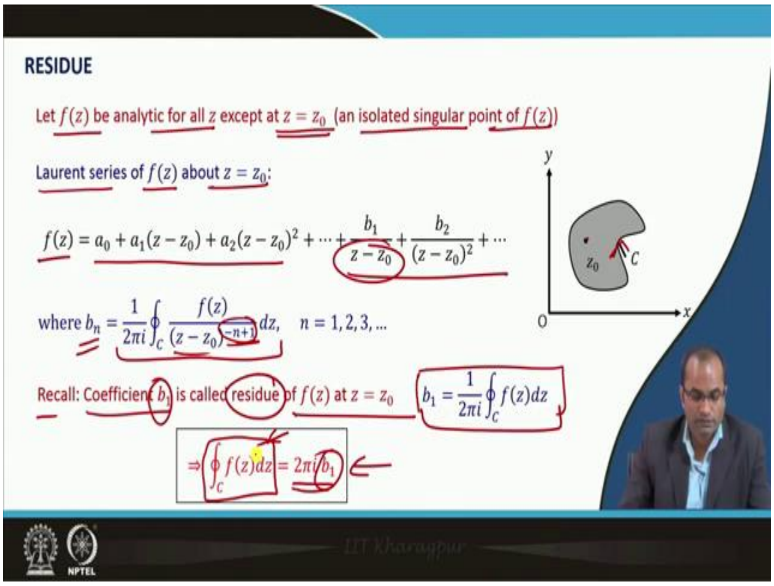
Engineering Mathematics – II
Professor Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur
Lecture 20
Residue

So, welcome back to lectures on Engineering Mathematics 2. So, this is lecture number 20 on Residue and this is the last lecture on this module Complex Analysis.

(Refer Slide Time: 00:25)

So, today we will be talking about what are the Residue and how to evaluate the residue and the most important theorem of this topic is the residue theorem which has application for evaluating a several complex integrals. So, you will also cover some portion for this evaluation of complex and integrals

(Refer Slide Time: 00:50)



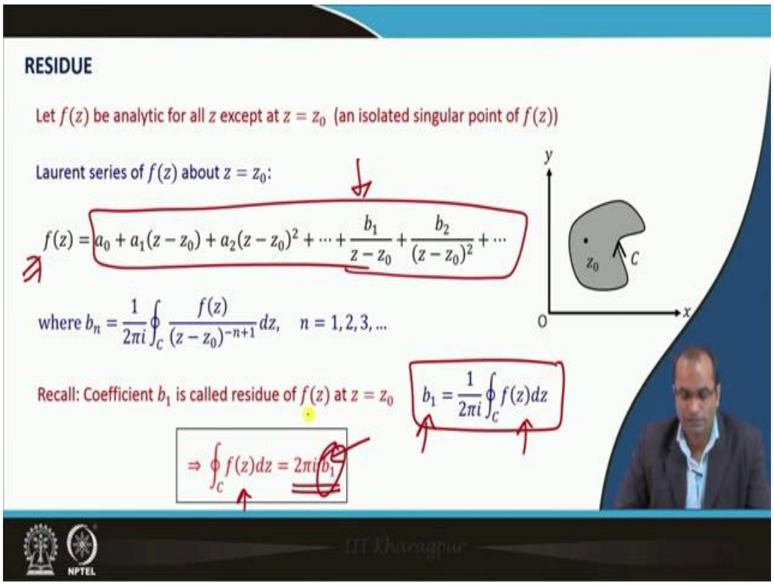
So, coming to the residue, so suppose this f z, a function which is analytic for all z except this z equal to z naught which is an isolated singular point of this function f z. So, the previous two chapters where we have discussed the singular points and also the expansion particular the Laurent series expansion that will be very useful to discuss this residue now.

So, the Laurent series of this f z about this z equal to z naught we can have a such term so the positive powers and the negative powers and where this bn this coefficient we can compute with the help of such integral that was already discussed in this topic where we have covered the Laurent series and just to recall we have this mentioned there, that this coefficient b1 the first coefficient or the coefficient of this 1 over 1 minus z. There it is called residue and now we will go more into the detail that what are different kind of and how to evaluate residues and then what are the applications of this residue.

But this coefficient b1 in this Laurent series expansion, the coefficient of 1 over z minus z 0, is called residue of this f z at this z equal to z naught point and remember this z naught is the isolated singular point of this z naught. So, now the b1 here which exactly directly coming from this integral it is 1 over to 2 Pi and f z, z minus z naught power this minus n plus 1. So, if n is 1 then this becomes 0. So, there is no term z minus z naught here.

So, this is a simple integral for this b1 equal to 1 over 2 Pi i and integral f z d z. So, indeed this is exactly the point why we are discussing this residue here. So, if we take a look at this, this integral over this C which is a curve which bounds exactly this point z naught is equal to 2 Pi i into b1.

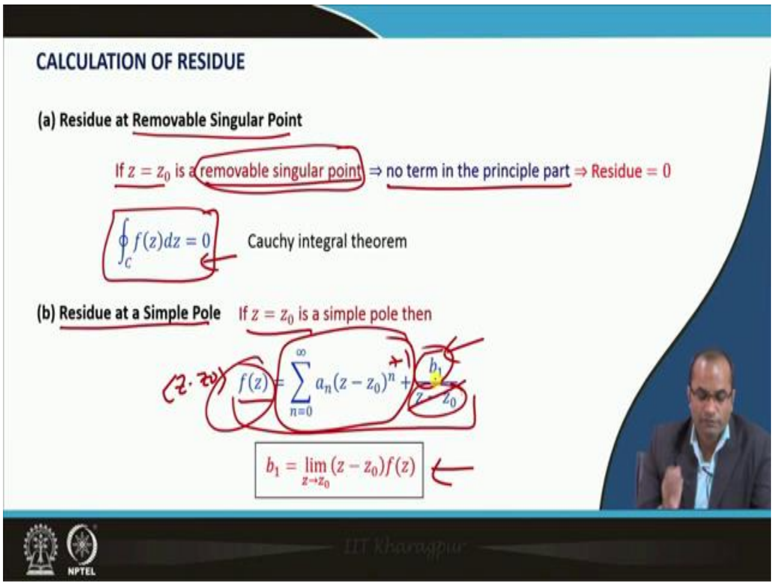
(Refer Slide Time: 03:28)

