Engineering Mathematics II
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Roots of Algebraic and Transcendental Equations (Contd.)
Lecture 25

Welcome back to lectures on Engineering Mathematics 2 and this is lecture number 25 on Roots of Algebraic and Transcendental Equations and we will continue. This is second lecture on this topic.

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So in the previous lecture we have already discussed Bisection Method and also the Fixed Point Iteration Method and in this lecture we will be talking about Newton-Raphson approach and also Secant approach for solving, for determining the roots of algebraic and transcendental equations.

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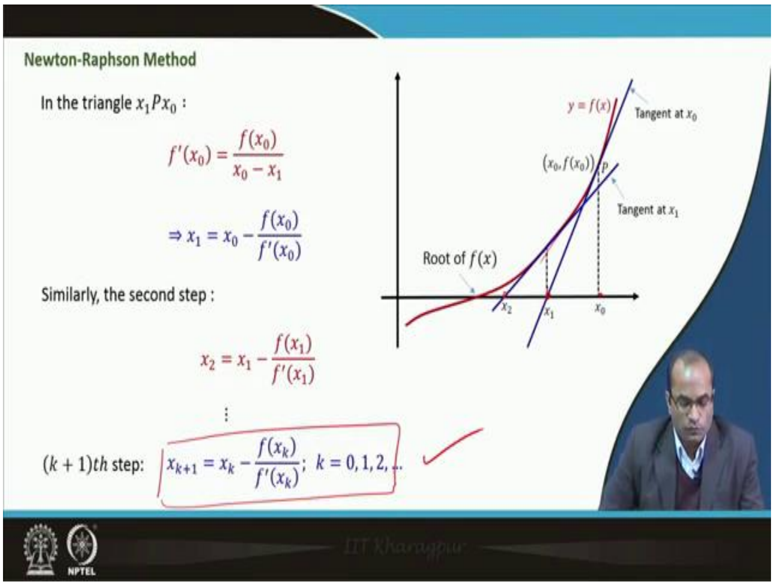

So coming to the Newton-Raphson method, suppose we have this y is equal to f x, the graph of this function f x and suppose this is the root here of this f x at this point where it crosses the x axis, so this is here x axis and then we have y axis there. So this Newton-Raphson approach or any other iteration method relies on the initial guess.

So here suppose we have this initial guess that is x naught and at this point then we have the f x naught and if we draw a tangent there at this x naught point, so this is the tangent at this x naught point and let us call this point p which is, which has the coordinate x naught and this f x naught and suppose this is the point x 1.

Then from this triangle p x1 p0 what do we observe, so from this triangle x1 P x0 we observe that this tangent at this x0 tan of this angle theta, if this is theta so we have here this tan theta equal to this f prime at x0 and this is equal to this distance here which is f x naught and divided by this x naught minus x1 so which is written here and if we simplify this now, for x1, so we are getting x naught minus f x naught divided by f prime x naught.

So if we start from this point x naught we can determine this x1 which is exactly the tangent at this x naught meets the x axis and this is our x1. We want to go in this way to this point to find the root of this y is equal to, f x equal to 0.So this x1 we can obtain with the help of this x naught with the initial guess using this formula. This is exactly the Newton-Raphson method.

So if we continue now with this x naught and again now draw a tangent at this x1 point which meets here this axis at this point which we are calling x2, then in a similar way now considering this new triangle, we can get x2 from x1. So just x0 is replaced here by x1 and then we are getting this new formula which is giving us x2 from x1.

So x1 minus f x1 over f prime at x1 and then we can continue this process. So here x2, then we will draw a tangent at this point which will be meeting here which is x3 and so on and finally as it seems we are approaching towards the root of the equation f x equal to 0.

Well, so in general then at k plus 1th step, this was the second step, this is first step so k plus 1th step what we will get or this is a general scheme now of this Newton-Raphson method which says that xk plus 1, the approximation at k plus 1, we can get from xk minus this f xk divided by f prime xk and then the k has to iterate so we get from x0 x1, then x1 to x2 etc. and we can approach towards the root of this equation f x equal to 0.

So this is one approach which, where we have seen the derivation of the scheme of Newton-Raphson through this geometrical interpretation which says that if we take any point x1 and then draw the tangent then the next point where this tangent cuts the x axis that will be our x1 point and again here draw the tangent, that will be the x2 point and so on. In this way we are getting this iterative scheme which is used for determining the root of algebraic and transcendental equations.

