

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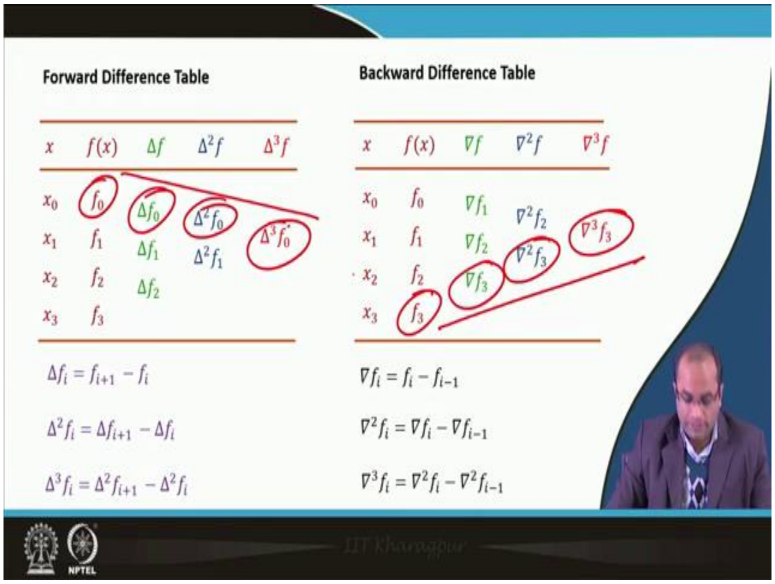

So, usually we will observe in this Newton's forward and backward difference formula. In the forward difference formula we will be using these values here of the forward difference from the forward difference table, these values will be used, whereas for the backward difference formula, these values will be used, whereas here these values will be used.

So, we will a form a same table and we can use either the forward difference formula, which you will derive now or backward difference formula, accordingly we have to select the values from the table. But the table the construction of the table will be actually the same, or the same table can be used for forward difference operators or from the for the backward difference operators.

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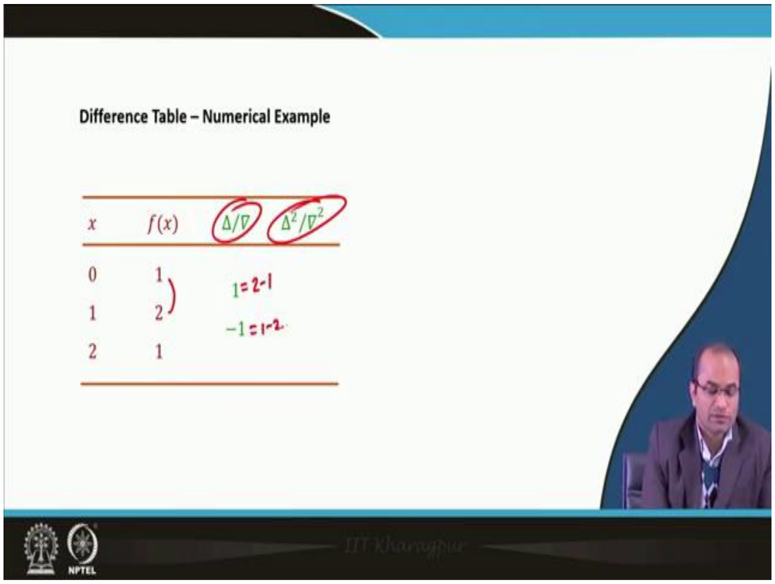

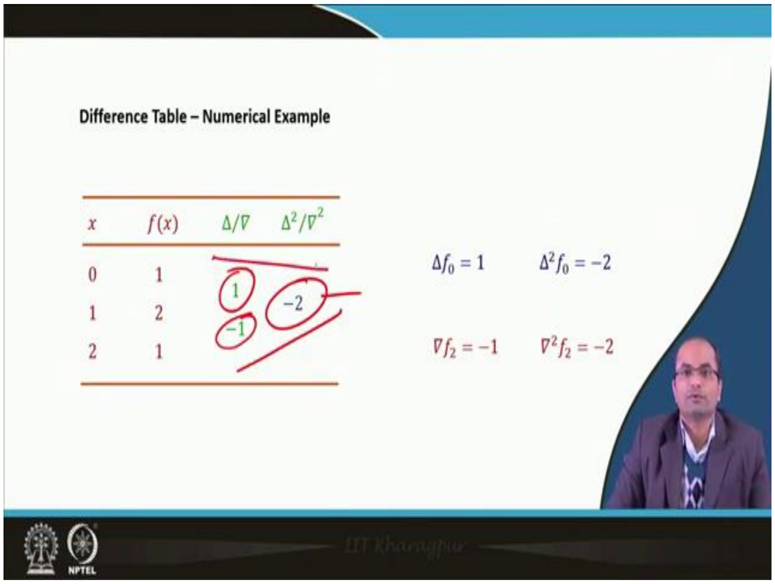

So, the difference table just a very simple numerical example again, so for instance 0, 1, 2, X values are given and then we have the Y values, so 1, 2, minus 1, and we want to construct that table which is valid for forward or the backward operators. So, we will go for the difference 2 minus 1, so this is 1 that is 2 minus 1 here this is 1 minus 2. So, this is minus 1.

And then again we will go with the difference so here this minus 1 and minus 1 so minus 2 will be coming as the second order operators. So, when we want to have the forward difference operator that means the delta F0 that 1 will be the value and delta 2 F0 that will be the value, when we are talking about the backward operator. So, backward F2 will be this 1 and backward the second or backward operator on F2 that will be this minus 2. So, we can use this values for the backward and those upper values for the forward operators.

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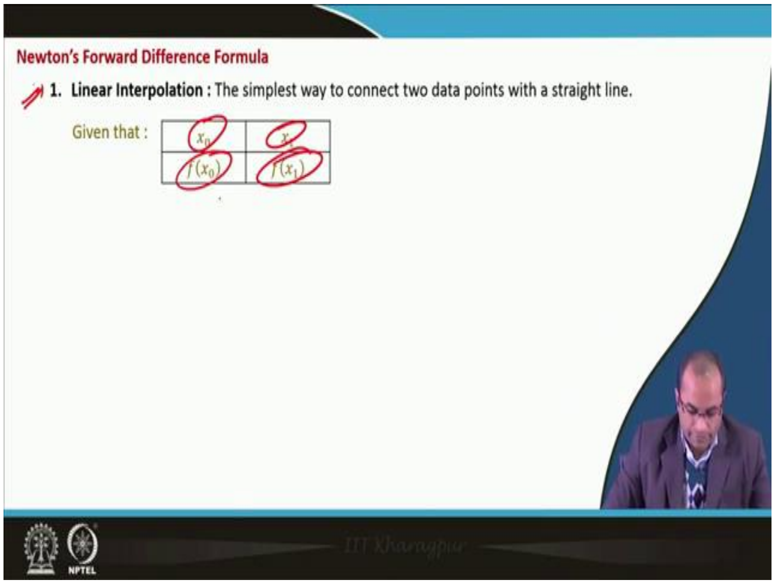

Coming to the Newton's forward difference formula we have now we will discuss first the linear interpolation case, a very simple one when two data points are given we can connect it through a point, then we will go with the quadratic one and then we can generalize for n plus 1 or n given data points. So, this is a simplest way to connect two data points with a straight line. So, given data suppose we have this X naught, we have FX naught and X1 and we have FX1.

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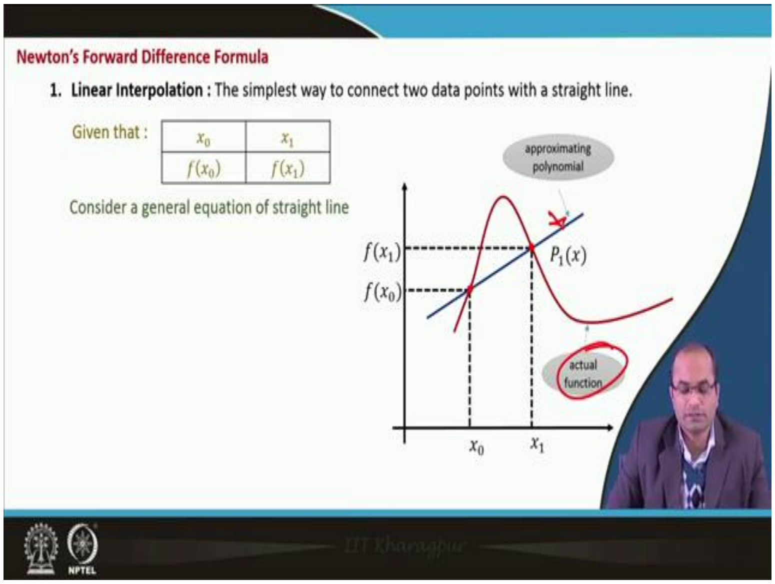