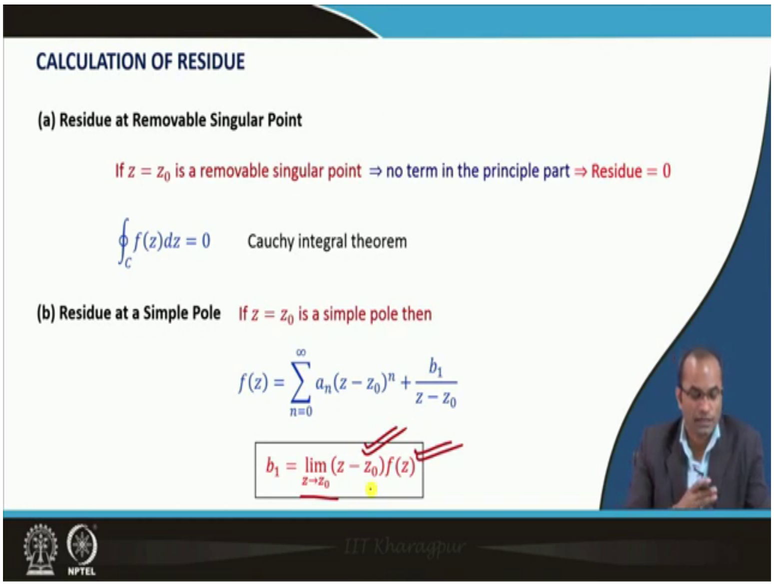


So, many complex integrals which are complicated to evaluate we can get the value with the help of this formula which says that this integral is going to be 2 Pi i into this b1. So, if we can find not using this formula naturally this b1 we will try, we will not use this formula in the expansion of also the Laurent series expansion if you remember we did not use this formula to evaluate these coefficients because computation of these complex integrals may be complicated.

So, by some other means we have expanded the function in terms of the Laurent series. Once we have the expansion we can get the b1 and once we have b1. We indeed we can get this integral. So it is goes other way around, to evaluate this integral we use this Laurent series expansion and then we can get b1 and we have this integral ready. So, but there are other ways also to evaluate this b1 the residue b1. So, we will now discuss in detail that how to get the residue at a point z equal to z naught which is isolated singular point of f z.

(Refer Slide Time: 04:41)





So, how to calculate the residue first we will discuss this and then later on using the formula we have just discussed we will evaluate some integrals. If z equals to z naught is a removable singular point and now the classification is coming into the picture. So, we have a singular point and if it is a removable singular point. Then we know that there is no term in the principle part of the Laurent series expansion and in that case if it does not have a principle part it does not have the negative powers of the z minus z naught.

So, naturally this b1 is going to be 0 and that means the residue is 0 which clearly says that this integral, the curve integral, curve is taken the curve enclose the simple closed curve which encloses this point z naught. So, this is going to be a 0 value of this integral which is

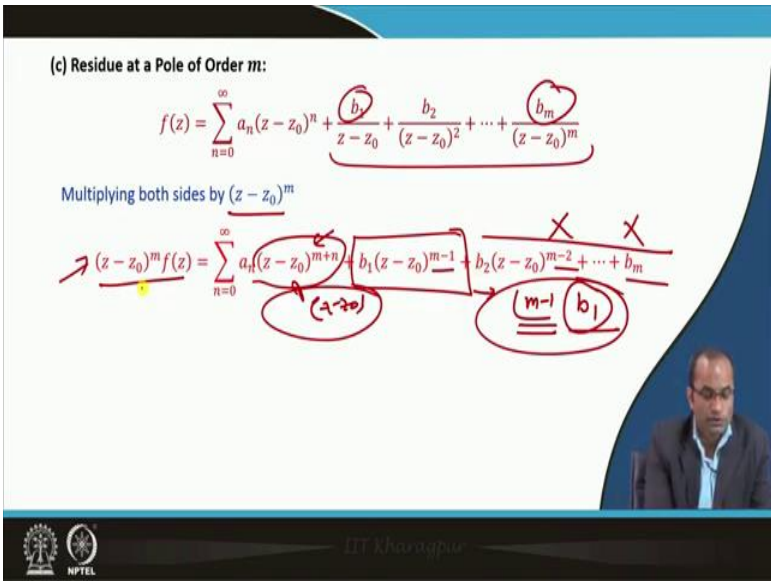
Cauchy integral theorem also can give us this result when we have a removable singular point that is the important point here to be noted.

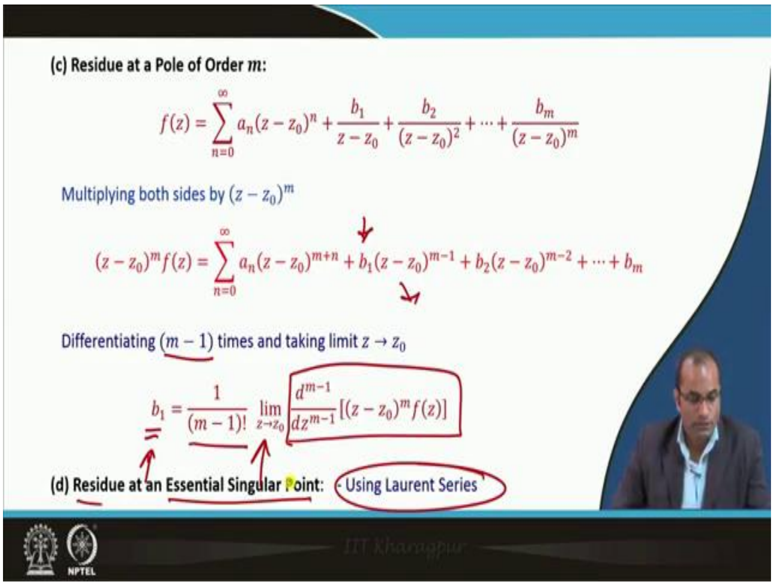
The second one if this point z is equal to z naught is a simple pole then how to get this residue. I mean always we can get with the help of the expansion but these are the some other ways we can easily get these residues. So, the, if we have a simple a pole then how to get the residue? In that case, this is going to be the expansion, the Laurent series expansion which we have discussed before that, I case of simple pole we have only just one term in the Laurent series expansion that b1 over z minus z naught and we are interested in what is this b1.

So, with the help of the limit it is easy because if we multiply here by this z minus z naught term here does not matter. It will be just increased and then this will cancel out. So, when we take the limit this everything will go to 0. So, this limit will give us directly b1, so to get this b1 we multiply this by z minus z naught and then take the limit z approaches to z naught.

So without expansion of this f z to the Laurent series we can get this b1 which is readily given here. So, if it is a simple pole we will just multiply by z minus z naught to this f z and take the limit and whatever number comes that is actually the residue.