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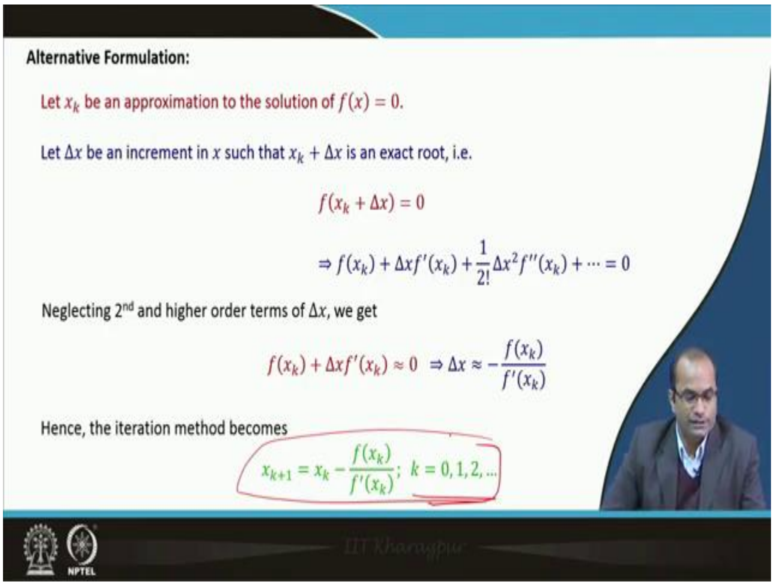

Now the alternative formulation which is based on more mathematics, so let us suppose this xk be an approximation of the solution of this f x equal to 0 and we also assume that delta x be an increment in x such that this x k plus delta x is an exact root. That means that f xk plus delta x equal to 0. So we are searching for such delta x so that f xk plus delta x becomes 0 meaning this x plus delta x becomes the root of this f x equal to 0.

Now we can use the Taylor series expansion of this f xk plus delta x. So f xk delta x f prime 1 over factorial 2 delta x square f double prime and so on, this will continue as an infinite series and if we neglect the second and the higher order term that means from here onward if we neglect this then we are not exactly getting equal but we are getting as an approximately equal to 0 for small delta x that will be a good approximation.

So we are getting that f xk plus this delta x f prime xk equal to, approximately equal to 0, not equal to 0 so we will not get exactly delta x which makes this xk plus delta x as the root of this f x equal to 0 but we will get a better approximation by having this delta x from this formulation.

So this gives us that delta xk is equal to minus f xk over f prime xk and the iteration method becomes, so we have started with xk and then with this xk plus delta xk we are getting a new approximation. So we are writing here xk plus 1, the new approximation is equal to xk minus this delta x which is f xk over f prime xk and then this k can go from 0,1,2,3 etc. So this is the alternative approach where we can see again a similar or the same formulation which was obtained geometrically on the previous slide.

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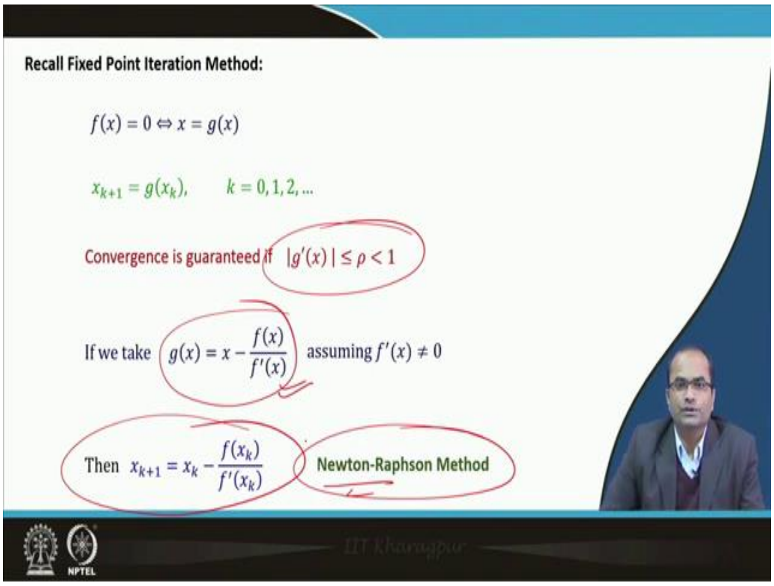

So just to recall, some remarks here, so recall from the Fixed Point Iteration Method, so f x equal to 0 was written as x equal to g x and then we have set up the iterative scheme that xk plus 1 is equal to g xk from this formulation g x is equal to g xk, and just remember that the convergence was guaranteed for this Fixed Point Iteration Method if the derivative is less than, strictly less than 1.

And what we will see here, that if we take actually g x is equal to x minus this f x over f prime x assuming that this is never 0, in that case this Fixed Point Iteration Method, that is this here becomes the Newton-Raphson method. So this Newton-Raphson method in a general sense, in a more general setting, it is a special case of the Fixed Point Iteration Method which we can observe just by fixing this g x as x minus f x over f prime x and then your Fixed Point Iteration Method will become the Newton-Raphson method.

So this is a special case, so naturally the convergence and all can be guaranteed if we have this g prime x less than 1. But we will study here the convergence because in this case when we are talking about this special case of Newton-Raphson method the convergence is better than the fixed point iteration approaches in general.

