Engineering Mathematics-II
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Lecture 19
Singularities

So, welcome back to lectures on Engineering Mathematics 2. So, this is lecture number 19 on Singularities.

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So, today we will discuss what are the singularities of functions of complex variables and then how to classify these singularities into different categories.

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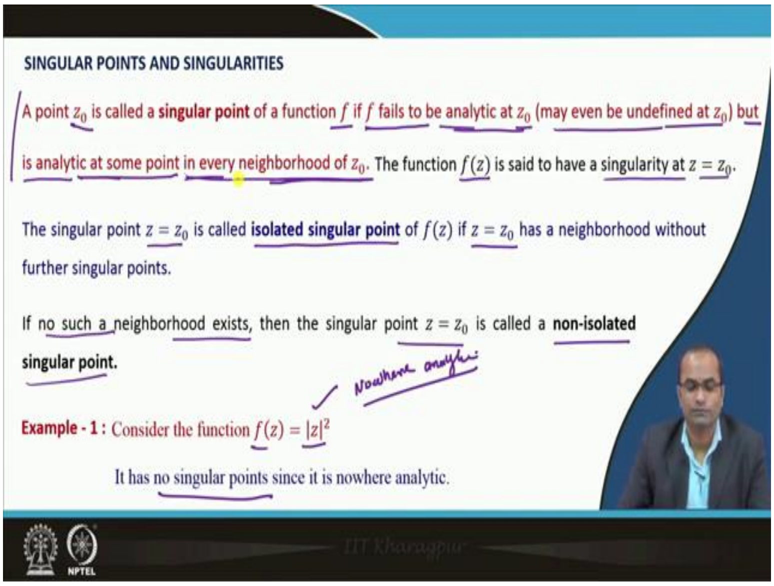

So, what is the singular point and what do we mean by singularities? So, a point this Z 0 is called singular point of a function fz, so it is a complex valued function, so the function of complex variables, if this functions fails to be analytic at z0, or this function may not be defined at z0, but this function is analytic at some point.

So, there should be at least one point where in the neighbourhood of this z0 where the function is analytic, then we call such a point a singular point. So, at this point, singular point either the function is not defined or it is not analytic but in the neighbourhood of this z0 at some point this function is analytic then we call that this is a singular point. And the function fz is said to have a singularity at this point z naught.

So, the singular point z naught is called isolated singular point of this function fz, if this z equal to z0 has a neighbourhood without further singular point. So, there must exist in neighbourhood in this neighbourhood of this z0 where the function does not have a further singular point then we call this isolated singular point.

If no such point in the neighbourhood exist that means every neighbourhood has a singular point then such a singular point is called non-isolated singular point. So, this is what we have discussed what is the singular point and further what is the isolated and non-isolated singular point.

So, consider for example, this function fz equal to mod z square or the absolute value of this z modulus of z square. So, we have already discussed this function and just to recall that this function is nowhere analytic. So, then what we can say about the singular point, so naturally this f fails to be analytic at any point but there is no singular point for this function and the reason is, because this function is nowhere analytic, nowhere analytic.

And therefore we do not have a neighbourhood where the function is analytic. So, we do not have some point in the neighbourhood where the function is analytic, and therefore, according to this definition itself there is no singular point for this function though every point, at every point this function is not analytic.

But this does not have a singular point, so it has no singular point since it is nowhere analytic. So, the second part is not satisfied that the function is analytic at some point in every neighbourhood of z0, so this is not possible here because the function is nowhere analytic. So, here there is no singular point for instance for this function.

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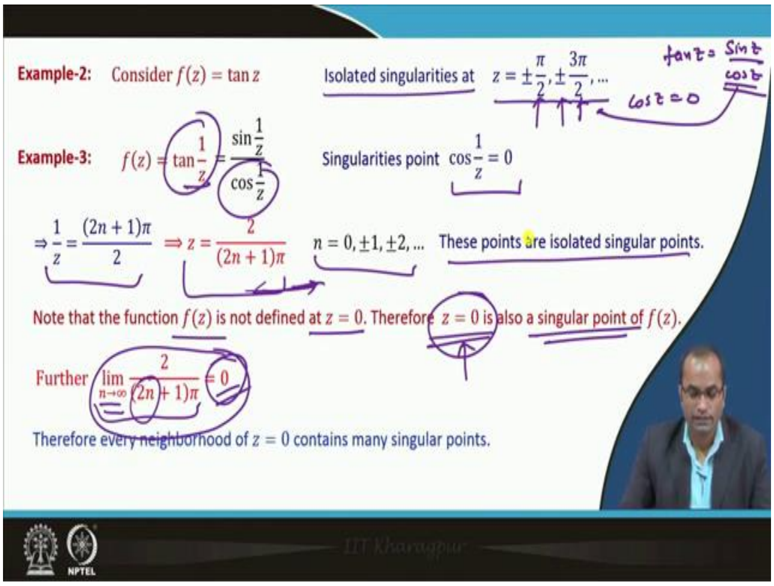

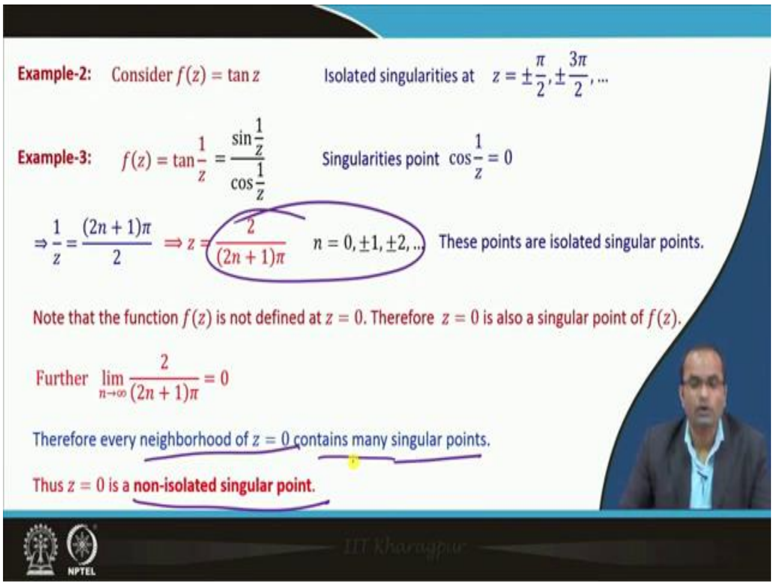

However, if we consider fz equal to tan z for instance, then we have isolated singularities at all these points because this tan z we can write as sin z over this cos z and this cos z is 0 this cos z is 0 at all these points pi by 2, 3 pi by 2 with the plus or minus at all these points we have the singularity of this function.