(Refer Slide Time: 07:29)



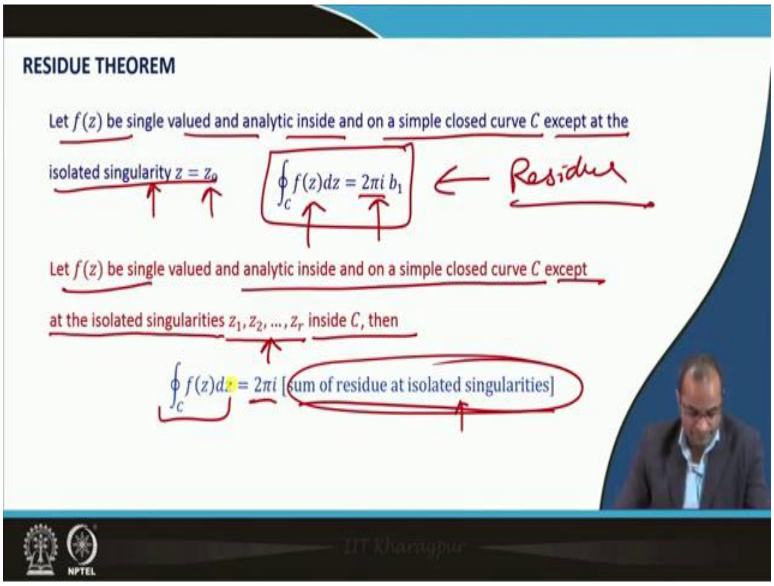


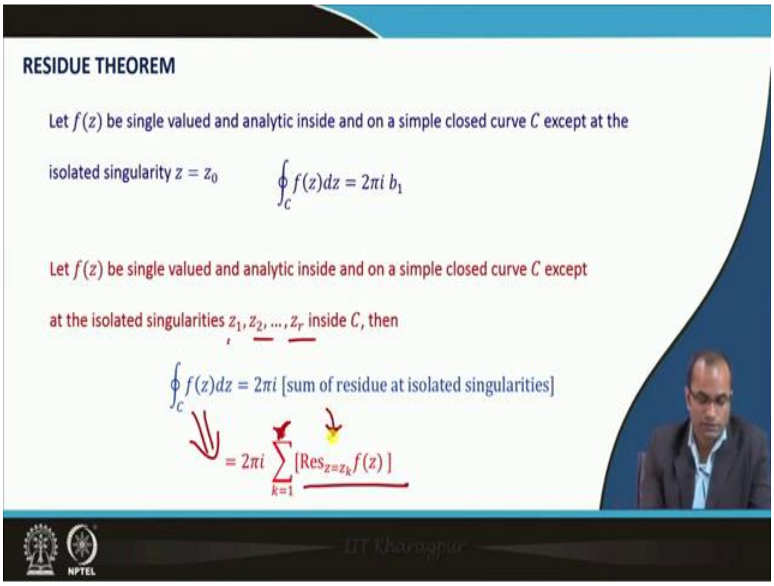
Now, if we have the pole of order m, so that means there are these first m terms in the Laurent series expansion with b1, b2, b3 and bm in this case what, how to get the residues, so how to get this b1. So, again with the, this limiting approach is going to help us. So, if you multiply both side with z minus z 0 power m. So, what will happen? We have this power m will increase here and then this is going to be m minus 1, m minus 2, the last one will be just bm.

So, this is the expression we will get for this one and now if we differentiate this m minus 1 times. So, all these terms will disappear only this term will come as a factorial m minus 1 and then we have here b1 all other these terms will, will be 0 and from here we will get b1 times this factorial 1 and here also we will get because the m plus n. So, more than m minus 1. So, here also you will get some powers of z minus z 0. But when we take the limit this portion will become 0 and here we will get just b1.

So, the conclusion here is how to get this b1 now. So we have already multiplied and if we differentiate m minus 1 times and then take the limit so we will get exactly b1 and this the b1 there will be a term here with factorial m minus 1 that will go to the right hand side. So, we have to differentiate this m minus 1 times and then take the limit. So, we can get this b1 directly if it is a pole of order m. Now, the residue at an essential singular point that we have to basically use the Laurent series expansion and then get the coefficient of this 1 over z minus z naught .

(Refer Slide Time: 09:43)



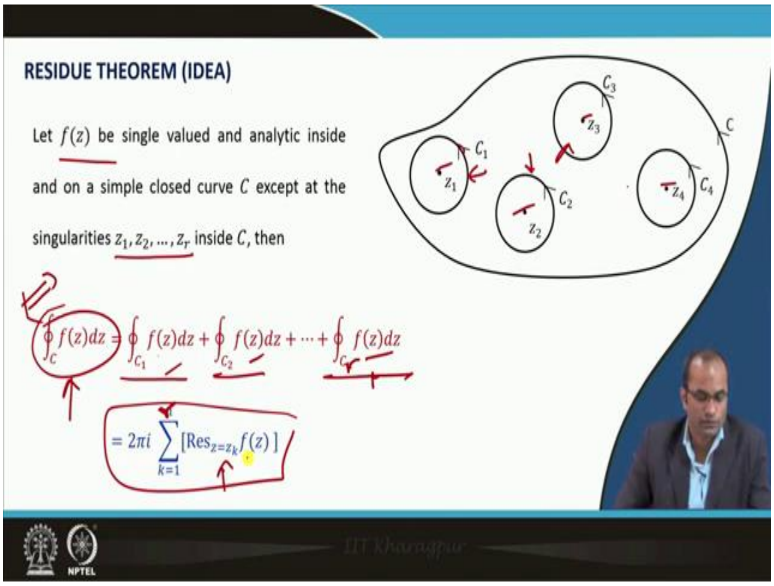


So, coming to this important theorem the residue theorem though we have discussed that integral that how that integral can be just evaluated 2 Pi i b1. So, just to recall we have discussed that this f z is single valued and analytic inside and on a simple closed curve C except at this isolated singularity is z equal to z naught. In that case we have seen that this integral value is nothing but 2 Pi I b1 this was the situation when we have just 1 this isolated singular point inside this closed, simple closed curve C.

But this can be generalized, so this is exactly the residue theorem but we can generalize this that f z is single valued, analytic inside a non-simple close curve C at and isolated singularities, except this isolated singularities z 1, z 2, z r. So, instead of one singularity z naught we have these r singularities inside C and in that case the result is just the 2 Pi i and the sum of residues at these isolated singularities.

So, we have r singularities that means we have to get these r residues and then we have to sum here these residues multiply by 2 Pi i and that will be the integral value over this curve and closes all these singularities. So, we have a this formula k from here 1 to basically r because we are taking z 1, z 2, z 3, z r and then these residue and the value of this integral can be, can easily be evaluated using this residue theorem.

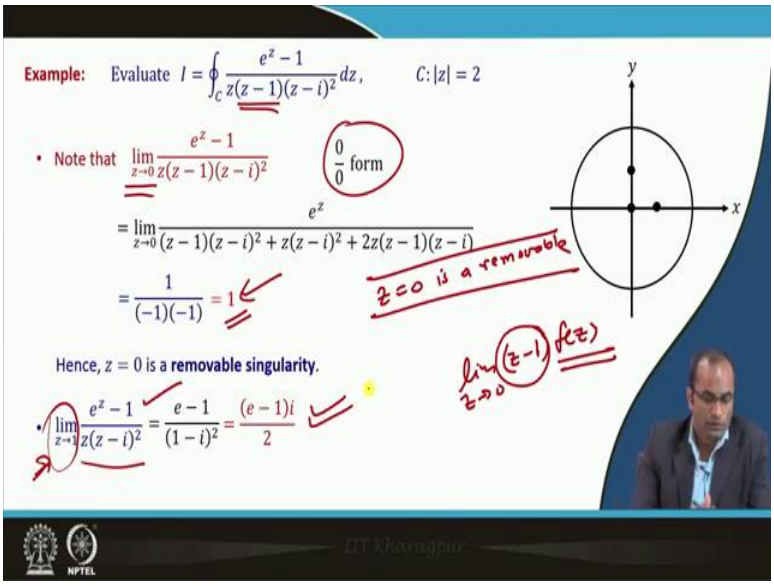
(Refer Slide Time: 11:36)