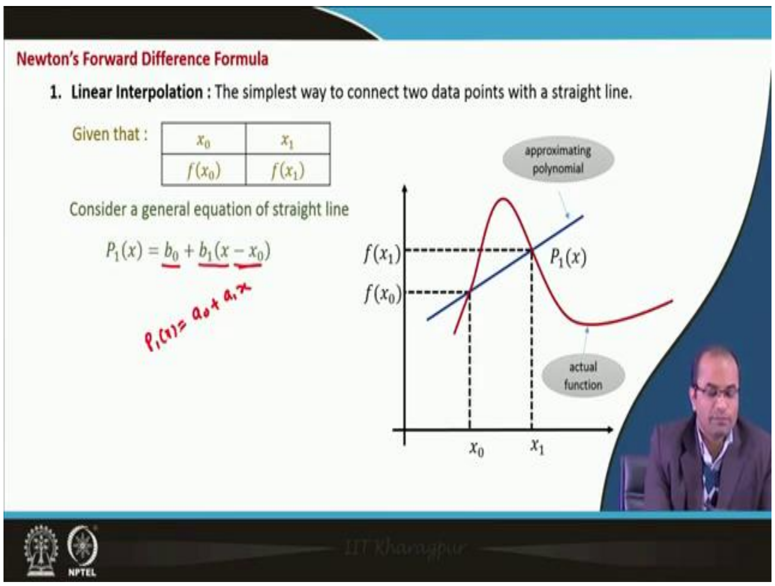

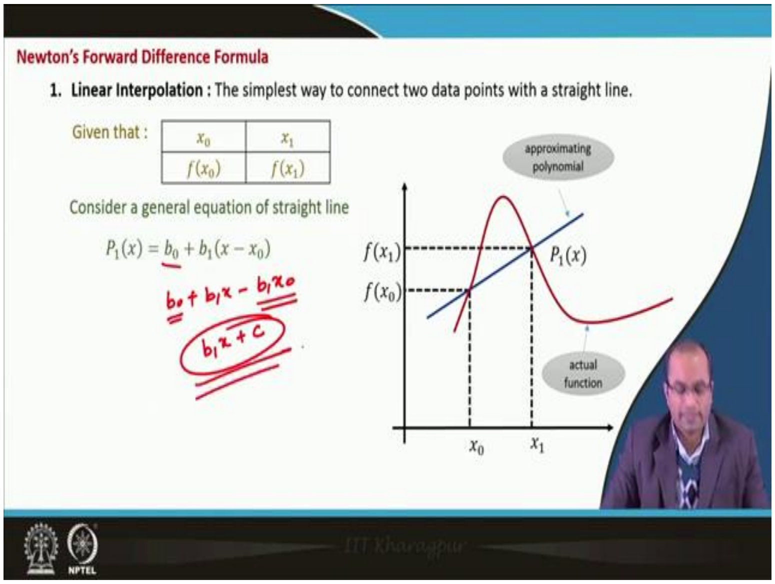

So, the, suppose we have the function here the F the actual function is given and then there are two data points given to us and we want to construct a linear interpolation, a linear polynomial or polynomial of degree 1, which passes through these two points, that means this line here.

So, we can take, we can consider the equation of the line, which we have written in this form in a little bit fancy way that b naught a constant B1 X and minus X minus X0, it is a linear function it is like we can write like A0 plus this A1 X. So, this is equivalent to, to this only. But we have just written in a slightly different form because one can notice that this is B0, then we have B1 X and then we have B1 X naught.

So, this B1 X naught and this B1 is a constant again, so we have B1 X and plus some other constant. So, that is the standard equation of the line. So, this is a standard equation only but we have just written in this way because of we will see in a minute that the evaluation of these coefficients will be much easier if we write this equation in this format.

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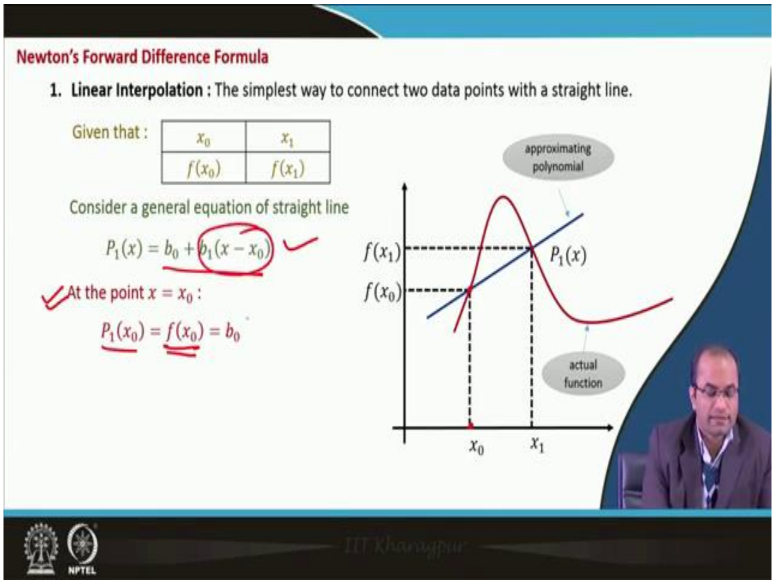

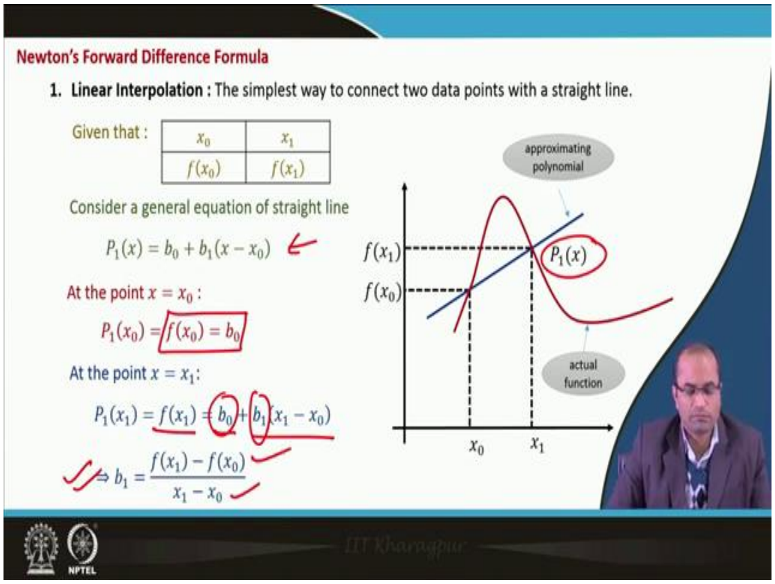

So, for at this point X is equal to X naught, we have both the values that this polynomial should match with the function values, that means this P1 at X0 should be equal to FX 0 and when we are putting here this X equal to X0, this term will get cancelled and we are getting directly B naught and this is the point we have taken this equation in this particular format. So, we are getting B naught equal to this F X naught.

And now at X is equal to X1 point. So, we will substitute here X is equal to X1 and this P1, X1 will be nothing but the value of the function, that is FX 1. So, here FX 1 is B naught and then B1

X1 minus X naught, so from here we can get the B1 also easily. So, FX 1 minus this B naught which was FX 0 and over X 1 minus X 0.

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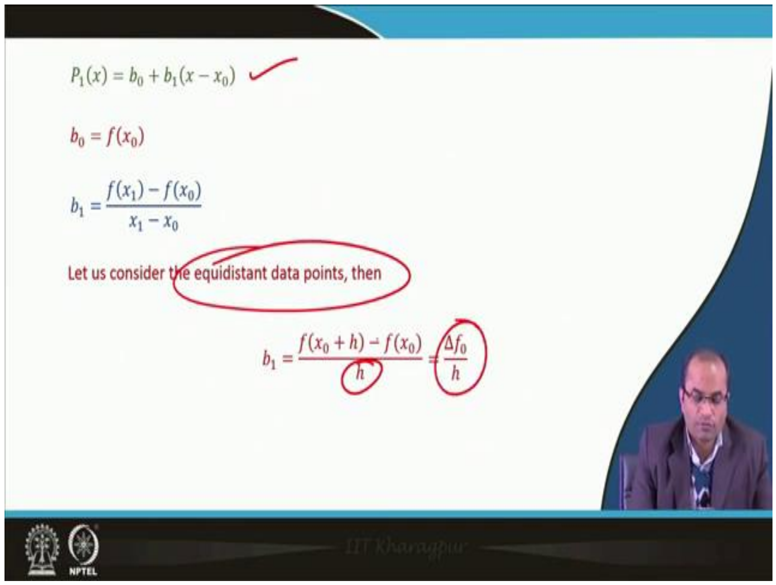

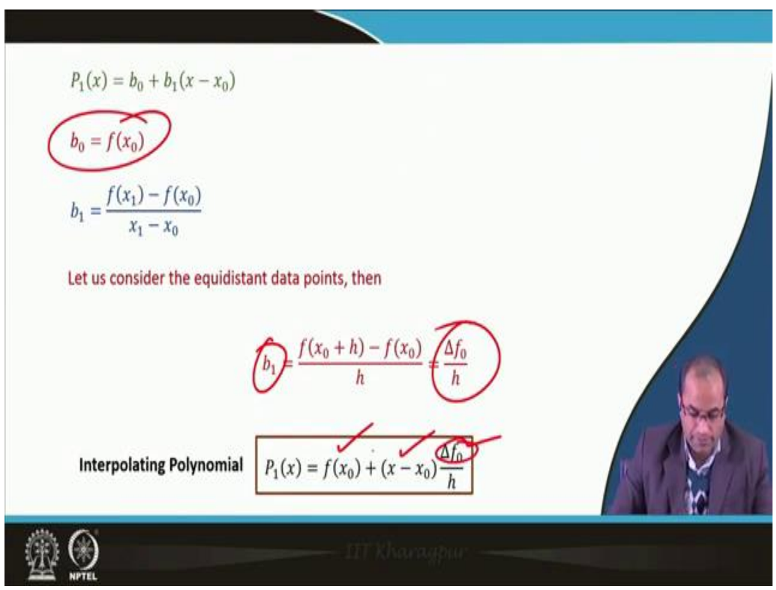

So, having these values of B0, B1 and this polynomial, so if we consider the equidistant data point. So, in the forward interpolation formula, we will consider we will assume that data points are equidistant. So, in that case we have some simplification here, that this X1 minus X naught we can replace by this h and then here we have FX naught plus h minus FX naught, which in terms of the forward difference operator we are writing now.

We are introducing the forward difference operator in our formula for FX naught plus h minus FX naught, we are replacing by this delta FX, delta F naught. So, having this now for B1 and the B naught was just FX naught our interpolating polynomial of degree 1 can be read as, so F X naught and then X minus X naught delta F naught divided by h. So, this is the forward difference operator which we will get from the table easily.