The function will become this infinity, so it is not defined as such so there is no point of analyticity, at all other points the function is analytic and therefore, we, the definition of this singular point is satisfied and all these points are singular point, because in the neighbourhood of these points the function is analytic, only at this point the function is not analytic, and therefore these are the isolated singularities of this function.

If we consider now instead of tan z, tan 1 over z in that case we can again write a sin by this cos, cos sin 1 over z and again the singularities or say singular points we can discuss by setting this cos 1 over z to 0 and again with the same argument what we see that this 1 over z is 2n plus 1 pi by 2, then this cos z will become 0 and then we have these singularities.

So, here the z comes to be 2 over, 2n plus 1 into pi and for all these n so this function is not analytic, and hence these are the singular points of this function. So, these all are the isolated singular points, because in the neighbourhood we can find the points where the function is analytic.

So, these are the isolated singular points, but what we should note that the fz is also not defined at z equal to 0 because this function is tan 1 over z, so at z equal 0 also the function is not defined and therefore, this z equal to 0 is also a singular point of this fz. Now, we will see, we will observe that whether it is a isolated or non-isolated singular points.

So, if we notice here that what happened to these points 2 over 2n plus 1 pi which are the isolated singular points. If we take n approaches to infinity, so n approaches to infinity this is going to 0. So, for large n as we are approaching n to infinity we are basically approaching to 0 and the 0 is also a singular point.

So, in the neighbourhood of 0, whatever, however, small neighbourhood we take about this z equal to 0 there will be a singular point coming from this sequence of these points, so that is the difference now for z equal to 0 we cannot find a neighbourhood where the function does not have a further singular point. Because the sequence is approaching to 0, so whatever, however, small neighbourhood we take around this z equal to 0 there will be some points which are singular again.

So, this is not a isolated singular point z equal to 0 is not a, not an isolated singular point, indeed it is a non-isolated singular point, because every neighbourhood of this z equal to 0 contains many singular points which are exactly coming with these sequence of points which we have discussed as singular points, so that is an example here where we can see that there are non-isolated singular points as well. So, in this case this z equal to 0 is a non-isolated singular point.

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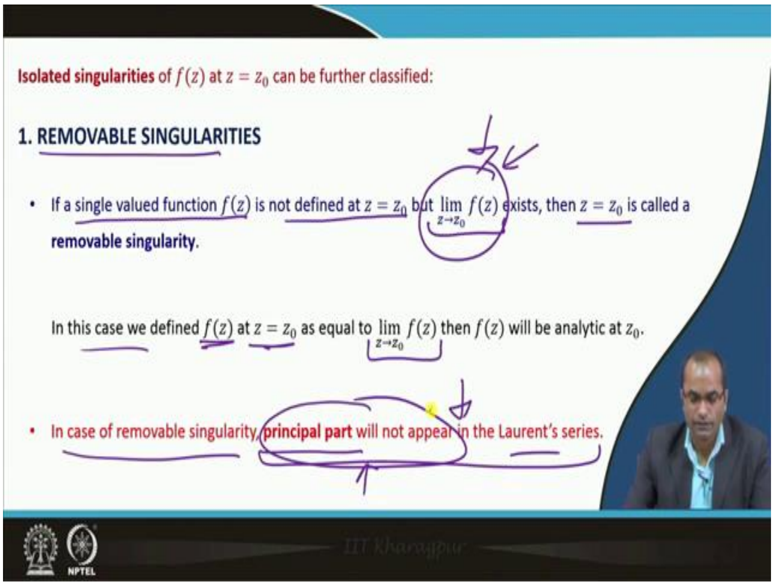

So, these isolated singularities of this fz at z equal to z0 can be further classified, based on the following description, the one is called removable singularities. So, what is removable singularity? If single valued function this fz is not defined at z equal z naught, if the function is not defined at this point z equal z naught, however, this limit when we take z to z naught fz exist then this point is called a removable singular point or it is called removable singularities.

Because in this case if we define the function at z equal to z0 with this limit which exist then this fz will become analytic at z naught point and that is the reason we call it as a removable singularity of the function. So, in case of removable singularity and this is the relation which we have with the Laurent series because the Laurent series plays very important role in this classification of this singularities.

So, the first we have discussed this removable singularity and for the removable singularity if we expand the given function about this z equal that z naught point then in the Laurent series the principal part will not appear that means the negative power of z minus z naught will not appear in the Laurent series and then we can so what is the process that around this z equal to z naught we will expand this function using this the idea of a Laurent series that means the plus positive and the negative powers of z minus z naught.

And if it comes out to be that the principal part meaning the portion where we have the negative powers of z minus z naught. So, if those terms do not exist in the Laurent series then we call that this is a removable singularity. So, either the direct we can get using this limit if this limit exist then we also call removable singularity or we will expand into this Laurent series and if there is no power the negative power of z minus z0 in the series then we call that this is a removable singularity. So, for each we have these two approaches, one we can classify directly by limit, another one using the Laurent series.