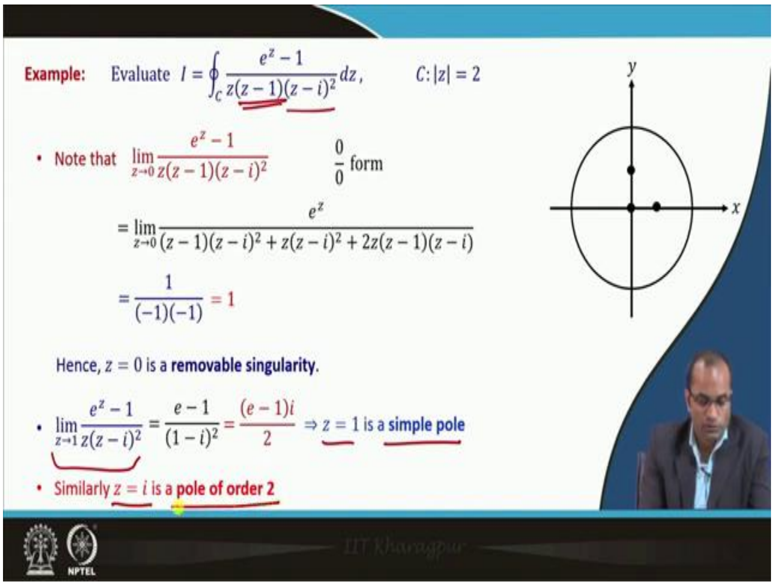


So, how these residue theorem is coming we will give just some idea here. So, we have this f z function which has these singularities r singularities that means this is the situations that we have a curve which encloses all these singularities here and for each singularities enclosing we have a curve c 1. So, these are isolated singularities so always we can have a, we can enclose this by another circles which does not have further singularities. So, these are the circles which encloses all these singularities, and having only one singularities in each circle and then by the Cauchy Theorem if you remember that this integral over the C f z is nothing but the integral of the sum of all these circles.

So, C1 over C2 over Cn and then for each we can use that we 2 Pi i and then the residue at that z 1 point here residues are z 2 point and so on. Residue at zn point or we are talking here n or r so this is going to be r and then here also we can go up to r. So, this is the residue theorem which is very helpful now to evaluate such integrals, the only difficulty that we have to evaluate these residues at different different points, but the evaluation of residues in each situation we have already discussed, so we can go for the evaluation part of the integrals.

(Refer Slide Time: 13:10)





So, for instance we want to evaluate this integral e power z minus 1 divided by this z, z minus 1 and z minus i. So, clearly what we see and the C here is the absolute value z greater than 2. So, the singular points are z equal to 0 here we will get 1 and then we will get i. So, these are the three singular points of this function of this integrant. 0, 1 and minus i, so this is 0 here and then we have 1 and then we have i there and this is the circle mod z equal to 2. So, in this case all these singular points are lying inside the circles.

So, we have to apply now the residue theorem meaning we have to compute the residue at all these three points, because all are lying inside the circle. So, these are the now for the first one, z equal to 0 we classify we are trying to classify. So, we first take just the limit what happened as z goes to 0. So, this is the integrant exactly the integrant here. So, this has 0 by 0 form because e power 0, 1 minus 1. This is 0 and then here we have also 0. So, 0 by 0 form we can deal with L'Hopital rule.

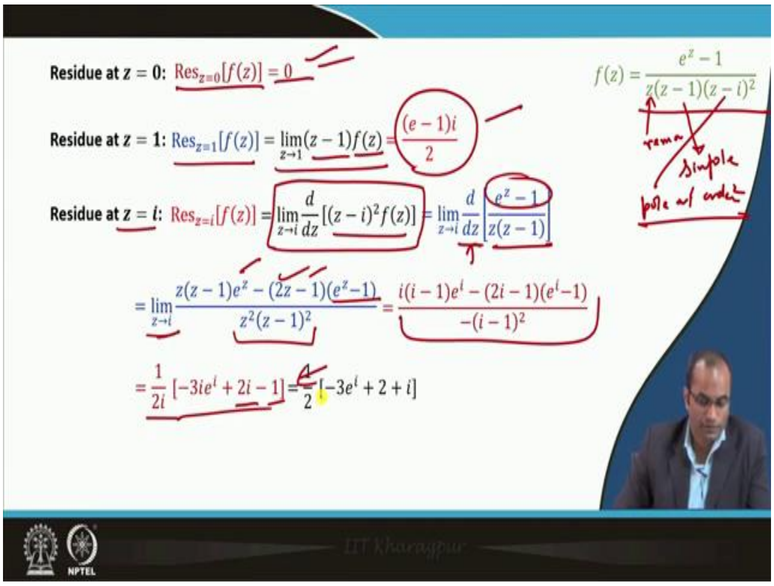
So, the numerator when we differentiate we will get e power z and from the denominator when we differentiate the product rule has to be applicable now. So, with respect to the first this z is differentiated. So, we have the remaining two, then the z minus 1 so we have z and z minus i whole square and then the third one is differentiated, so we have the two times and z minus i and z into z minus 1 remain as it is.

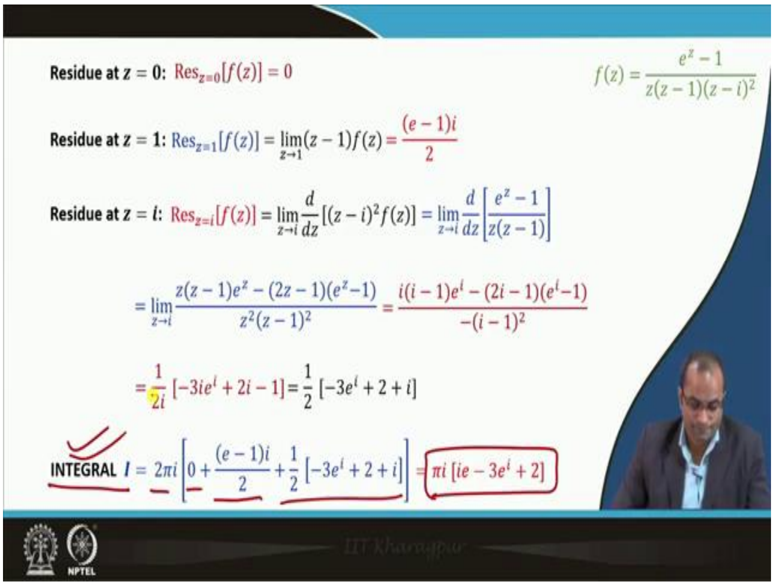
So, here by doing so now we can pass the limits and we observe that this is 1 over minus 1 into minus 1 that is 1. So, the limit here is 1, so the direct limit is coming as limit z approaches to 0, this is the value is 1 here that means this is the removable singularity z equal to 0 is a removable singularity and now we will discuss the other one. So, for z minus 1 now because this can be also observed here because here the L'Hopital rule was applicable and as a result finally we got this value as 1 but this is not going to be the case when z approaches to 1.