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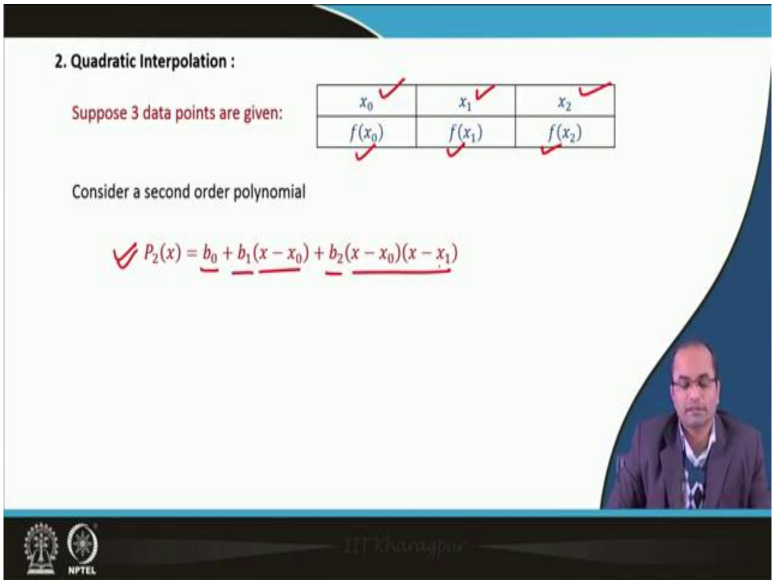

Now, going further to have more general formula, so let us first derive for the quadratic interpolation, that means now suppose are 3 data points, instead of 2 we have 3 data points we have X naught, we have X1, we have X2 and their values FX naught, FX 1, and FX 2, are given.

So, now this time to fit these 3 points by a polynomial, we have to consider a second order polynomial, because 3 points are given, that means we will consider now a second order polynomial again in the same format, that means a constant term B1, with the X minus X naught, B2 the second order term here with X minus X naught and X minus X1.

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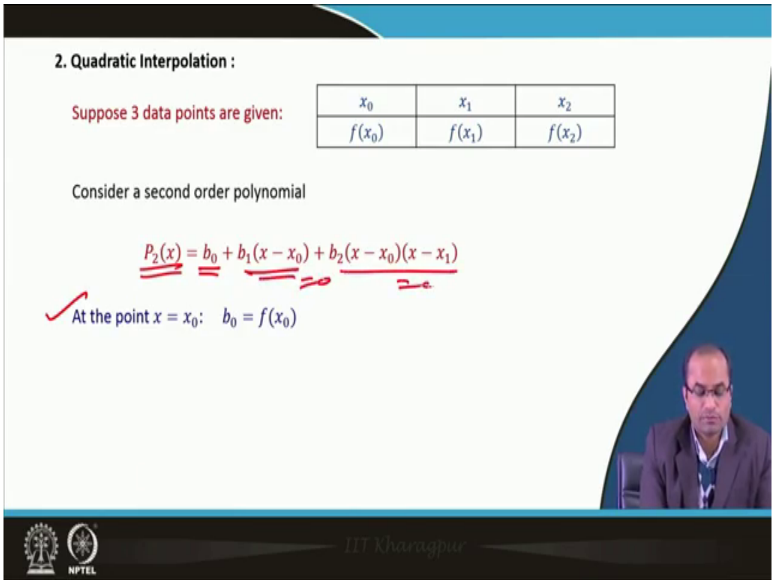

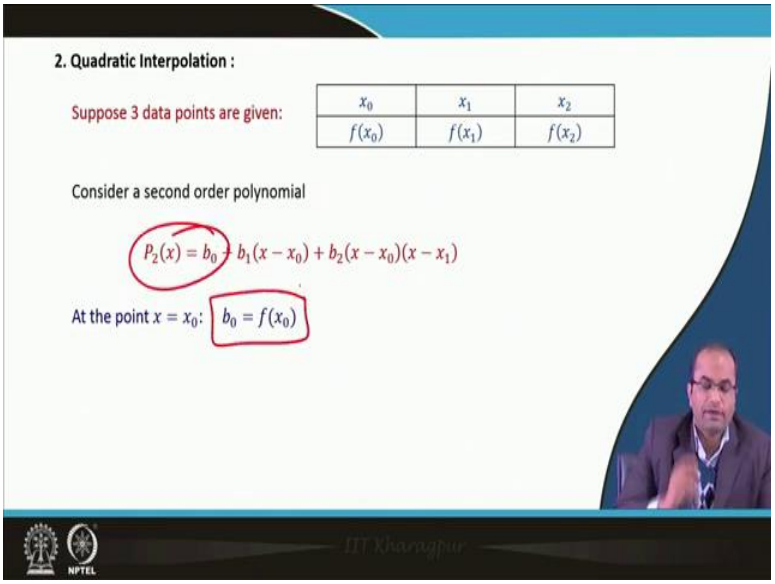

So, at the point here X is equal to X naught, if we substitute there what we will get we have B naught and this P2 at X naught will be the function value at X naught. So, the, because this term will go to 0 and this term will also become 0 at X is equal to X naught. So, directly we will get B naught is equal to F X naught.

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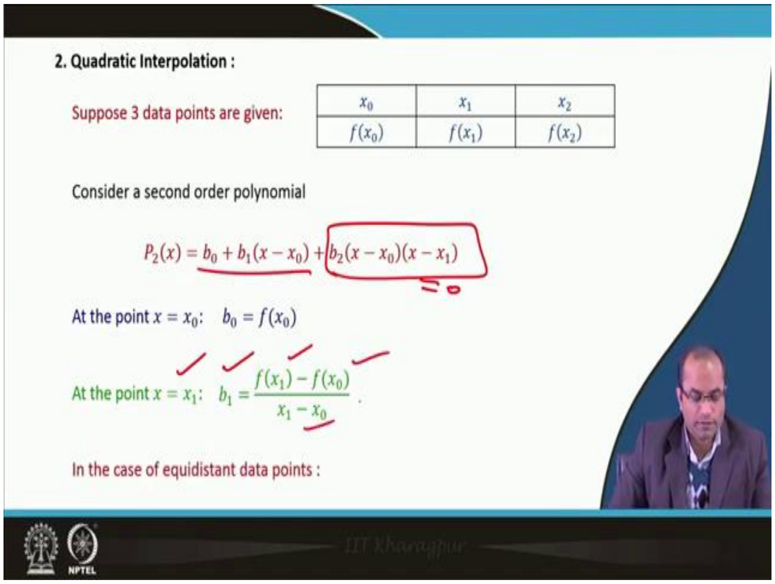

And now we will go for the second point that is X1 so X is equal to X1 if we put this term will become 0. So, we have only the first two terms there, so from there we can just compute this B1. So, B1 will be FX 1 and this B naught is FX 0 over X minus X1.

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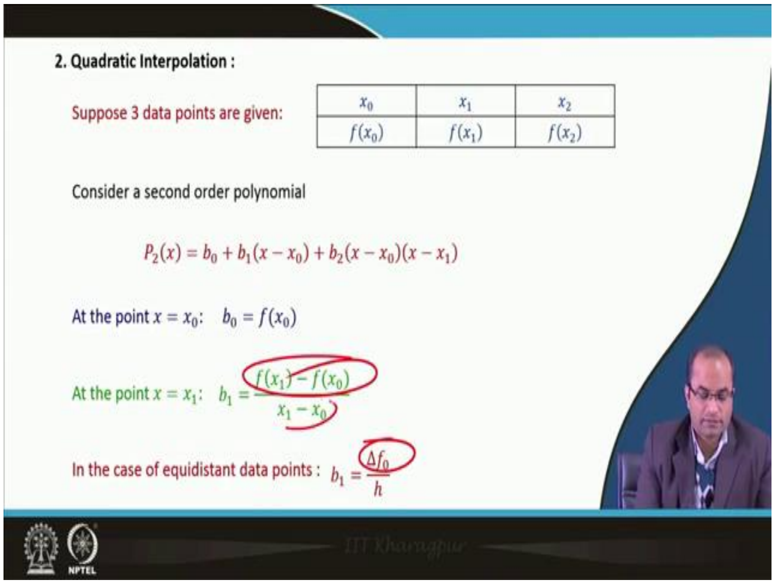

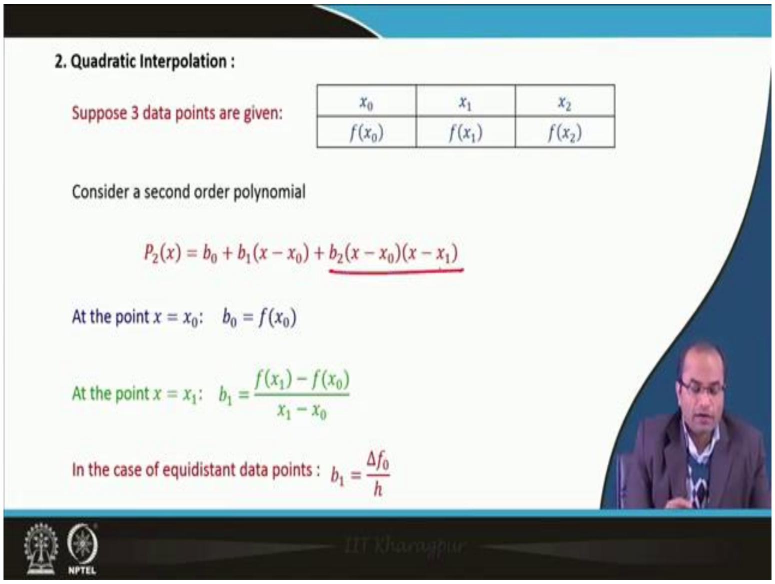

And in case of the equidistant data point, which we are considering. So, this is nothing but the delta F naught and then this is h. So, B1 we have got now delta F0 over h.