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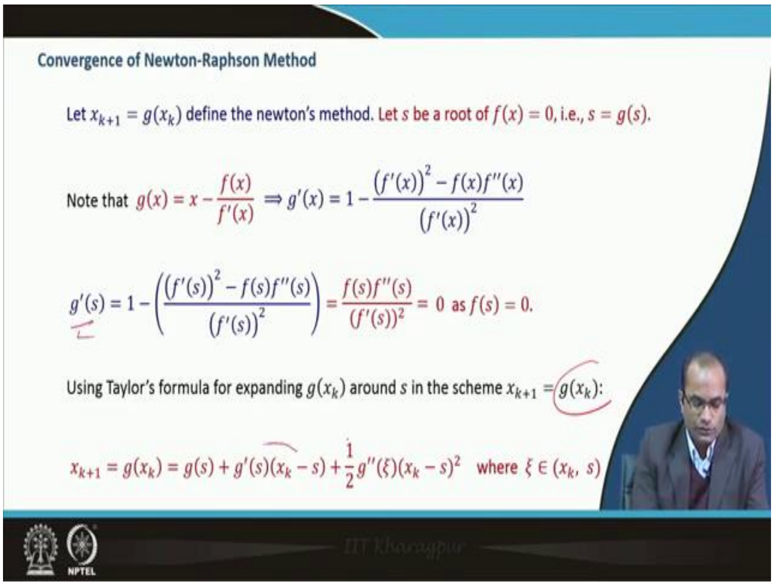

So let us talk about the convergence of this Newton-Raphson method. So let this xk plus 1 is equal to g xk defines the Newton's method, we had just discussed that. This g k can be set for the Newton's method and in this case here, let s be the root of f x equal to 0. So meaning is that s is equal to g s. So this s is the fixed point of this function g. Or in other words this s is equal to g s, so if s is the root of f x equal to 0.

Now we note that this g x is x minus f x over f prime x for Newton-Raphson method and that is what we have discussed on previous slide as a special case of Fixed Point Iteration Method. So having this g x now we can get the g prime, the derivative of this g x. So g prime is the derivative of this x which is 1 and then the derivative of this quotient where the quotient rule apply.

So f prime x square will be coming and then we have to take the derivative of this f prime x and then this f prime x will go up so this is square and then we have the minus sign, this f x as it is there and the derivative of this f prime x which is f double prime x. So this quotient rule we got the derivative of this g there.

Now if you want to evaluate this g prime at s, so the g prime at this point s which is, s is the fixed point of this g, we have 1 minus f prime, s here, f s, f double prime s and f prime s whole square. This is how we can get the derivative of g at the point s.

Now just simplifying this, because this whole square can go there, we are making this common denominator. So this will cancel out with this and finally we will get only the f s, f double prime s and f prime s whole square.

Now here this f s is equal to 0 because that is the root of this equation, s is the root of this f x equal to 0, that means f s is 0. So this is 0, then everything here, numerator becomes 0 and finally we have this 0. So the g prime s, so g prime exactly at that point where we have this root is nothing but 0. This is what we have observed now from this Newton-Raphson approach.

And now we can use the Taylor's formula here for expanding this g xk around this point exactly, the point s there. So using this Taylor series expansion here we have g s, we have g prime s then this x k minus s, g double prime, this is the Taylor's formula. So we have just stopped here by putting this xi which lies between this x and s and then square term so the xi is in the interval of this x k and s. So expanding this now and we know already that this g prime s is 0. So this term will get cancelled.

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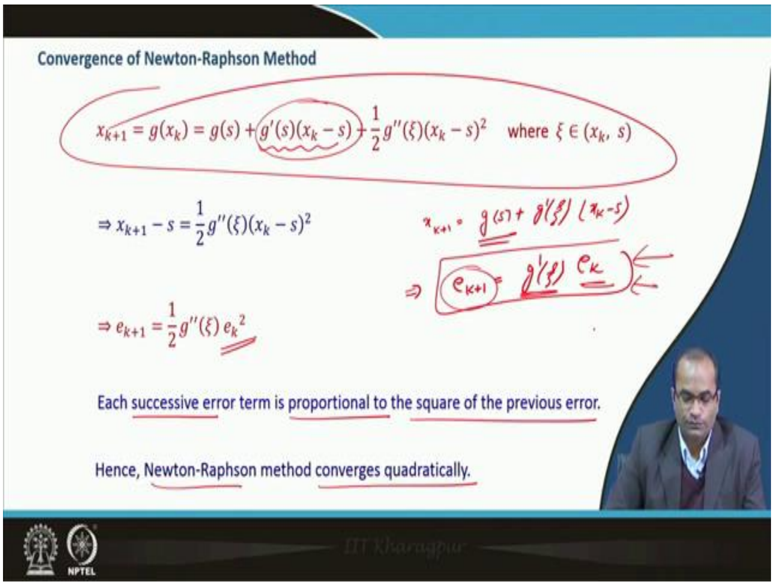

So let us continue. So having this g prime s 0, this gets cancelled and we have x k plus 1 g s is equal to s because that is the fixed point so we have minus s there and the right hand side then we are getting half g prime at xi and xk minus s whole square. Or in terms of the error we can write down, so this is the error because s is the actual value of the root and xk denotes the approximation, approximate value so here we have e xk, the right hand side we have ek whole square and then we have here half g double prime xi.