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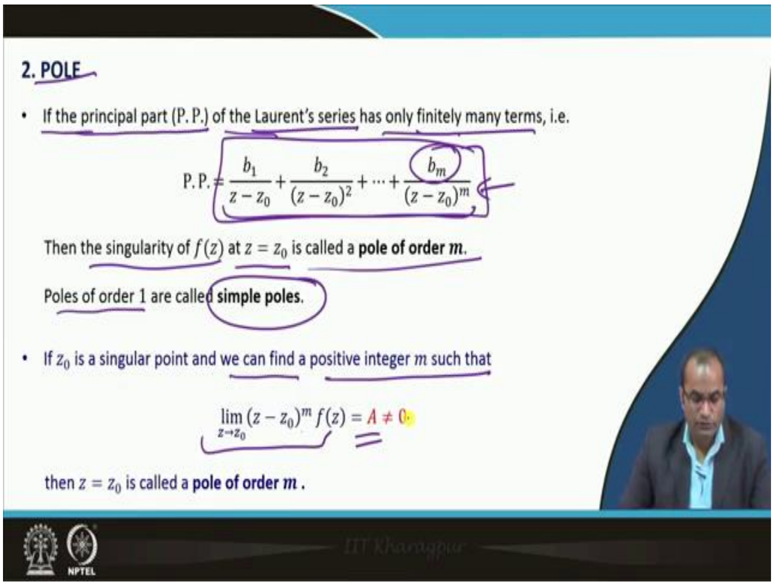

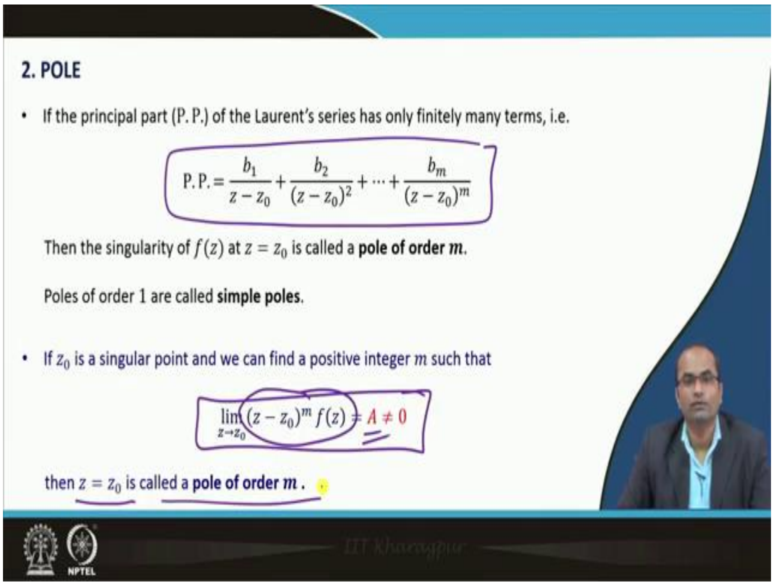

So, the second category is the pole, we call as a pole that is the name of the singularity the classification of the singularity. So, again if the principal part of this Laurent series has only finitely many term. So, there could be three situations and we have the three classification here, either the Laurent series has no principal part that means no negative powers then such singularity will be treated as removal singularity.

If it has only finitely many terms then this is, these are called the poles and the later on we will see when we have infinitely many terms it is called the sensual singularity. So, let us discuss this pole there, so if the principal part of this Laurent series has only finitely many terms that means the situation in the principal part is somewhere like this that you have these finitely many terms b1 z minus z0, b2 the coefficient we have the second order term and so on we have these m terms.

So, in this case the singularity of fz at z equal to z0 is called the pole, so it is a pole of order m. And the poles of order 1 are called simple poles, so another classification if we use simple pole that means it is a pole of order 1. And if z0 is a singular point we can also find a positive integer m such that this when we multiply this fz by z minus z0 power m which is clear from this Laurent series as well.

So, when we multiply this by z minus z0 power m we will have something bm there that means the constant a which is not equal to 0 and then we call z equal to z0 a pole of order m, so again here we have the two approaches to classify, one is using Laurent series so if we find the Laurent series has only finitely many terms we will classify such a point as pole of order m, or the another approach we can if we can find a positive integer m such that this z minus z naught power m fz is some constant A which is not equal to 0 then this z equal to z0 is called a pole of order m.

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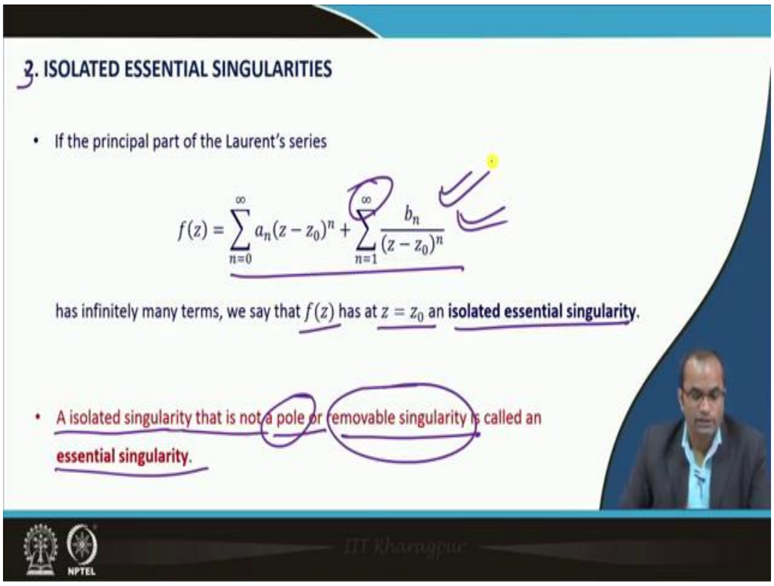

Well, so the third one. So, this is the third one now is called the isolated essential singularities, so in this case if the principal part of the Laurent series, so this is the Laurent series. If it has infinitely many terms, so infinitely many terms then we called that fz has at z equal to z naught an isolated essential singularity.