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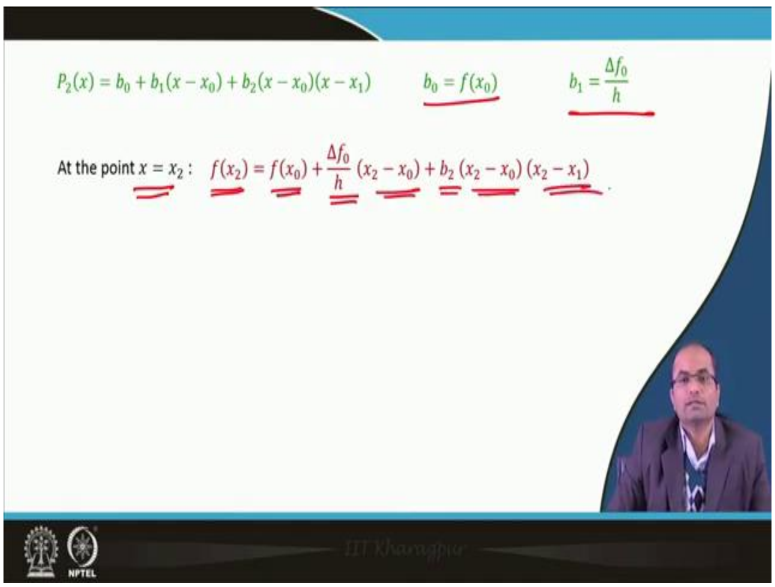

And now the B2, so having this B naught and B1 so remember this is the same values what we got also for the first order polynomial. So, now for the point X is equal to X2 so we will substitute there the polynomial value and the function value are same. So, FX 2 is equal to FX 0 we have delta F0 multiplied by this, we have B2, we have X2 minus X0, we have X2 minus X1.

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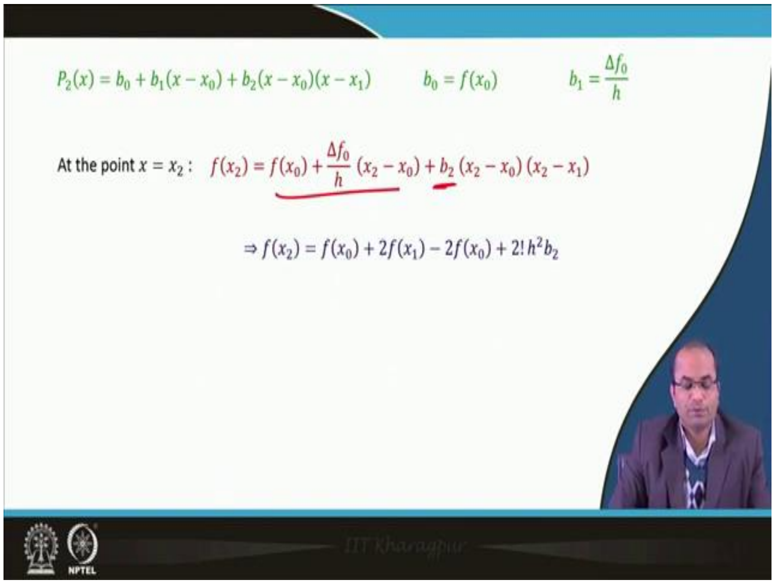

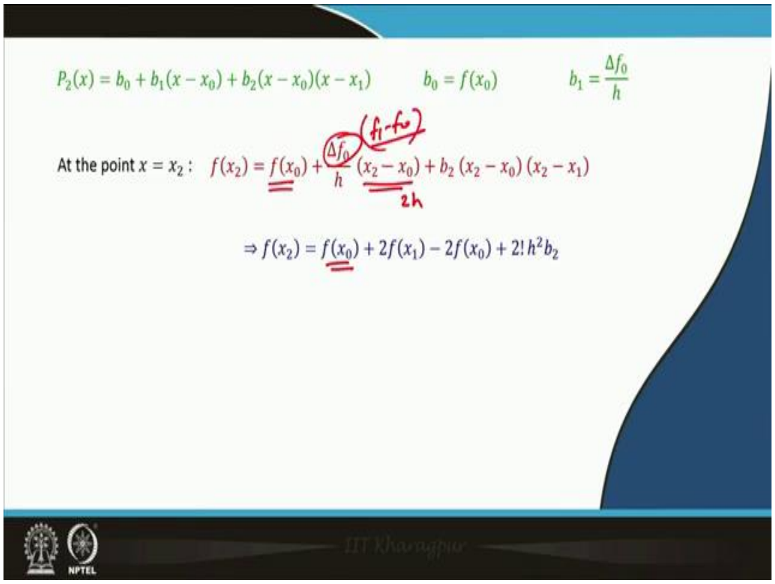

So, from here or now everything is known other than this B2, so we can compute this B2 and so this is FX naught which is written here and then we have here delta F naught first we have written this as F1 minus F naught and this is 2 times h. So, basically h, h will get cancel we have 2 times and multiply it to this F1 minus F naught.

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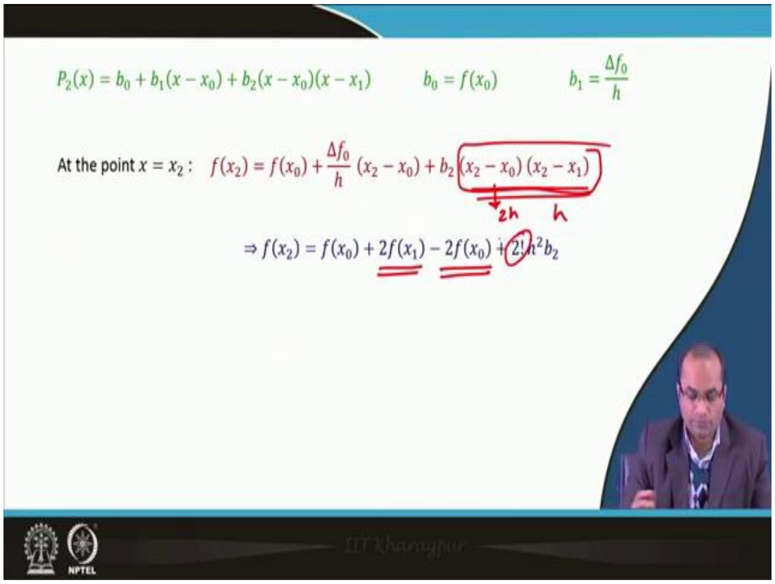

So, we have 2 times FX 1 and minus 2 times FX naught and then from here what we will get we will get h square because of this h and h here indeed we have this 2h and here we have just h. So, we will get 2 h square and we just for the generalization purpose we have written this factorial 2.

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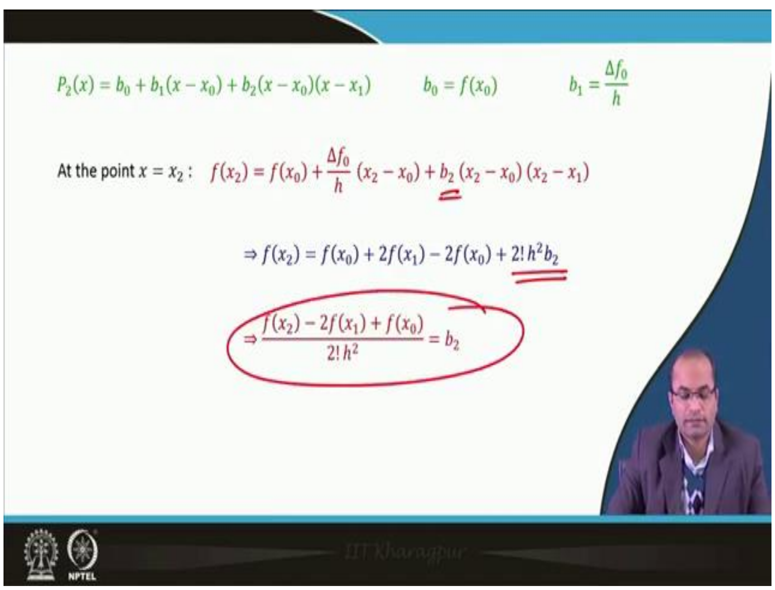

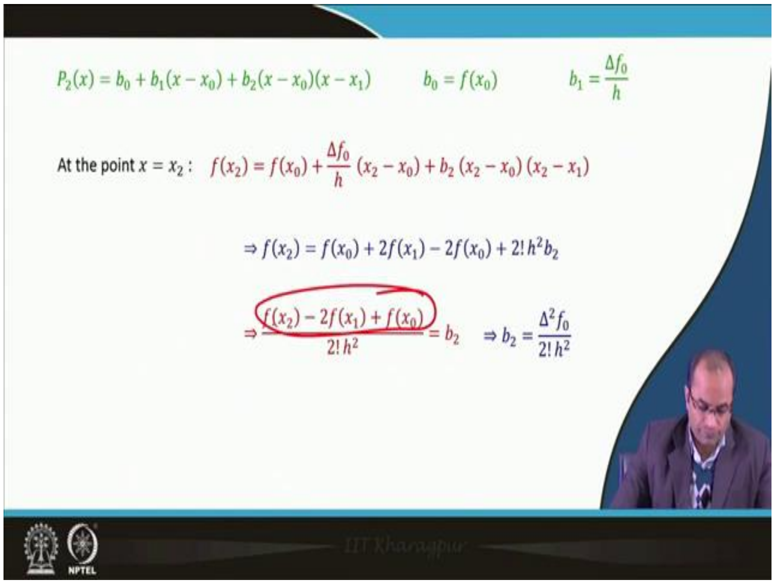

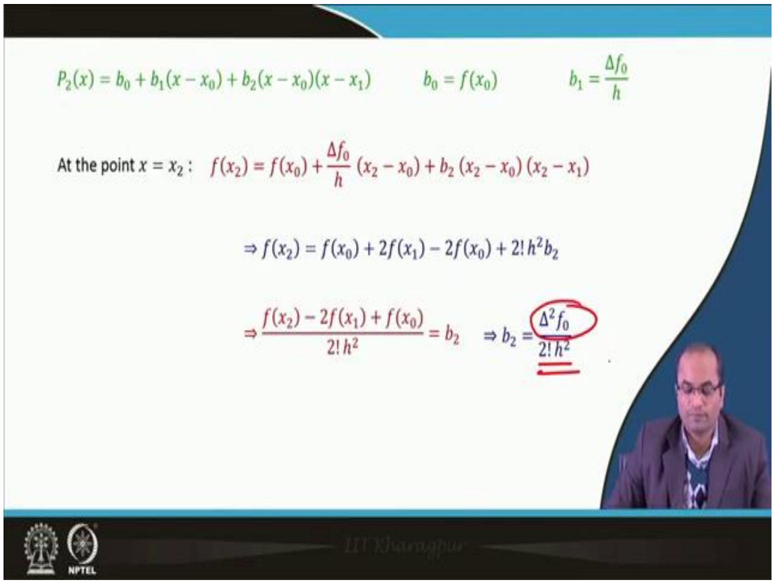

So, this is h square with this B2 and which we can simplify and we will get the coefficient B2 in this form and remember the formula of the second difference operator. So, we can write this delta 2 F naught over factorial 2 H square, so we got the B2, we got B1 we got B naught. So, we can just put everything together.