So having this relation what we can observe now that the error at k plus 1th step is equal to this value g double prime at some point xi and ek square. So this is important that whatever error we have at kth step that will be squared and then we will get the next error. So if it is a small value we will get very, very small value for e k plus 1 because we are getting here the square of the previous error.

So each successive error term is proportional to the square of the previous error and that is a good news because it shows the quadratic convergence towards the actual root. So the Newton-Raphson method converges quadratically. On the other hand, if this term is not 0, for example we take any other Fixed Point Iteration Method. This formulation will remain as it is but this g prime will not be 0 and our formulation will be like xk plus 1 is equal to this g s and then we have g prime, this xi there we can stop here itself, xk minus s.

So this is now the relation we will get that e k plus 1 is equal to g prime xi and ek. So this relation we will get if this g prime s is not equal to 0 and this relation shows that e k plus 1 will be some multiple of ek. There is no quadratic term appearing now. That means we will have the linear convergence.

So linear convergence is slow as compared to the quadratic convergence and therefore this Newton-Raphson method which is a special case of fixed point iteration approach gives the convergence of order 2 and therefore this a very well-known method which is used for computing or determining the root of the equation f x is equal to 0.

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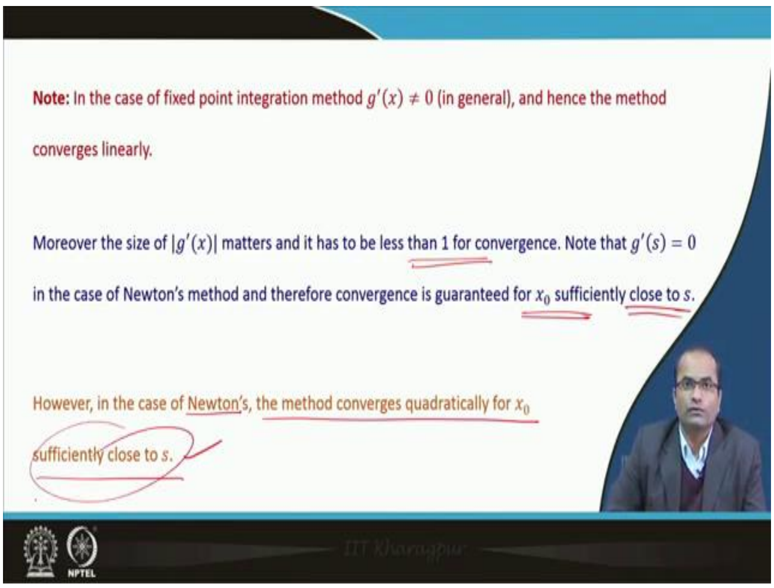

So in case of the Fixed Point Iteration Method which we have just discussed, that in general g prime x is not equal to 0 and hence the method converges this linearly and moreover the size of this g prime x matters and it has to be less than 1. That is what we have observed in Fixed Point Iteration Method for convergence.

Note that this for the fixed point, for the Newton's method we had g prime is 0 at least at this point s and therefore the convergence is guaranteed if we take this x naught the initial guess sufficiently close to s because the function is continuous and if g prime is 0 then there would be a neighborhood also where this g prime will be also close to 0 and that therefore less than 1.

So this convergence is guaranteed if we take x naught initial guess sufficiently close to s that is argument already coming from the Fixed Point Iteration Method. Moreover in case of this Newton's, the method converges quadratically. This is what we have seen for x naught sufficiently close to s. So this is the advantage of this Newton's method that it converges quadratically to the actual root of equation f x equal to 0.

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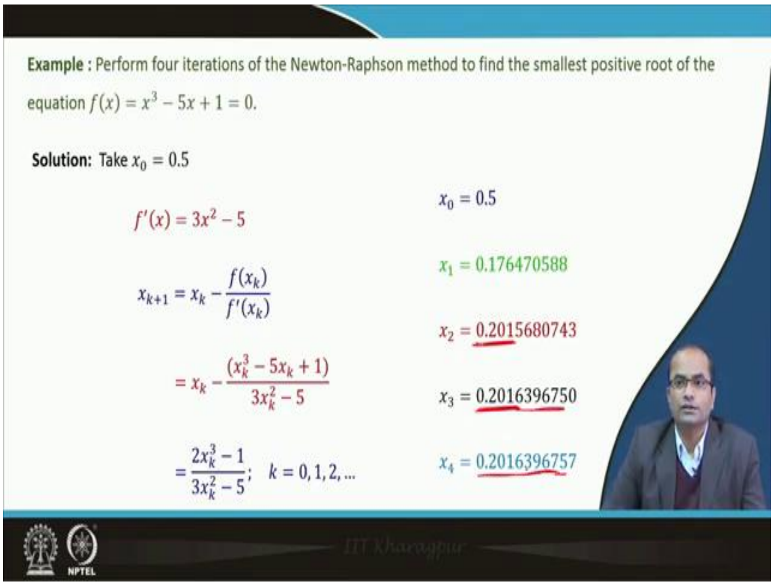