So, this z minus 1. So, what we are considering now, the limit z approaches to 1 and z minus 1 and the f z which is the integrant there. So, that means the z minus 1 will get cancel from this function and we have e power z minus 1 z into z minus i square. So, if we take this limit now, so we can substitute directly and we got this value e minus 1 i by 2 as the value of this limit.

So, it is clearly then the z is equal to 1 is a simple pole. So, this another singularity is classified as a simple pole and similarly we can talk about the z equal to i. So, in this case z minus i the whole square we have to multiply and then check the limit. So, this is going to be a pole of order 2.

(Refer Slide Time: 16:58)





So, all classification is done here the z equal to 0 is a removable singularity the z equal to 1 is a simple pole and z is equal to i is a pole of order 2. Now, for each we have to evaluate the residue. So, in the first case since it is a removable singularity the residues is going to be 0. So, removable singularity there will be no term with having negative powers in the Laurent series and therefore, the residue is going to be 0 and the residue at z is equal to 1.

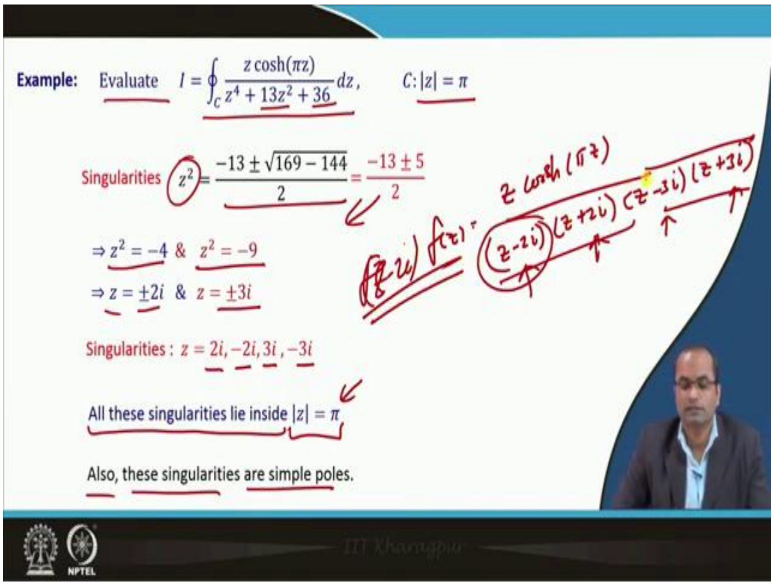
If we compute, now this was a simple pole, so we have to get basically this limit, this is what we have discussed earlier z minus 1 f z and z approaches to 1 we have to evaluate and this we have evaluated before also, so it was e minus 1 i by 2, this calculation says. z equal to i we know it is a pole of order 2 and then this we have to get this derivative ones for this z minus i square f z a term and if we do so this derivative here of the square means this z minus i square will be removed from this function now.

So, e power z minus 1 z z minus 1, we have to care the derivative and then we have to take the limit at z approaches to i. So, in this case z is equal to i and here this quotient rule will be applicable. So, the square of this and then we have to differentiate this that is e power z with z minus 1 and then again e power z minus 1 will remain and here the derivative will be 2 z minus 1 with minus there.

So, this is just the quotient rule, the simplification substituting this limit z is equal to i we are getting this number there. So, which can be further simplified here to get a simple minus 3 i e power i 2 i and then minus 1. So, this is the residue at z equal to i. So, we have these three residue, one was 0, the other one e minus 1 i by 2, the third one this one.

So, to get the integral value there, so the integral i will be 2 Pi i and the sum of all the residues. So, 2 Pi i and then we have 0 the second residue and this residue of this at the point where it is a pole of order 2 and now if we just simplify a bit more we will get this value of the desired integral. So, this was one of the applications which we have seen these residues play important role for the evaluation of these integrals without using the residue this might be little bit more complicated.

(Refer Slide Time: 20:11)



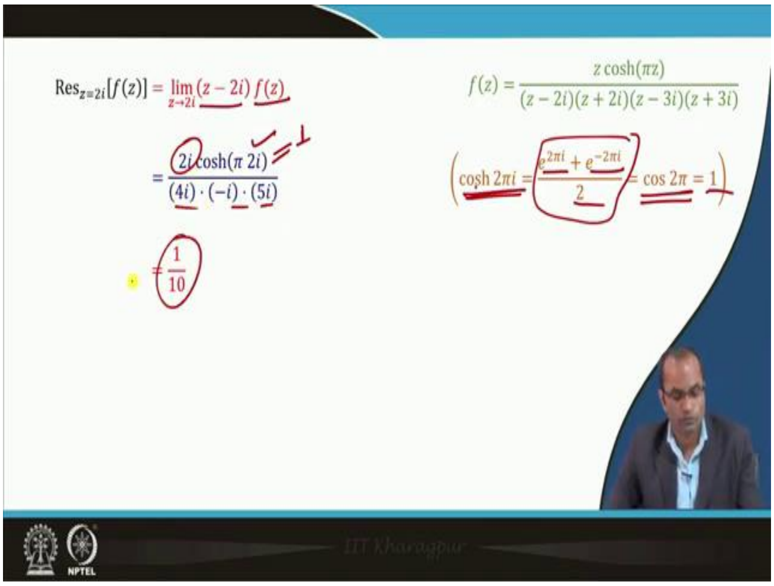
So, another example, which we will see in this lecture now we will evaluate this function this integral where we have the z 4 13 z square and 36 and the mod z is Pi. So, again we have to look for what are the singular points and then we have to classify the singular points and accordingly we have to get the residue and then the residue theorem. So, all these points to be used again.

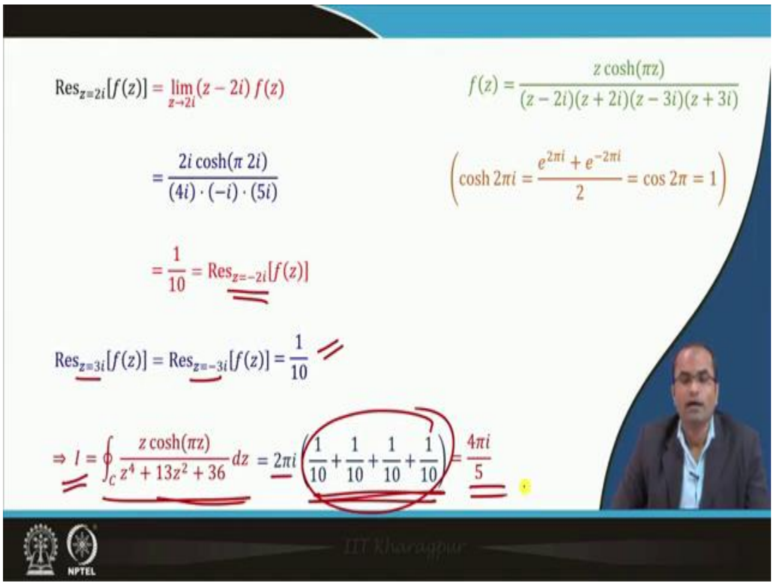
So, coming to the singularities so this is a quadratic equation in z 2. So, we can just solve it and what we observe that z square is equal to minus 4, z square is equal to minus 9 these are the two equations coming out for z square. So, that leads to that z is plus minus 2 i and z equal to plus minus 3 i are the zeroes of this polynomial which is appearing in the denominator.