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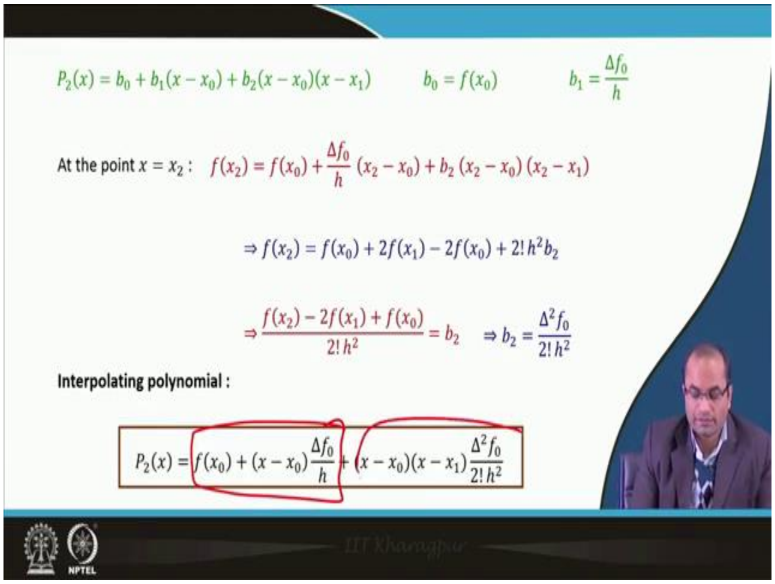

And we have this interpolating polynomial and the interesting thing is that it is just continuing. So, here it was same form, when we derived the first order polynomial and this term is added now to the formula. So, we got the second order polynomial which passes through that given 3 points, we can continue this derivation for the 4 points, 5 points or in general we can write down.

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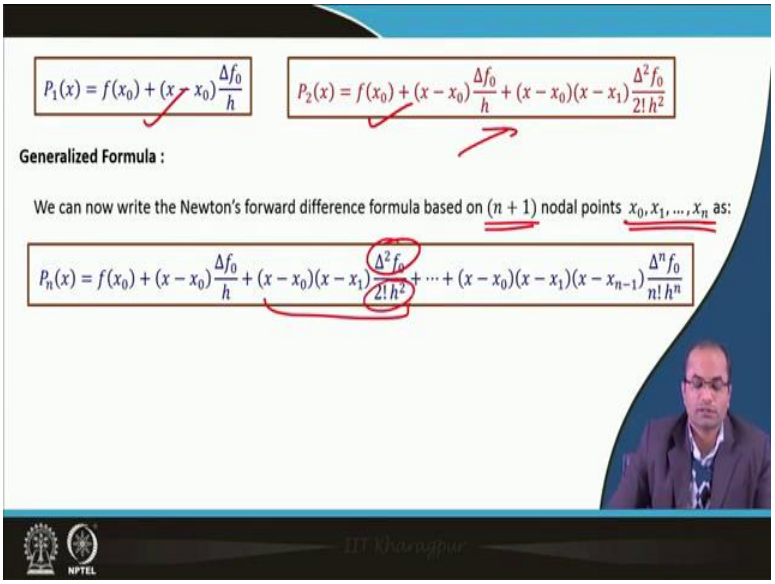

So, for if the 2 points are given, we have the first order polynomial, if 3 points are given we have second order polynomial and the general formula if there are n plus 1 distinct this nodal points are given, then we can have this nth degree polynomial, where the formula one can see is just the extension of what we have derived, this is the second order term and we will continue with this.

So, the next term will be X minus X naught, X minus X1, X minus X2 and there will be third order derivative their factorial, the third order operator and then factorial 3 and h q.

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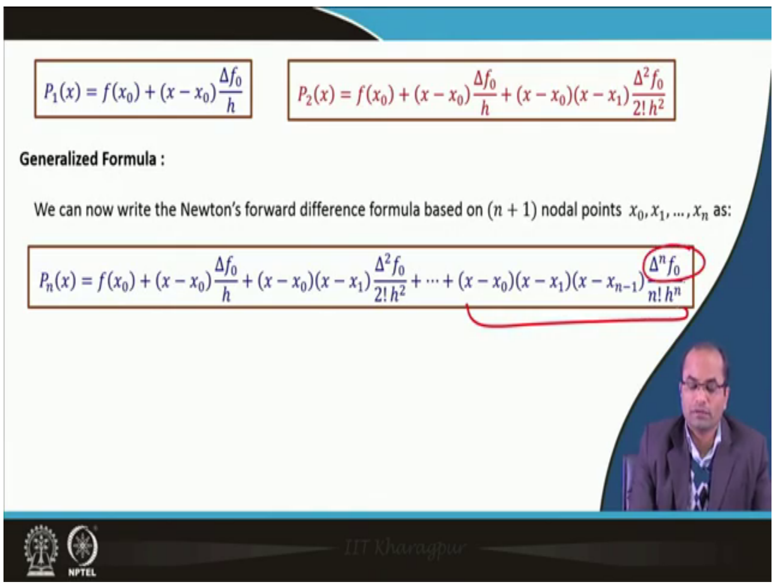

So, the last term in this way we can get this nth order operator here, on F naught and then.

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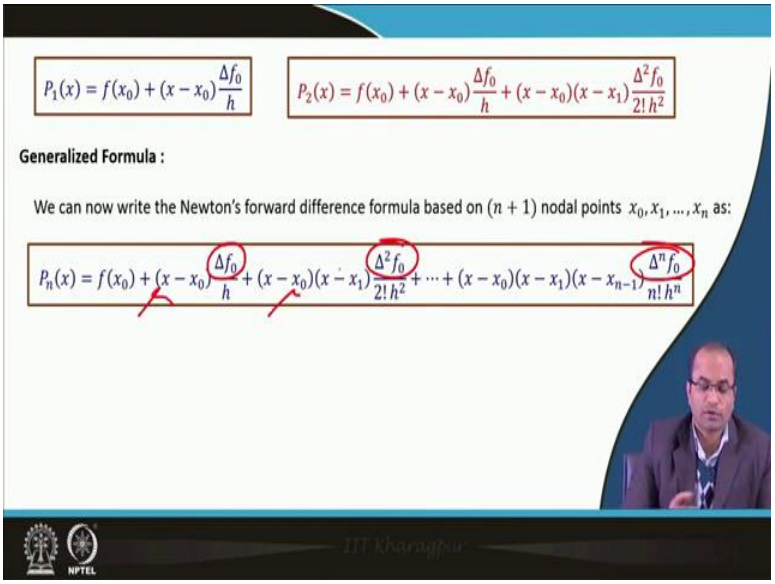

So, what is interesting here that from the table we can get all these values delta F naught, delta 2 F naught, delta n F naught and the rest is already there in the formula. We can get a polynomial easily.

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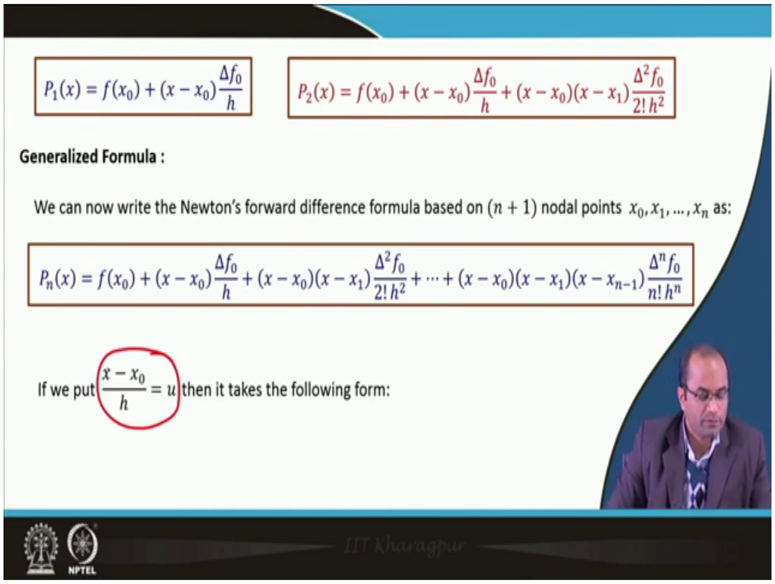