So, this is the third classification which we have, which we are discussing now. And further, another way we also define this isolated essential singularity which is not a pole or which is not a removable singularity is called essential singularity. So, isolated singularity which is not a pole and not a removable singularity we will call as essential singularity, because either this Laurent series will have no term or it will have finitely many terms or it will have infinitely many terms. So, these three classification we can have for the isolated singularities.

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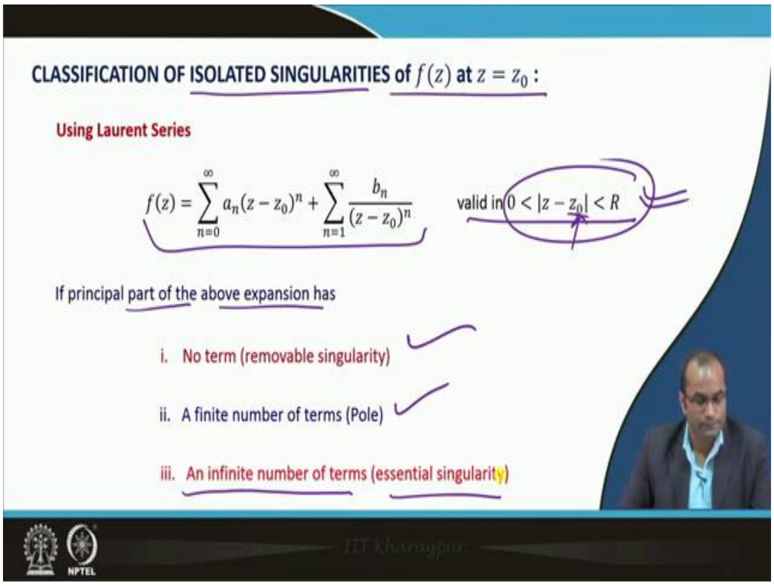

So, just to summaries what we have done these isolated singularities at z equal to z naught, using the Laurent series we can classify. So, we will expand the Laurent series in this region around z equal to z naught point the singular point which we want to classify, so we will do this expansion there and if this principal part of this above expansion has no term we call removal singularity, if it has finite number of terms we call pole and depending on the number of terms we also say the pole of order m or pole of order 1 are called simple poles. And if it has infinitely many terms then we call that this singularity is an essential singularity.

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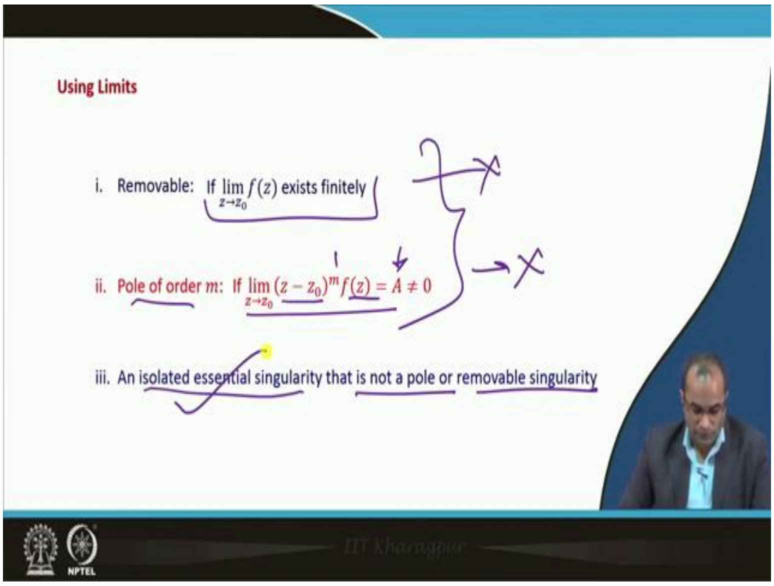

So, using limit also we can classify, so for the removable we have observe that if this limit exist as a finite number, pole of order m if we can find an m where z minus z0 power m and fz equal to A, some finite number exist then we call this is a pole of order m and for the isolated singularity we just classify which is not a pole and not a removable singularity with the earlier approach here with the limits if we see that this is not a removable singularity, this is not a pole then this is going to be an essential singularity.

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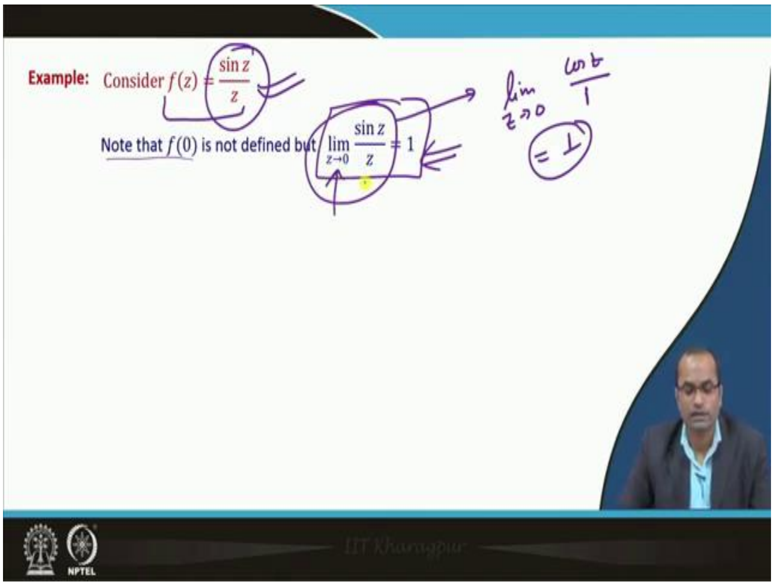