So we can go through the some numerical examples. So we have here; perform four iterations of the Newton-Raphson method to find the smallest positive root of the equation x cube minus 5 x plus 1 equal to 0. This is exactly the example which we have discussed earlier for the Bisection Method as well as the Fixed Point Iteration Method. And the smallest positive root was something 0.201.

So if we take the initial guess, for instance, here 0.5 and then we have to compute this f prime x which is directly coming from here 3 x square minus 5 and then we can set up this Newton's iterations xk plus 1 is equal to xk minus f xk over f prime xk. So xk then this is here in this f x we have written, replaced x by xk. So we have xk cube minus 5 xk and then plus 1, then we have here 3x square so 3 xk square and then we have minus 5.

Or we can simplify this. So we have finally this formulation that xk plus 1 is equal to this 2 xk cube minus 1 divided by 3 xk square minus 5 and then k can go for 0,1 etc. So taking this initial guess x0 is equal to 0.5. We will substitute here and then we will get x1 which is coming to be 0.176470588.

Using this x1 here now we will get x2. So we will get 0.201 and that is what we can see the convergence is very fast. We have done the same example in, using other, this Fixed Point approach Method or Bisection Method and after 6-7 iterations we were getting value which close to this 0.20 and now in the second iteration itself we are getting the value which is close to the actual root.

If we go further up to these 3 digits there is no problem. There is no change because this is exactly matching with the actual value also and for example if we go x4 we are matching to many, many digits. We are matching up to these so many digits. So just after 4 iterations we are getting a very good value, a very nice approximation using this Newton-Raphson method which was not the case earlier when we have solved this with Bisection approach or the Fixed Point Iteration Method.

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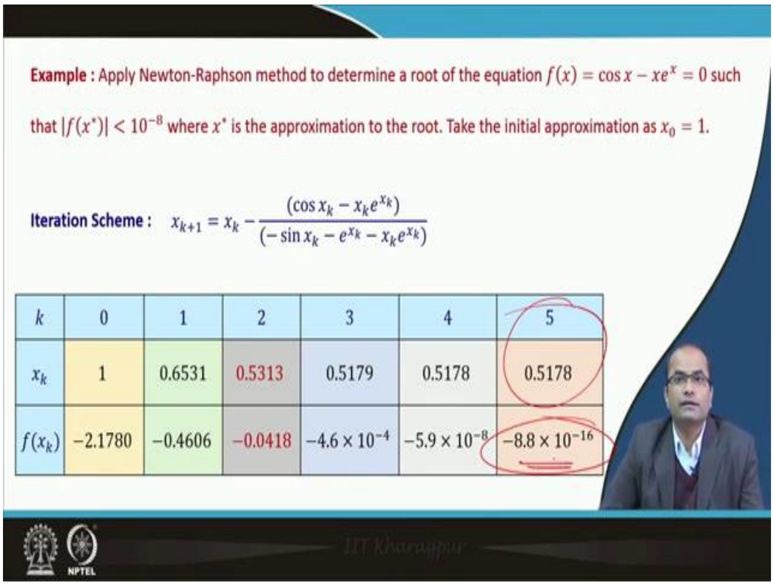

Apply, so another example, apply Newton-Raphson method to determine a root of the equation this cos x minus x e power x equal to 0 such that f x star is less than equal to 10 raised to the power minus 8 where x star is the approximation of the root. So we will stop the iterations when f x star, so x star denotes here the approximation will be the absolute value will be less than 10 power minus 8. So we don't know exactly how many iterations we have to do, but this is the stopping criteria.

This is kind of error we have set because this has to be 0 for the actual value of the root but we are talking about the approximations here. So we will continue the iterations until we achieve this accuracy of order 10 raised to power minus 8. It is given that; we take the initial guess, x naught equal to 1. So iteration scheme, we have xk plus 1 is equal to xk, this is f xk which is cos x minus x e power xk. And then its derivative, the cos will become minus sin and then here the product rule. So once you will get e power xk n and we will also get xk e power xk.

So this is the general scheme of the Newton-Raphson method and then we will take k here and then we will take xk and then the value f xk which will indicate where to stop exactly. So when k is equal to 0 that means the initial guess which is 1, the value at this, 1 of this f x. So here the f xk is evaluated and it is coming minus 2.1780 so naturally it is a initial guess and it is nowhere approx...