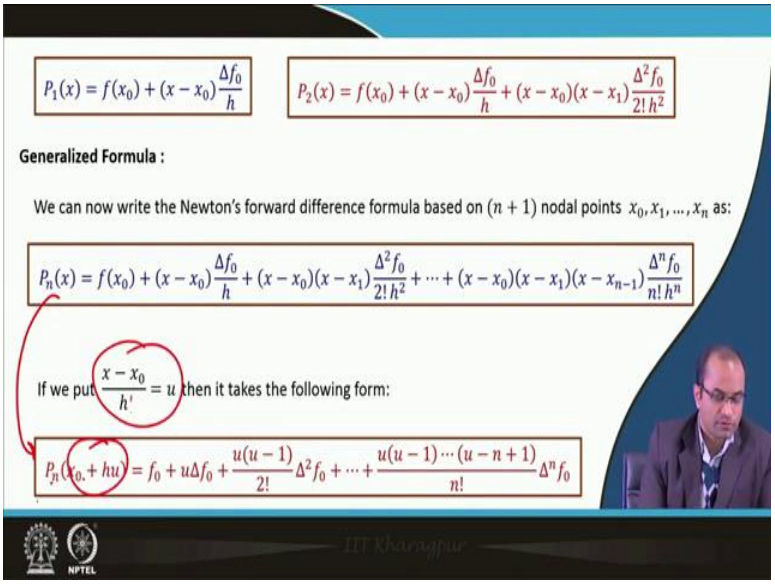

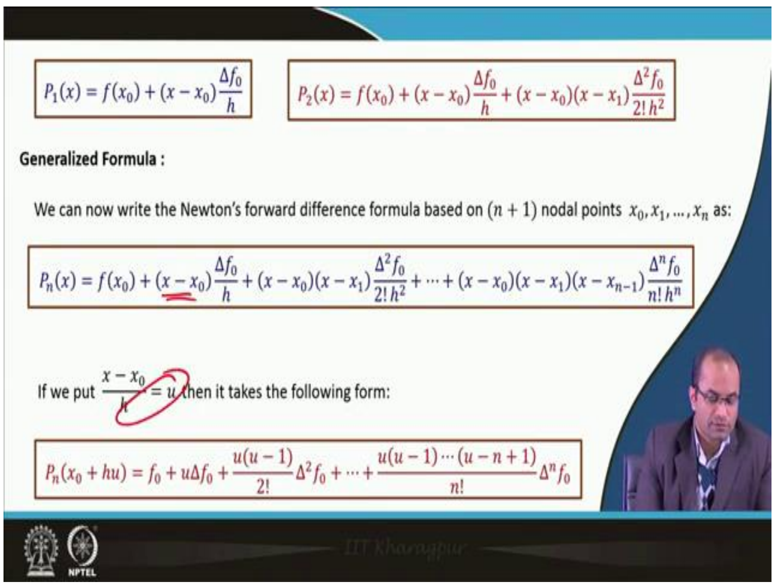

There is another form also in the literature one can find that if we substitute this X minus X naught by h as u, another just a change of variable kind of thing. So, X minus X naught over h, if we substitute as u, then this formula can be written as in this form Pn and this X is nothing but from here we can get hu plus X naught.

So, X naught plus hu and then we have this F naught, which is a short notation for F X naught and then here X minus X naught is hu. So, h h gets cancel, so we have basically only u, and if the operator.

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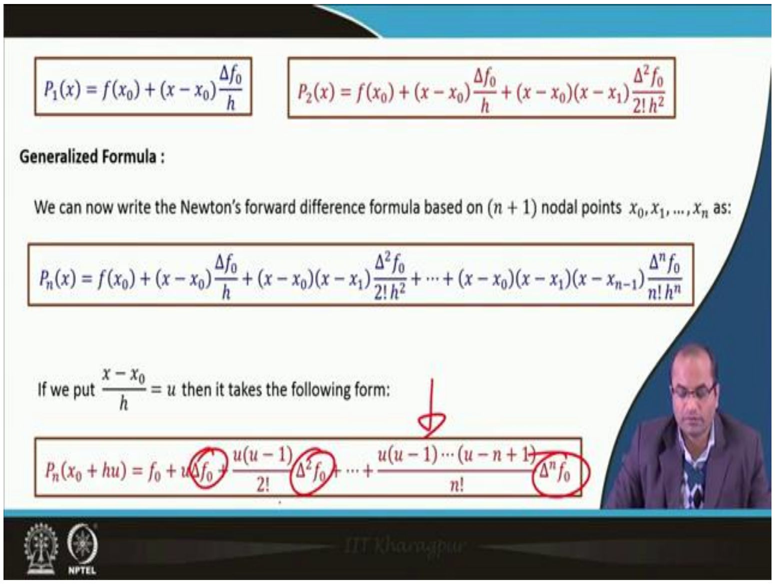

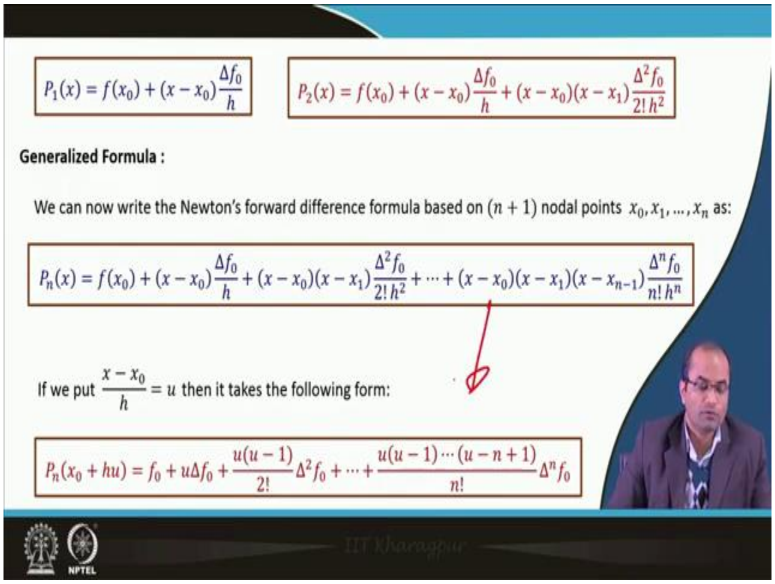

So, what we do see here that in this case now it is little bit simplified, we can use this from the difference table and the rest is just u u minus 1, the h is disappeared now from everywhere. So, this is another way of rewriting this forward difference formula.

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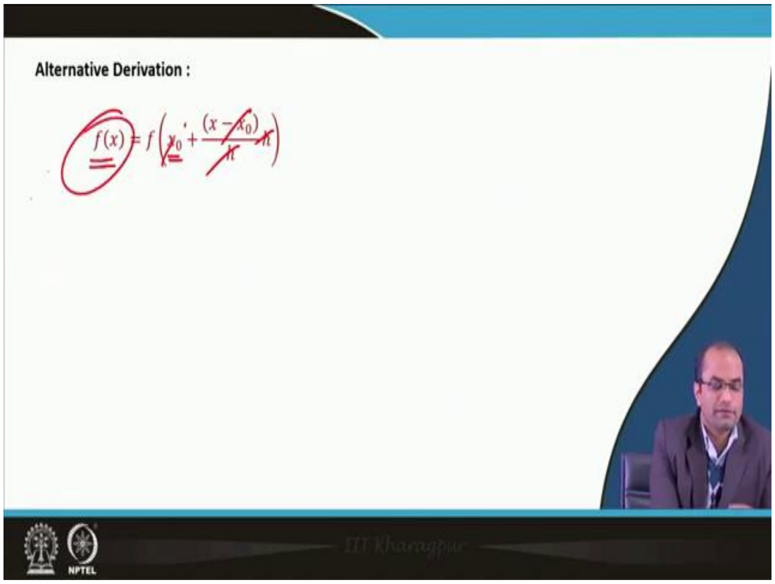

So, the same derivation there is just a quick view to derive this in a different way, so FX we can write down as X naught plus X minus X naught over h. So, just see that h, h gets cancel and X naught, X naught, get cancel and we have a still FX there.

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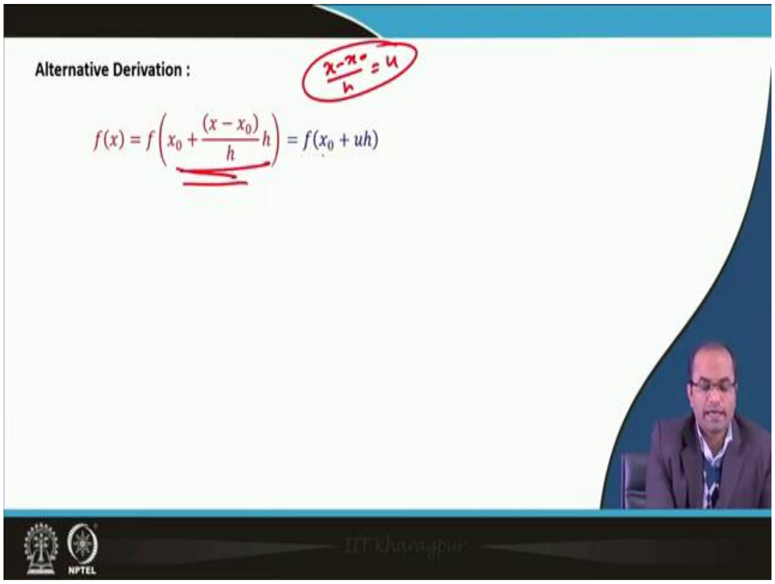

So, having this now and we know the notation only that X minus X naught over h, we are taking as u. So, taking this we have this one X naught plus uh.