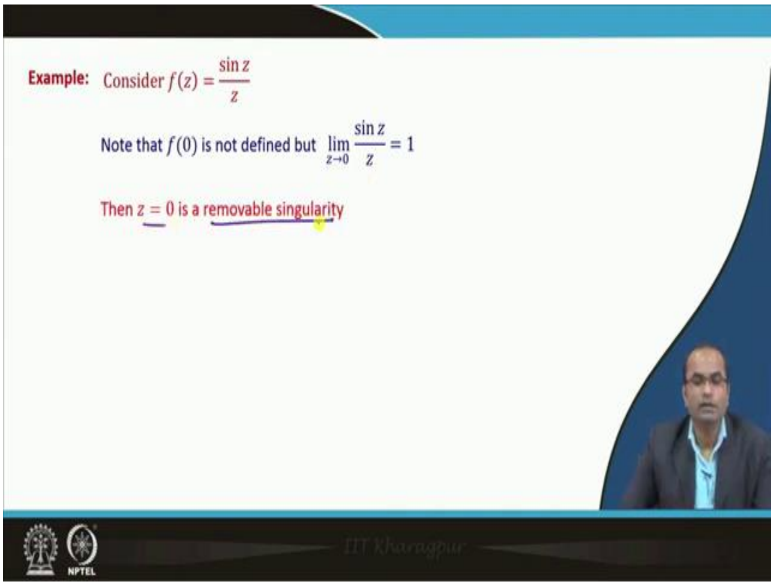

So, with this classification now we can go through some of the examples, the one the sin z over z we discussed, so we consider this fz as a function fz over z and in this case we note that this fz is not defined but if we consider this limit here as z approaches to z naught, sin z over z. So, this that there be the L'Hopital rule is applicable, so here we have 0 by 0 form and then the limit z approaches to 0 we have the cos z and then (())(17:15) cos 0 is 1 so this limit is 1.

So, this limit exist and that was the, that was the only critical point at z equal to 0 but here we see that not the critical point, the singular point and there we see that this limit exist at when a z approaches to z naught for this function sin z over z and we therefore we call that this is a removable singularity as per the description before.

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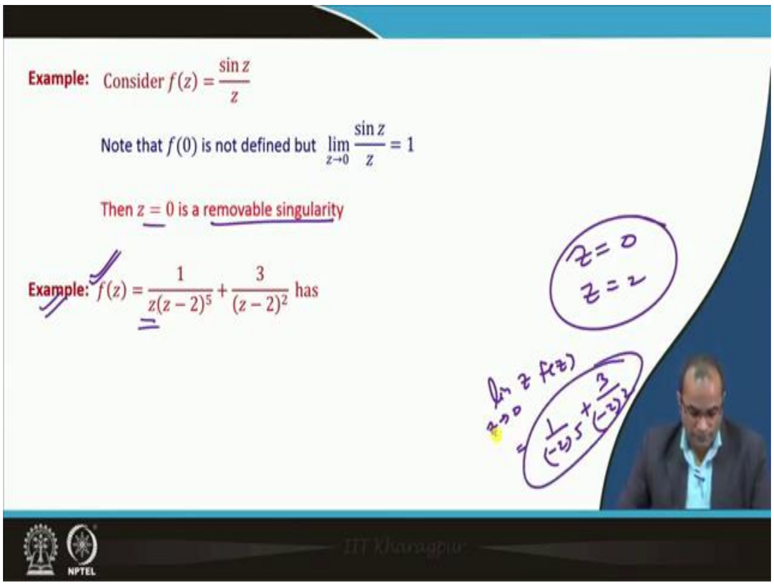

The second example we take fz 1 over z, z minus z naught power 5. Then we have this 3 over z minus 2 square and then we have here the z minus z square. So, we are discussing now the singularity of this function so naturally z equal to 0 and z equal to this 2, these are two singular points, but we want to classify them whether they are removable singularity pole of what order or the essential singularity.

So, in this case this z equal to 0 it is clearly when we multiply by z there and then take the limit z equal to 0 something finite will happen, so if we multiply this z fz and then take the limit as z approaches to 0, so what will happen. So, here 1 over this minus 2 power 5 and then 3 over minus 2 square, so this is a finite number and therefore this is a pole of order, order 1.

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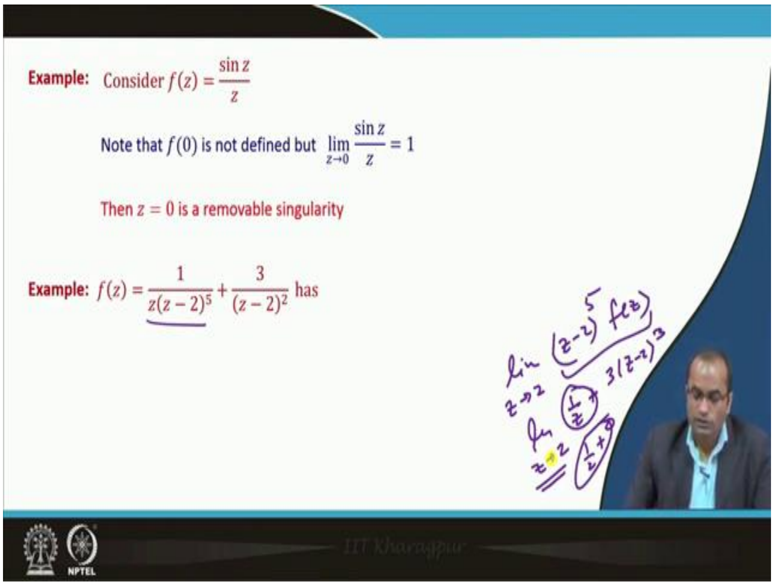