So, all this points are going to be the singular points for this integrant. So, we have the singularities now 2 i minus 2 i 3 i and minus 3 i these are the four singularities and for each we have to evaluate residue. So, clearly what we will also see here that all these singularities, again are coming inside this mod z equal to Pi because Pi is more than 3 and now we are going here up to 2 or 3.

So, all these singularities are lying inside this circle of radius this Pi and again with now we have the idea we do not have to check each and every case because your function is now are like z and then cos hyperbolic Pi z and divided by we have the z minus 2 i e then we have z plus 2 i term, we have z minus 3 i term, we have z plus 3 i term. So, these each of them whether z equal to 2 i or z equal to minus 2 i or z equal to 3 i or z equal to minus 3 i, all these points are simple poles because if we multiply by z minus 2 i for instance to this f z here this z minus 2 i and then take the limit and the limit will exist, similarly, for z plus 2 i and so on. So, all these are a simple poles

(Refer Slide Time: 22:47)

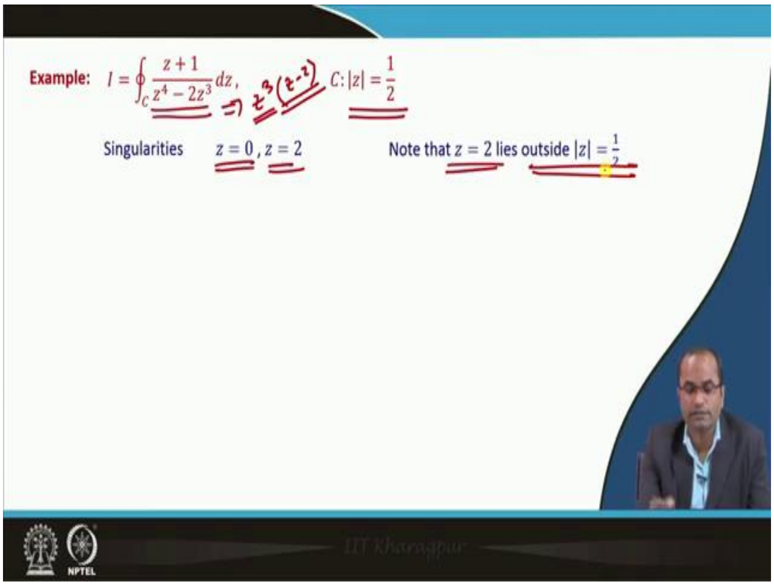


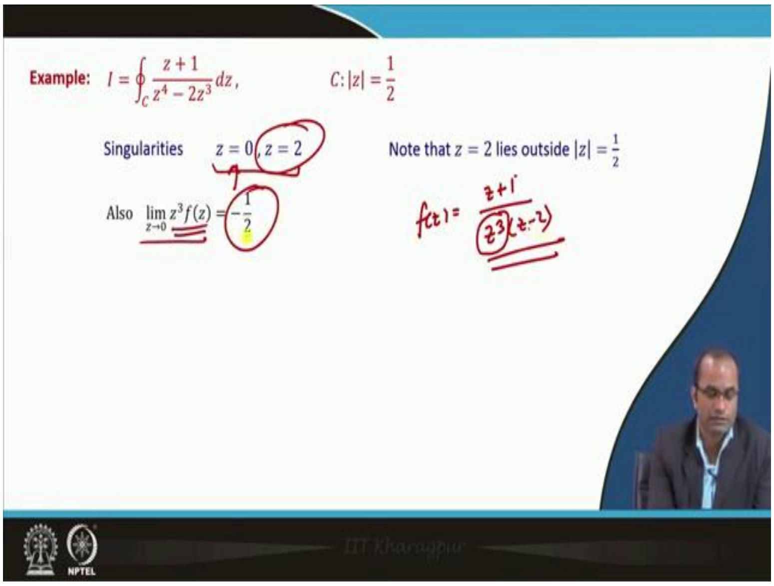


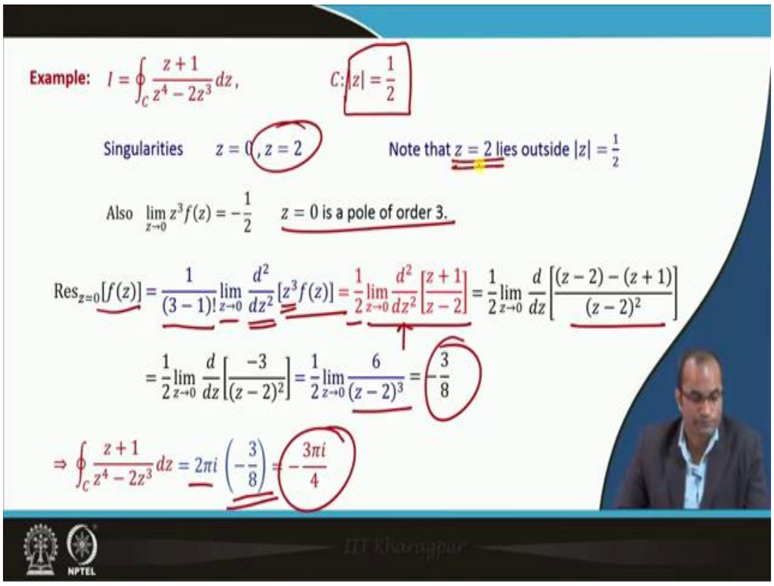
And now so the residue if we want to get for instance is z equal to 2 i for this f z, this was f z here. So, we have to multiply by the z minus 2 i and then f z. So, if we do so we have this result after putting the limit also for this z equal to 2 Pi everywhere and this cos hyperbolic 2 Pi i. So, cos hyperbolic 2 Pi i by definition we have e power this 2 Pi i e power minus 2 Pi i divide by 2 and this is nothing but the cos 2 Pi which is 1.

So, this is 1 there into 2 i. So, we get, then here also i i get cancel, so finally, we are getting this 1 over 10. So, the residue at this z is equal to 2 i is 1 over 10 and what we will, we can observe that if we compute the residue at z equal to minus 2 i also it is coming the same number as 1 over 10 and again if we compute now z equal to 3 i or z equal to minus 3 i they both are also coming to be exactly equal to 1 over 10. So, all these residues at all these simple poles the value is 1 over 10. So, we can now evaluate this integral which is 2 Pi i and the sum of all these residues which each of them is 1 by 10 so we have 4 by 10 here and then 2 Pi i. So, the final value of this integral is 4 Pi i by 5.