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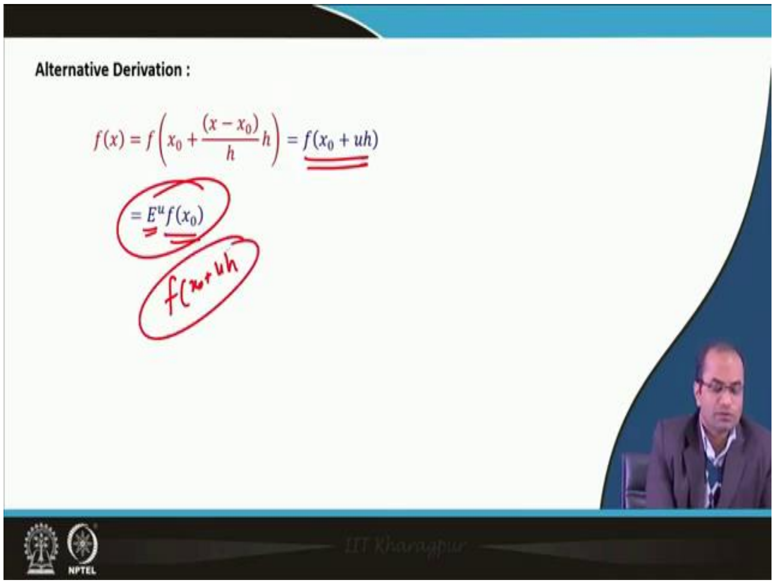

By this shift operator we can say it is a E (())(28:08) operator this u times, so E power u and F X naught because this was exactly what we have learned there, that this value will be X naught plus this u times h. So, this is a quick derivation sort of which will lead to the formula which we have just derived.

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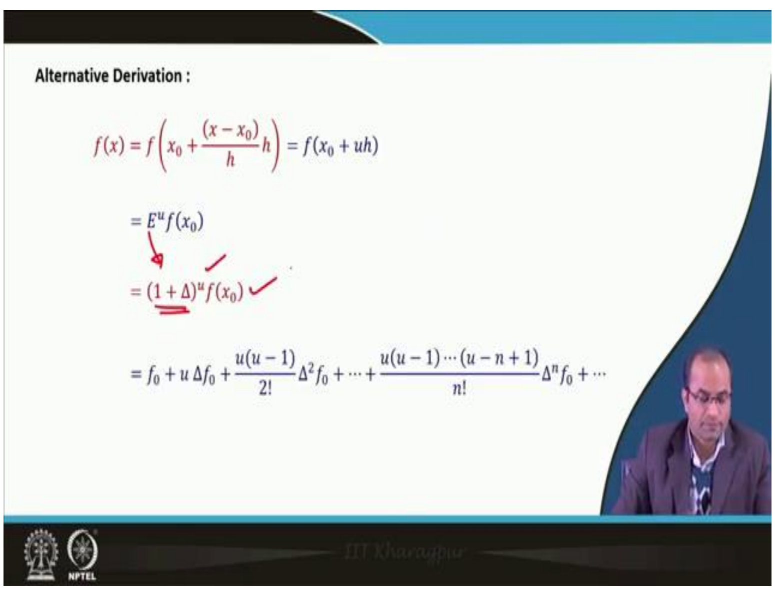

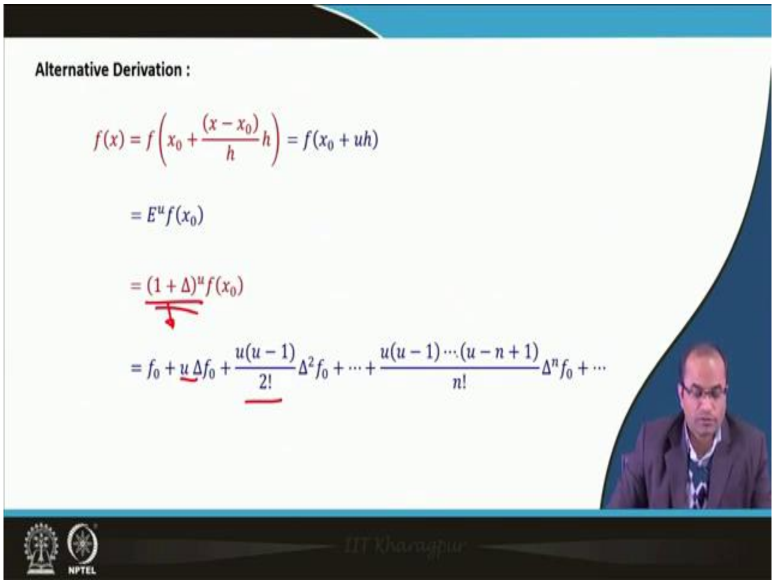

And that relation also we have seen that E, we can write 1 plus delta again this power this u will come FX naught and then we can expand this 1 plus delta power u, it is a kind of volume, binomial expansion. So, we have this expansion here 1 plus this u times delta and then the u, u minus 1 factorial 2, u, u minus 1 u minus 2 over factorial 3 and so on.

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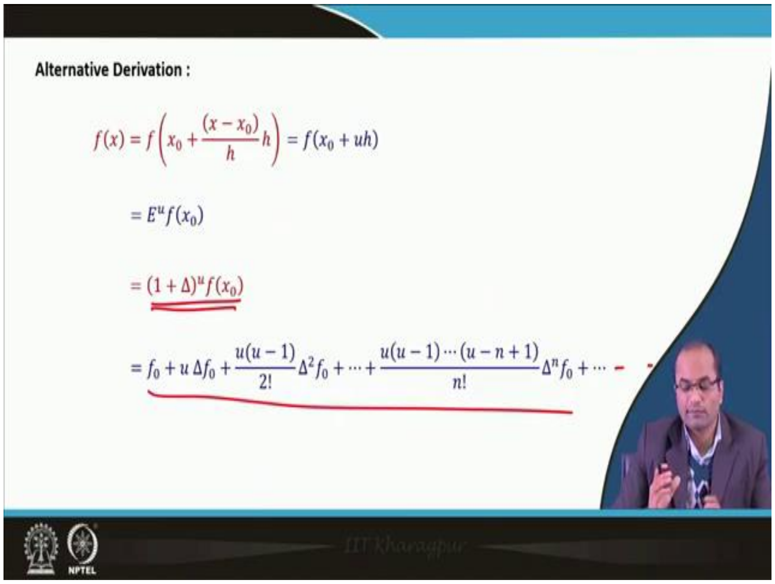

So, having that now we got exactly this formula but they are infinitely many terms. So, we have to truncate this.

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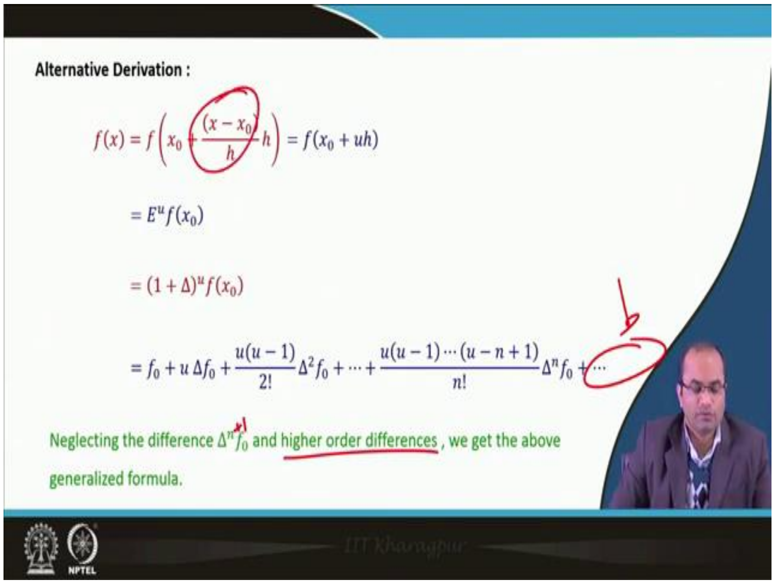

So, neglecting the differences the n plus 1 and higher order differences, we get the above formula. So, if we neglect all these differences which are coming, which are bigger than this, n plus 1 we can get the desired formula, which we have just seen above and replacing again u by this relation, we can convert into the form X and h.

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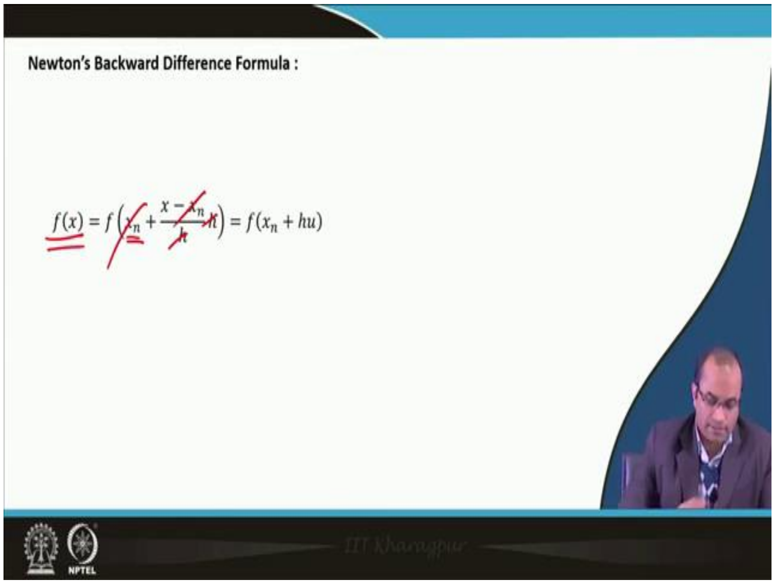

So, there is a Newton's backward difference formula having this Newton's forward difference formula, this is much easier now to talk about this. So, again we will use this quick approach to derive this formula. So, we have F X and now we will start with this F xn minus plus this F X minus n over h. So, again h, h gets cancel, this, X, X gets cancel we have a still FX.