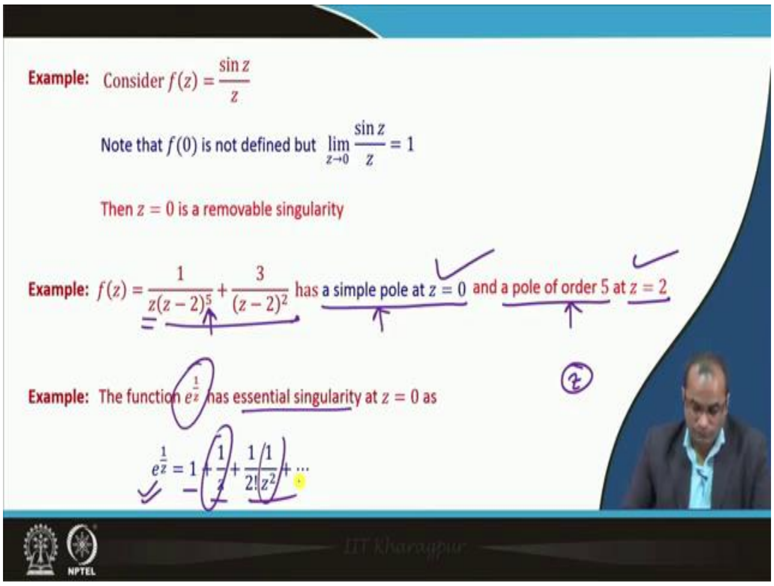

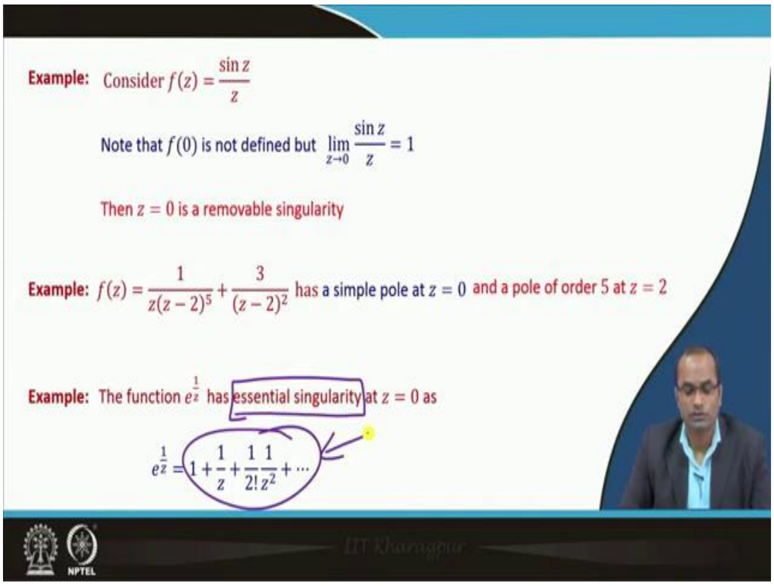

And now for the second part what we will do now because if you multiply by z minus 2 whole square and then take the limit z to 2 and this z minus 2 some power we will see what power we should take, so that we get some finite number, so if we take for instance the power 1 or 2 in this case this will be 3 but the first term will have something going to infinity so this is not going to help.

So till that time we go to the power 5 here because if we go beyond 5 then it will become 0, this limit will become 0 so that also we are not looking for, so if we take this power 5 so we just 1 over z there and then here 3 this z minus 2 the cube will be there and if we take the limit z approaches to 2 now.

So, due to first term it is 1 by 2 and the second term is 0 so we have something finite which is not equal to I mean 0 or infinity. So, here what we observe now that this function has at z equal 0 a simple pole a pole of order 1 and at z equal to 2 it has a pole of order 5, so is has a simple pole at z equal to z0 and a pole of order 5 at z equal to 2. Just looking at the given function itself we can conclude without evaluation if the scenario is that much simple that what kind of classification we have for the singularities.

Another example we will consider here, that this function e power 1 over z and what we observe that this has a essential a singularity at z equal to 0 and the reason is clear that this exponential 1 over z if we expand in terms of z in terms of powers of z, so what we observe that the expansion has 1 plus 1 over z, 1 over z square, 1 over z cube and there will be infinitely many terms in the expansion and so the negative, the principal part has infinitely many terms of this 1 over z.

And therefore, this z equal to 0 is the essential singularity in this case we do not have to check with any the limiting case whether it can be a pole or anything that is not possible, so here it is a essential singularity therefore.

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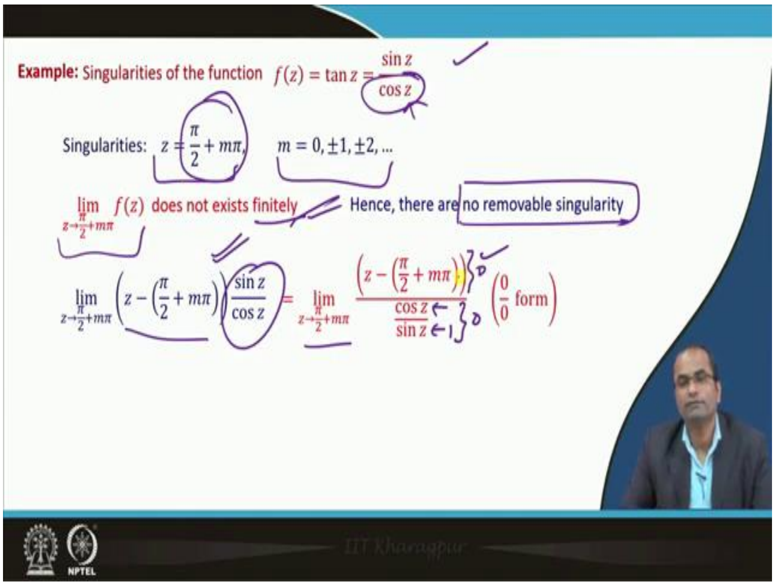