We are close to the set value of the accuracy. So we go further. This is the first iteration and then xk, so x1 is evaluated from this scheme. We are getting 0.6531 and here the f xk is 0.46 so it is still far from the desired accuracy. We go further. We take the second iteration, the value of xk and then we have f xk which minus 0.0418, again far from the desired accuracy. And after third iteration we compute xk from this scheme.

We have 0.5179 and the error reduces to 0.46 10 raised to power minus 4 which is always a quadratic so it will, one can check that this is the quadratic pattern, one can see that the next error is square of this with some constant. After the fourth iteration it goes really to 10 raised to power minus 8 but here we have 5.9 but we want this value to be less than 10 raised to power 8, so this is greater than 10 raised to power 8. So we have to go further, perhaps the next iteration will give exactly the desired accuracy.

So if we compute the values, after this fifth iteration we are getting 0.5178 and the accuracy here we are getting 8.8 10 raised to power minus 16. So it is even smaller than the desired accuracy but that is fine because we cannot stop here after fourth iteration. This error was more so we have to go once more here for one more iteration and this shows exactly that we have achieved the desired accuracy. So here f prime, f x is less than the given 10 raised to power minus 8.

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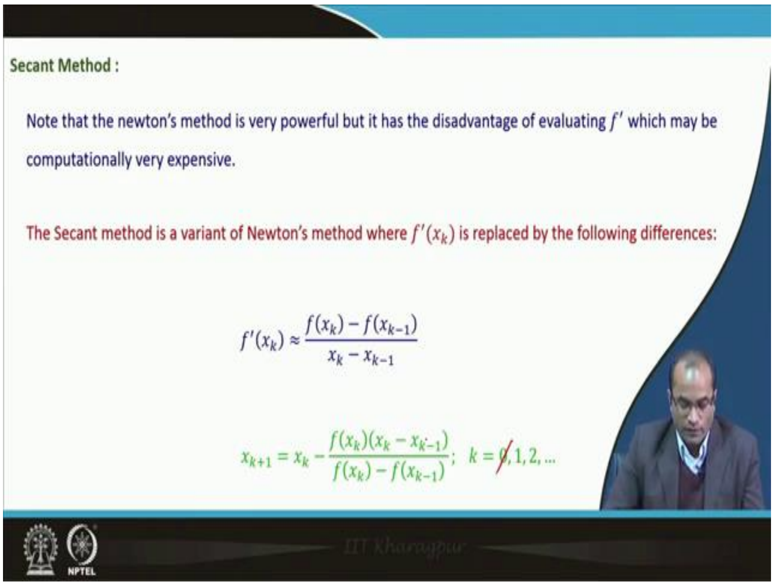

Well, the Secant Method, so it is the another variant of this Newton-Raphson method only so here we will not discuss much. Not that in the Newton's method is very powerful but it has a disadvantage. What is the disadvantage? Disadvantage of evaluating f prime because in the formulation, in the denominator f prime is coming which has to be evaluated at every iteration and this becomes usually computationally very expensive because for many functions getting a closed form of this derivative is not possible.

Then you have to also suffer for some kind of numerical approximations whatever. The Secant Method is a variant of Newton's method which avoids this evaluation of this f prime xk so indeed this f prime xk is replaced by this quotient which is the approximation of, approximation of this f prime, the derivative. So if we use this f prime x by this quotient f xk minus f xk minus 1 over xk minus xk minus 1 then we can avoid this derivative and our formulation becomes exactly xk plus 1 is equal to xk minus this f xk, xk minus xk and then this difference f xk minus f xk minus 1.

So using this formula here we can avoid the, we can avoid the evaluation of the derivative term but here we need for instance to get this x1, if we want to put here x0, then, so the first value we can get, that is x2. That means k has to be, has to be 1. So that means here x1 minus x0, so we need x1 and x0 to start these iterations unlike Newton-Raphson method where we just take one initial guess and then get the next one. Here we need two initial guesses and then we can compute the third and so on.

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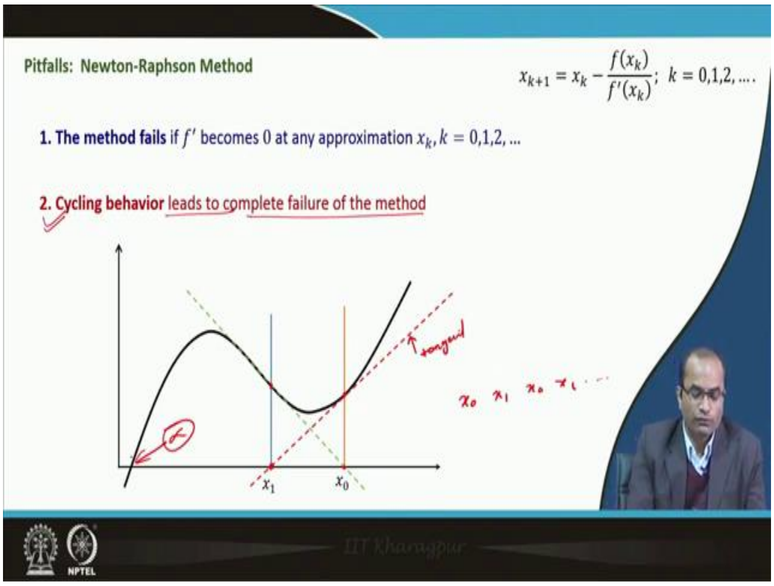