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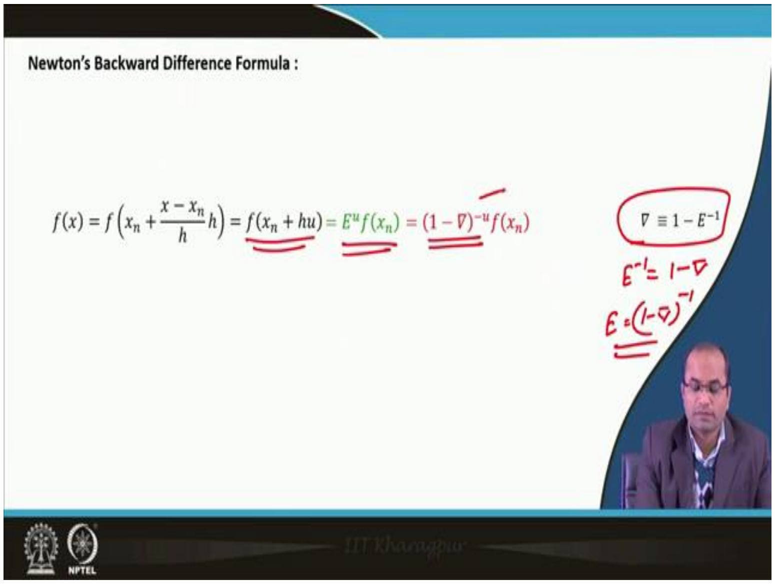

And that can be written as F Xn plus hu. So, this again can be written as E power u F Xn and which this relation we will use that backward operator is 1 minus E or this E minus 1 is nothing but 1 minus this backward operator or E is nothing but 1 minus backward operator power minus 1. So, this relation we can use there and so we have 1 minus this backward operator power this minus u and then F Xn.

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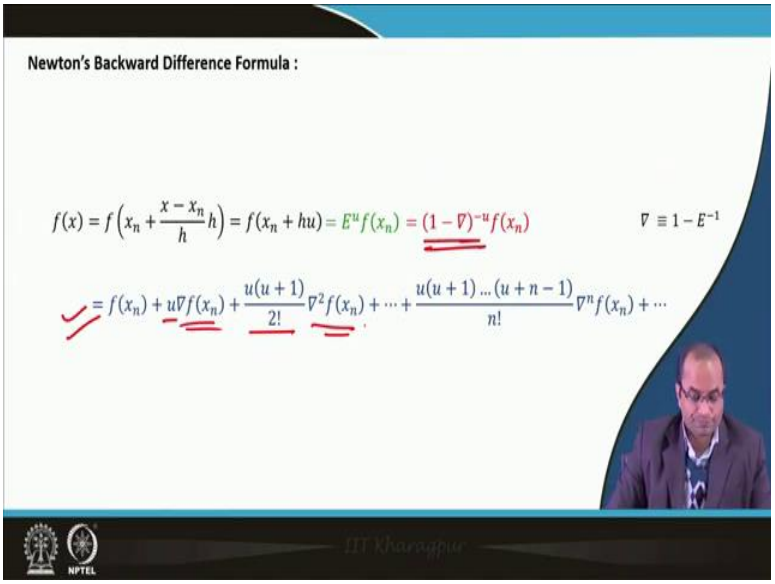

So, again this expansion of this will lead to the u, u, u, plus 1 factorial 2 u, u plus 1 and so on factorial n and then we have these operators there. So, now we have the backward operators.

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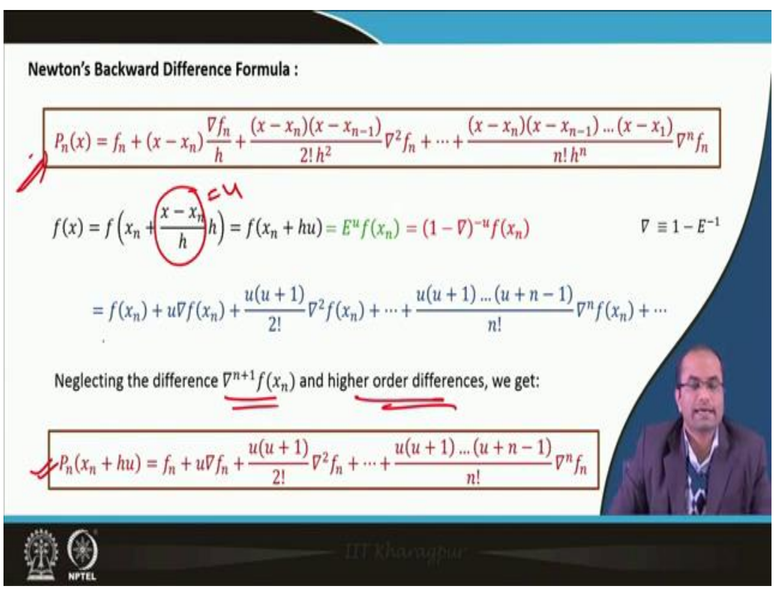

And again truncating the or neglecting the differences n plus 1 and the higher order we will get exactly the formula, which we are looking for. So, this is the backward difference formula, which can be used now in terms of u, and if you want to get back to X. So, we can just use this relation that this is equal to u and we can rewrite everything in terms of this X and h. So, this is the backward difference Newton's difference formula.

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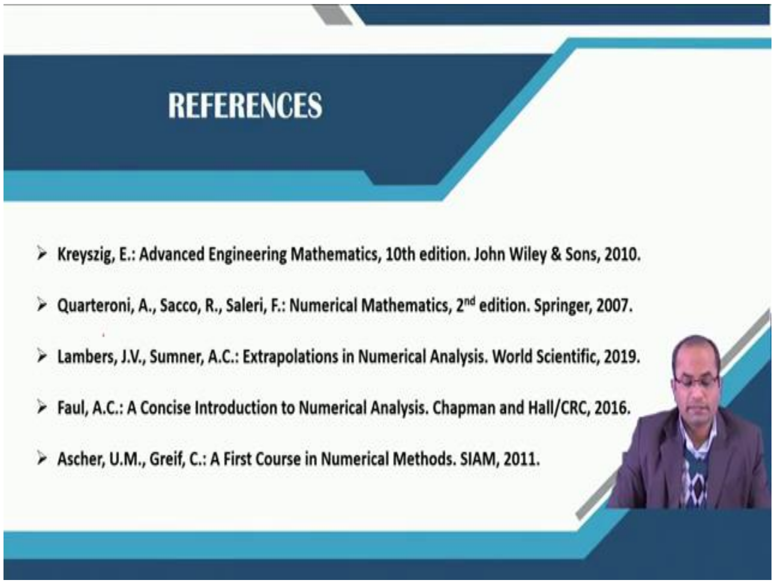

These are the references we have used for preparing the lectures.

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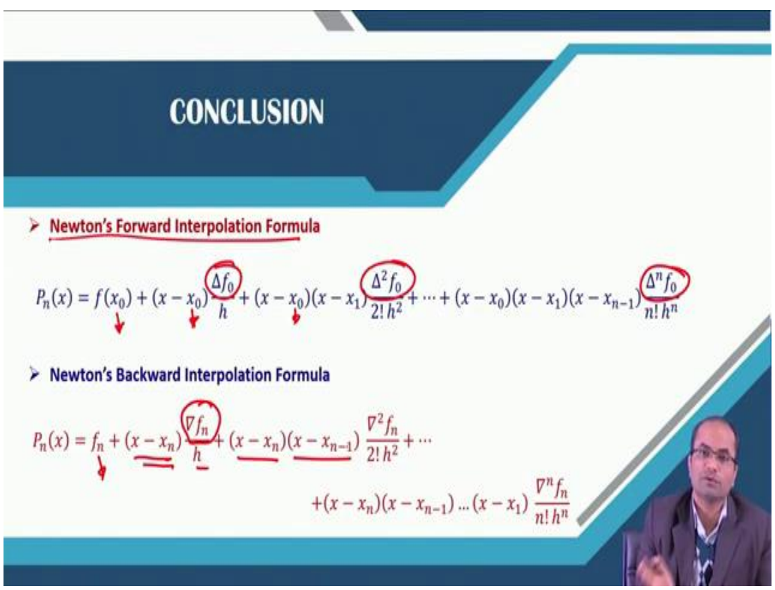

So, just to conclude in this lecture we have discussed the Newton's forward difference formula, which uses these forward differences, delta F naught, delta 2 F naught, delta n F naught and we have also discussed the Newton's backward difference formula the difference mainly that instead of this X naught, we go from the backward direction.

So, we take the F at Xn and then X minus Xn and instead of the forward we have the backward operator, operator on Fn divided by h and then here X minus X and X minus Xn minus 1. So, we go from the backward direction.

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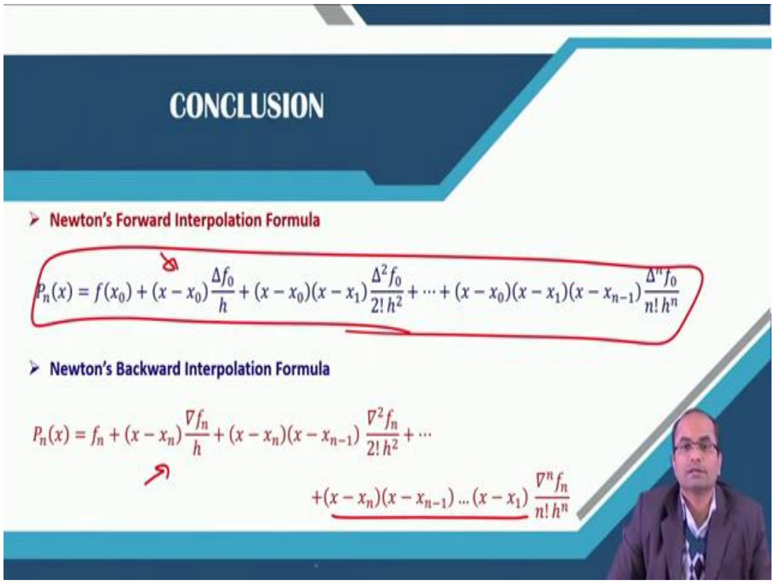

Having remember this one formula, other one can be written, because it is taking from the forward direction, this is taking from the backward direction and similarly, the other higher order terms can be interpreted. So, that is all for this lecture and I thank you for your attention.