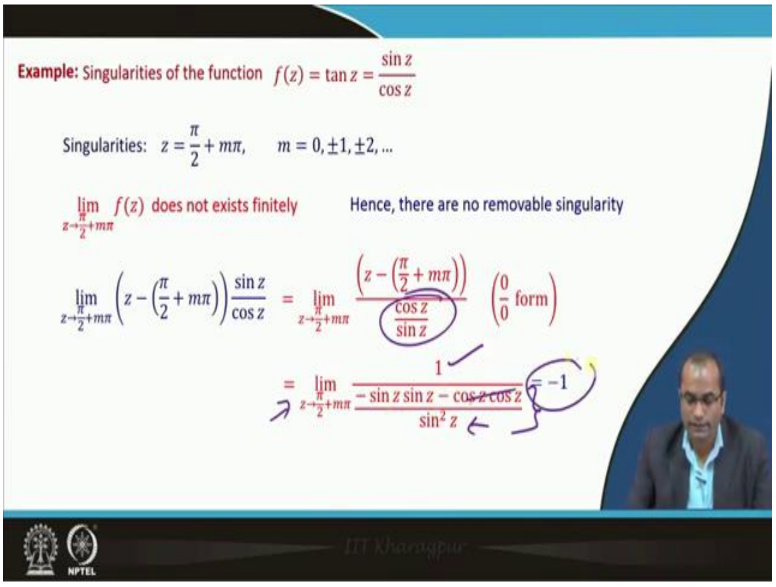

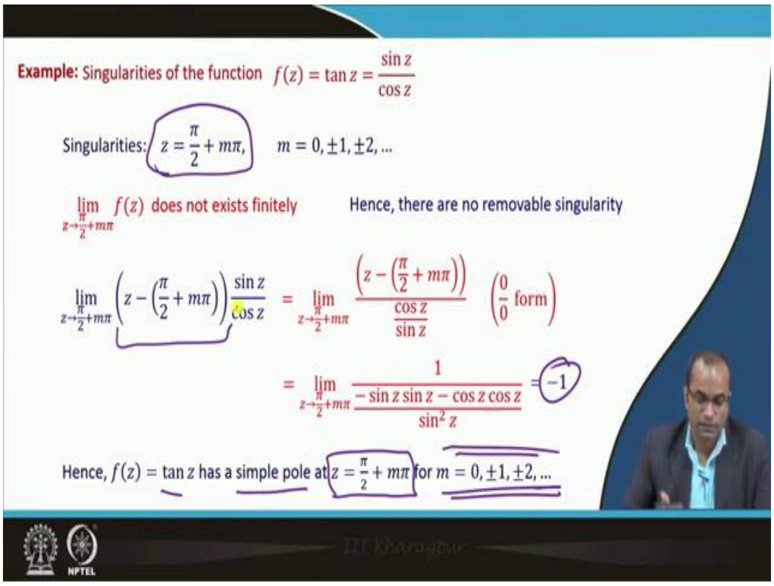

Now, if you want to discuss the singularity of the function tan z which is sin z over cos z, so we have already discussed that this are the singular points for this function and now we will classify them that what kind of singularities does this function has. So, here if we take just the limit as z approaches to this pi by 2 plus m pi of this fz we will notice that this limit does not exist, so exist, does not exist finitely because of this cos z that will go to 0 so we have the infinity there.

So, naturally this does not exist, so it is removable singularity that is clear just by taking this limit itself, because the numerator the sin z will approach to 1 and this will approach to 0 so that will go to infinity so this not exist, so will try now with this multiplication of z minus this z naught, so the z naught is this pi by 2 and m pi the general z naught we are considering.

So, if we multiply this by multiply sin z over cos z by this z minus z naught, so in that case what we observe that we have a 0 and this we have written as cos z over sin z so this cos z is 0 again and this is going to 1 so we have the 0 term here and 0 in the numerator so have this 0 by 0 form and therefore we can apply this L'Hopital rule and see what comes out, so from the numerator we are getting 1 when we go for the differentiation and the differentiation of this cos z over z by this (())(23:30) rule we have this expression.

So, in this case when this z goes to pi by 2 plus m pi we observe that this is going to be 1 and then here we have this 0 and then again 1 there, so minus 1 we are getting as a value of this expression that means this limit exist now finitely we have the number 1 minus 1, it means that all these points here z equal to pi by 2 plus m pi they are basically simple pole.

So, this tan z has simple pole at this point or all these points indeed because these are infinitely many points. So, we have classified with the help of this limit their, though we can do with this expansion Laurent's expansion as well, so their classification can be done with the Laurent series.

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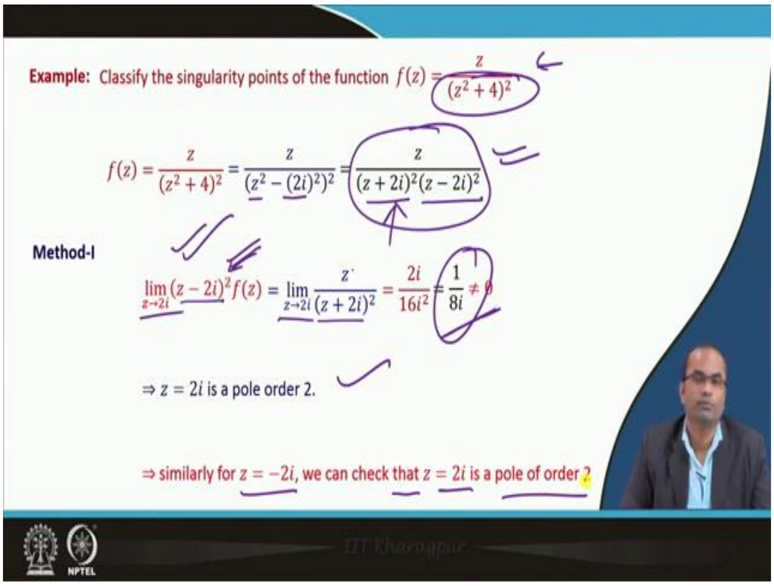

Now, you want to do classify this singular points of, of this function fz equal to z over z square plus 4, so first we will find out what are the zeros here for this function means where the these singular points, so this is we can be written as z square minus this 2i whole square and then we have the square there.