(Refer Slide Time: 24:28)







So, another example now, here we have this f z, z plus 1 over z 4 minus 2 z cube and the circle is mod z equal to half. So, now the circle of radius half. So, if we discuss the singularity here so we have actually z cube and then z minus 2. So, z is equals to 2 is a singularity and z is equal to 0 is also a singularity. So, these are the two singularities and we should note that this z is equal to 2 here now is outside this z e is equal to half. So, at least one of them is outside the circle. So, this is of no concern to us in the residue theorem now we do not have to compute the residue at z is equal to 2, we need to just compute the residue at z equal to 0.

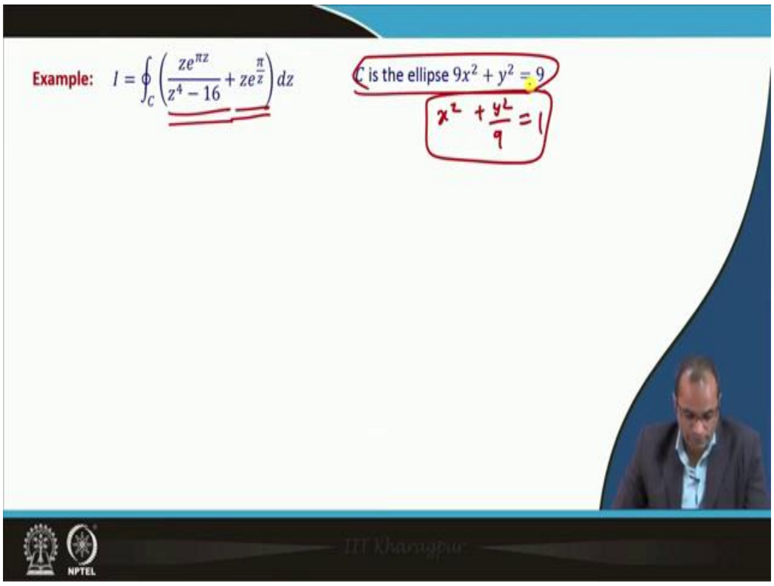
For z is equal to 0 the function f z is z plus 1 and then we have z cube and z minus 2. So, clearly it is visible here that we have to check with this z cube f z. So, this is going to be a pole of order 3 if this limit exist. And obviously when z 0, so 1, so minus half is coming there straight away.

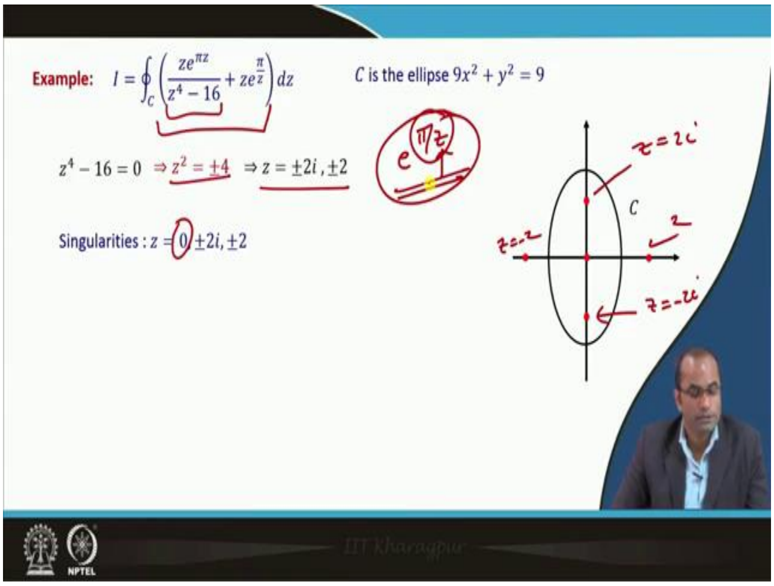
So, this is a pole of order 3 and then we have to compute the residue for this pole which is of order 3. So, the residue of this f z function at z is equal to 0 will be with this formula which we have just derived before 1 over this m minus 1. So, here 3 minus 1 and then again this m minus 1 at derivative that is the second derivative and then the limit z approaches to 0 we have to consider of this function z cube f z.

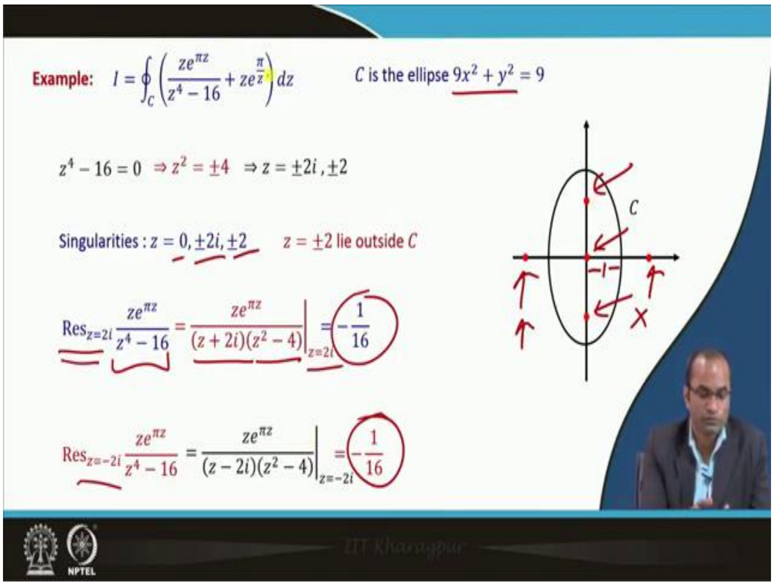
So, here we have 1 by 2 and then this limit. So, the second derivative for this z plus 1 over z minus 2 that is the portion when we multiply this to z cube. So, the z cube gets cancel and we have to take the double derivative simply for the functions. So, the first derivative is given here, then once again we have to first simplify this and then take the derivative again and finally take the limit. So, it is coming as minus 3 by 8 the residue at z equal to 0.

So, the value of this integral is going to 2 Pi i the value of the residue which is minus 3 by 8. So, the integral value is minus 3 Pi i by 4. So, what we have observed that one of the singular points of this f z was outside the region of this integration or the, this curve. So, we do not have to now compute naturally the residue at this point.