So some pitfalls of this Newton's method where it fails that we will discuss in the next two slides. So the first, the method fails if f prime becomes 0 at any approximation xk because the formula right away here tells that this f prime xk is in the denominator so if at some at approximate value of this xk, if this f prime becomes 0 then that will be the point of failure. We cannot compute the root of the f x equal to 0 so this is one pitfall.

Another one sometimes what is observed, that the cyclic behavior leads to the complete failure of the method. So what is the cyclic behavior? This is rare but it can happen which is demonstrated here with this figure. That suppose we have this x naught here and then we have this tangent at this x naught which meets, so that is the next point here x1 where it meets the x axis and then we have to draw the tangent at this point and this tangent now crosses the x axis exactly at x naught.

So what is happening now? You will get x naught, you will get x1, then x naught then x1 and so on. So there will be a sequence here x naught, x1, x naught, x1 and this is the cyclic behavior and we can never approach to the root which is, for instance, here. So this also can happen at some point of time if such a cyclic behavior occurs for some special functions.

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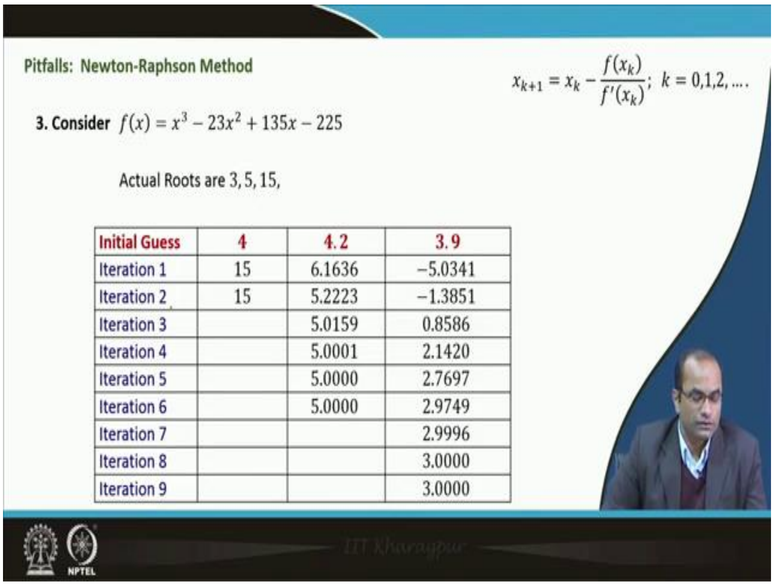

So some more, so consider for instance this f x, x cube minus 23 x square 135 x and this minus 225, the actual roots for this f x equal to 0 are 3, 5 and 15. So what will happen here? That is demonstrated in this table. So here the initial guess, if we take initial guess as 4 for instance then after iteration 1 itself we are getting 15 and then 15 and then 15, we are actually on the exact root just after first iteration itself. But that is not interesting. The point is that we have taken 4 which was close to 3 or 5 which were also the two roots of the given f x equal to 0 equation.

But this is converging to the root which was much far from the initial guess which is again a drawback because we expected that if we take the initial guess 4, it should go to 3 or 5. On the other hand if we made a slight variation here by taking the initial guess 4.2, so if we take 4.2 then it is actually converging to 5 or for instance we take 3.9 then it is converging to 3. But if we take 4 here it is straightaway going to 15. So such situations can also happen in this Newton's, Newton-Raphson method.

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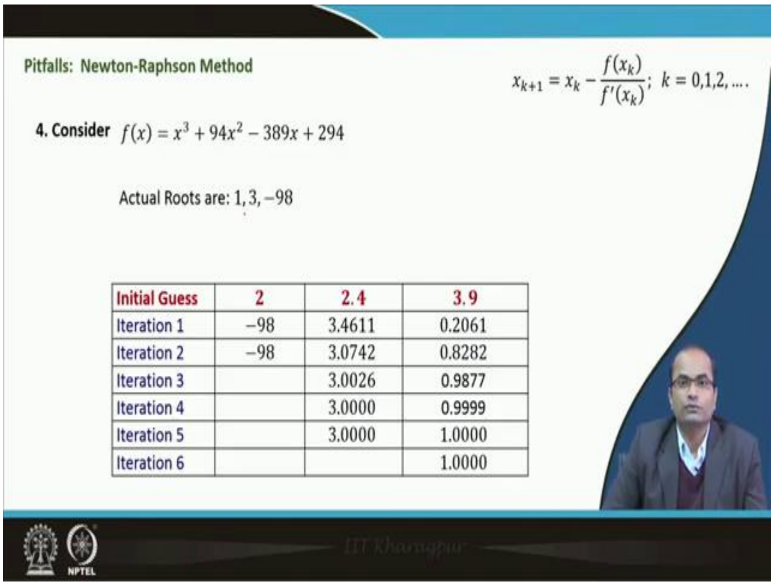

Another occurrence here for example we have x3 then 94 x square and then this x and this 294. In this case also our roots are 1, 3, and minus 98. If you see the behavior, if we take the initial guess 2 here, immediately after even first iteration it is going to minus 98. So it is going to some root but that is minus 98 and we can never expect by taking the root which is close to 1 and also close to 3, it is going to somewhere else which is minus 98.

But if we change here, for example 2.4 it is converging to 3, if we take 3.9, it is converging to 1. So again a strange behavior; by taking 2.4 which was close to this 3, we are getting 3 in this case but 3.9 is taking us to the 1. So one cannot, sometimes one cannot predict that taking initial guess something to which root it will actually converge to? So these examples demonstrate that pitfall of the Newton-Raphson method.

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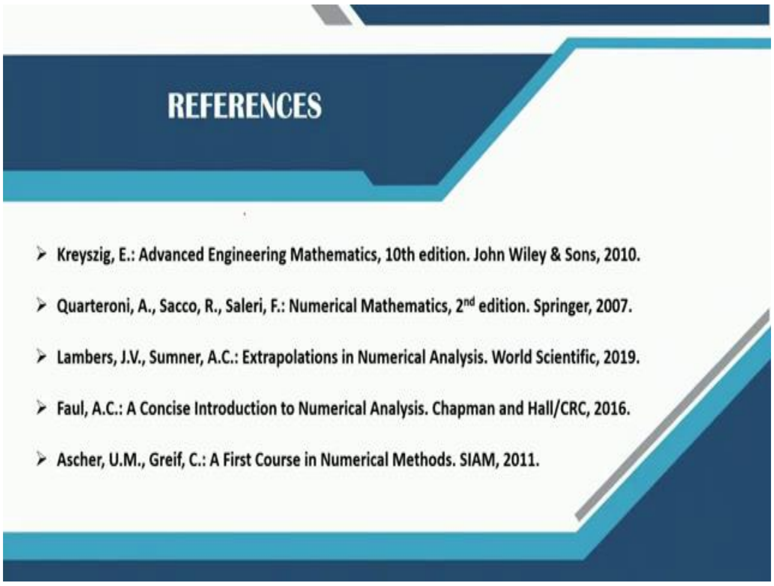

Well, these are the references we have used for preparing this lecture.

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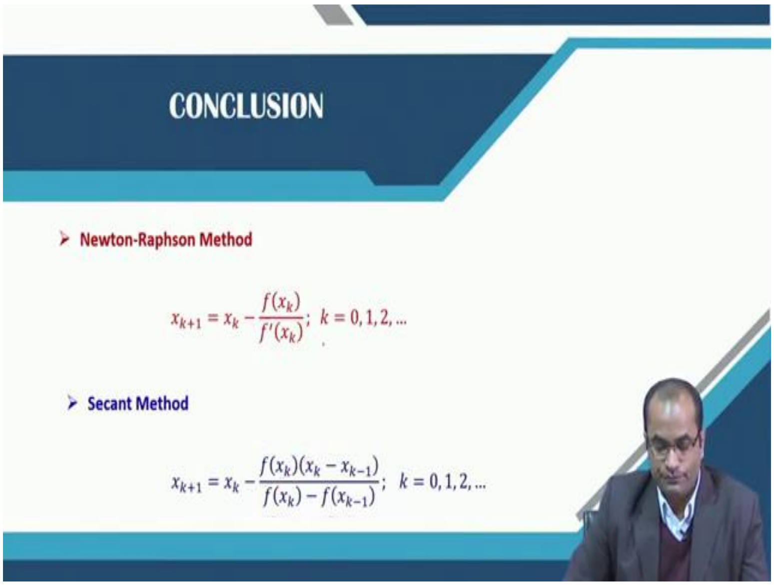

So we have discussed the Newton-Raphson method, a very powerful technique for determining the root of, or the roots of the algebraic and transcendental equations and this formulation here is based on this formula xk plus 1, xk, f xk, and f prime xk. There are some pitfalls which we have also discussed, when for instance this becomes 0 there was cyclic behavior and there was a problem with the taking initial guess and it is converging to very unexpected root.

The Secant Method, the problem of this Newton-Raphson method was calculation of this derivative term which can be avoided in the Secant Method and instead of this derivative, approximate value is taken as this quotient here. So this also works well when it is difficult to compute this f prime x. So that is all for this lecture and I thank you very much for your attention.