(Refer Slide Time: 27:37)







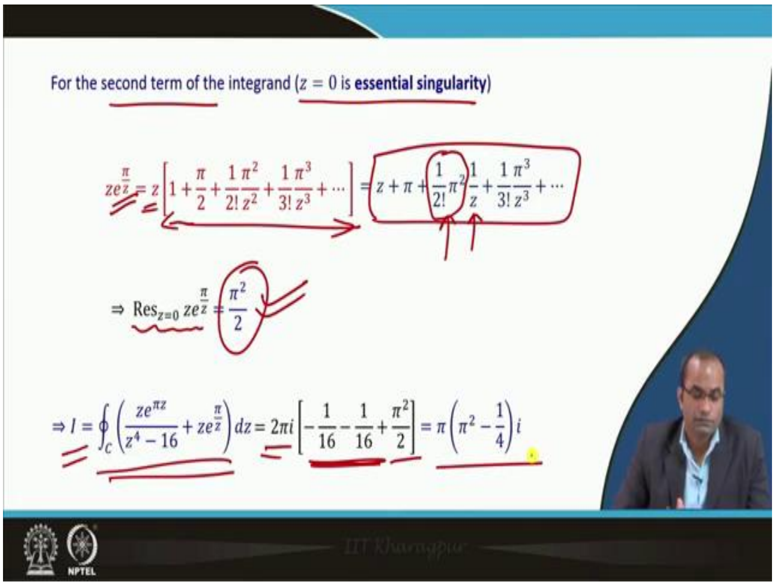
So, another example where we have the integrant with some of the two functions z e power Pi z and then z 4 minus 16, then we have here z and e power Pi z, then d z there and c is a circle with ellipse, which is x square plus this y square by 9 equal to 1 in the standard form. So, this is the ellipse there and if we look at the first portion here because of z square minus 16 we have z square is equal to plus minus 4 and therefore z is plus minus 2 i and we have plus minus 2. So, these are the four points were this function break down.

These are the singular points and we can easily see now. So, this is z equal to 2, here we have z equal to minus 2. Then we have z equal to 2 i and then we have z equal to minus 2 i. All these points and there is another point here z equal to 0 that is also singular point of this f z of this integrant because we have e power Pi over z and as z goes to 0 we have here infinity and everything goes to infinity now.

So, that is also the function is not defined. So that is also a singular point, so we have several singular points here 0 plus minus 2 i and plus minus 2 all are here. These two they lie outside because z is equal to 2 and here this distance is coming as 1 only. So, this is outside the domain. So therefore, these two will not be considered. We have the three points where the residue has to be evaluated.

So, first for the first function the z e power Pi z over z square minus the 16. So, if we compute the residue there at z equal to this 2 i and z equal to 2 i. So, z plus 2 i z square minus 4. So, except this z minus 2 i term the rest is considered here and then we will get this limit z equal to 2 i we got minus 1 over 16. Similarly, for minus 2 i also we can evaluate this and the same number minus 1 over 16 will come, so these are the two points, there we have evaluated.

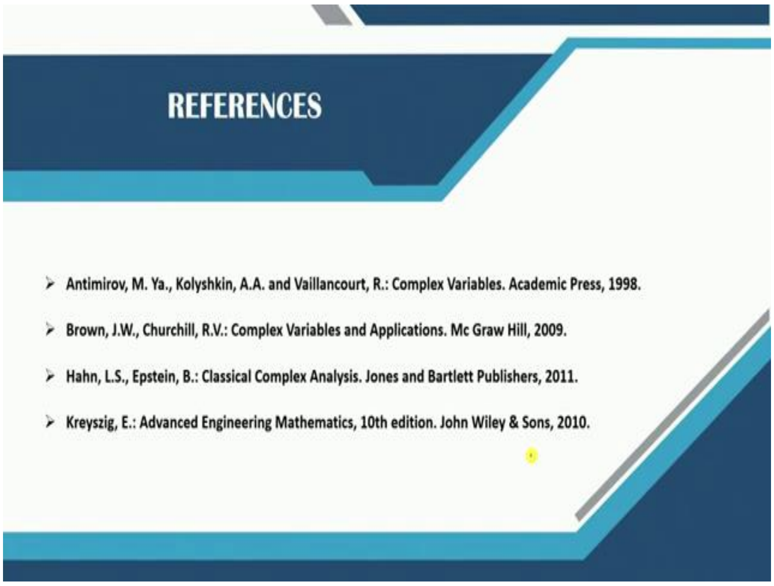
(Refer Slide Time: 30:27)



z equal to 0 also we have to see what is, which is appearing in the second term and this is an essential singularity as we have seen it is appearing in e power Pi z. So, we have z and then we have the expansion for e power Pi z. So, finally this is the expansion for this given function and as far as this 1 over is concerned this is the coefficient here for 1 over z and therefore the residue of this function z e power Pi z is Pi square by 2 because this is going to be the b1 term the first term where the negative power starts.

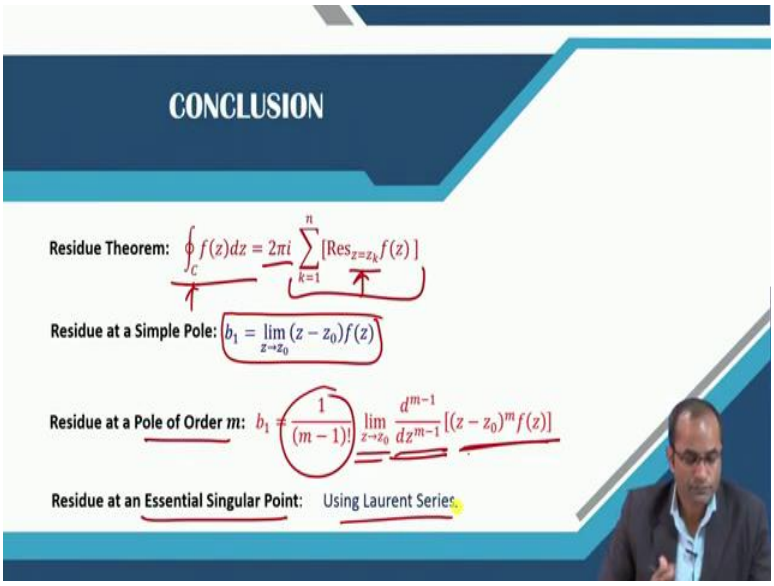
So, this is the first term 1 over z and this is 1 over factorial 2 Pi square that is the residue we have and the integral then this integral which is to be evaluated now. So, we have 2 Pi i the sum of the two residues and then the third one here were the functions is having essential singularity Pi square by 2. So, this adds to this Pi Pi square minus 1 by 4 and i. So, that was the example were we have also considered one of the singular point as essential singularity.

(Refer Slide Time: 31:44)



And now these are the references we have used for preparing this lecture.

(Refer Slide Time: 31:51)



Just to conclude, so we have discussed the residue theorem which is simple that 2 Pi i and the sum of the residues. We have to consider and these points are the singular points z k point which lies inside this boundary C. Residue at a simple pole can be easily computed by just taking this limit z minus z naught and f z. If it is a pole of order m then we have to take this m minus 1th derivative of this function z minus z naught power m f z and then take the limit also we have to do divide this 1 over factorial m minus 1.

If it is essential singular points we have to use the Laurent series expansion and then we can find the coefficient of this first term where the negative powers starts. So, that is all for this lecture and I thank you very much for your attention.