So, this z over z plus 2i whole square and z minus 2i square, so now the singular points are clear we have the minus 2i and the 2i these are the singular points, so if we consider this limit z minus 2i square so in this situation now it is clear that if we take the square there and then take the limit z equal to 2i we are expecting to get a finite term now.

So, the remaining portion now when we this z minus 2i square will get cancel, so we have z over z plus 2i whole square and z approaches to 2i, so it is 4i and then square so 16i square that means we are getting 1 hour 8i so it is a finite number, so here with this 2 it works to get the finite number indeed if you increase this power 3 then it will be 0, if you decrease that power to be infinity, so this is the only possibility to get something nonzero and finite number for this limit.

So, we have this pole of order 2 and similarly we can observe that as z equal to minus 2i also has a pole of order 2, so not equal to 0 so it is a pole of order 2 and similarly z equal to minus 2i we can check that this is also a pole of order 2.

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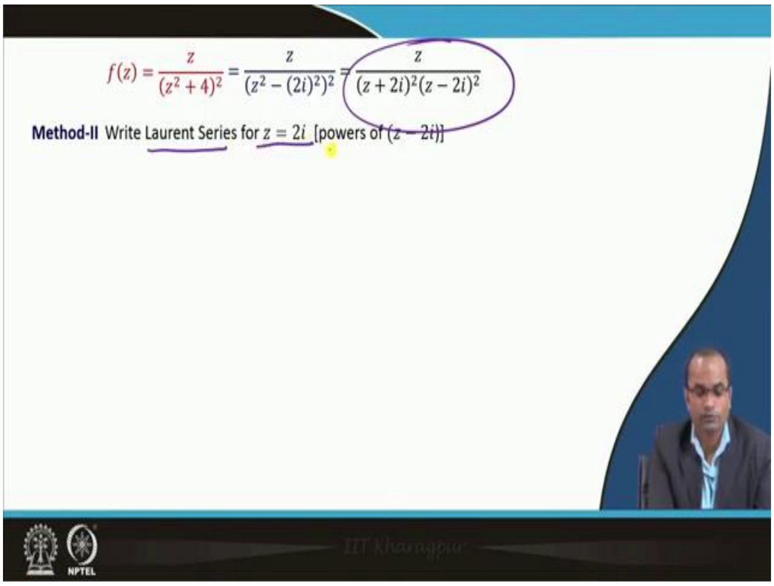

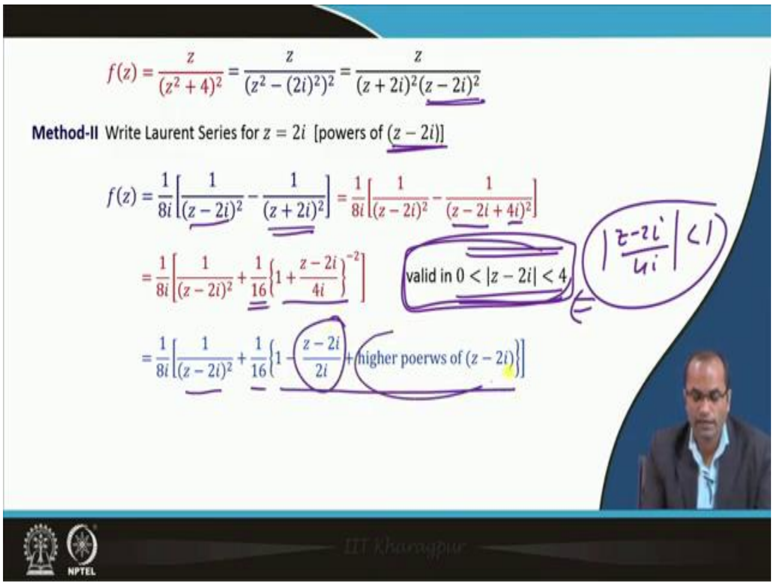

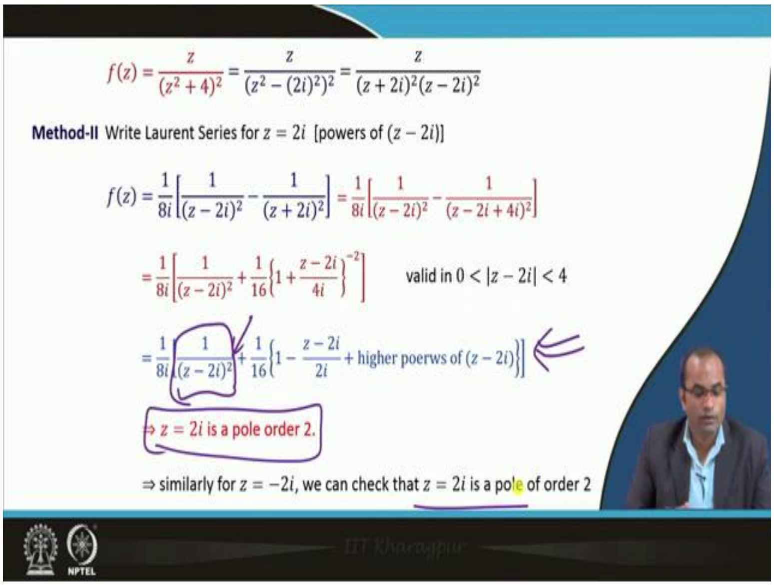

So, having this partial fraction, we can also observe looking at the series expansion for this the Laurent series expansion for this function and then again we can observe that these are the poles, so this is method 2 we are going to now have. So, for the Laurent series at z equal to 2i that means the powers of z minus 2i we are going to have this expansion.

So, z minus 2i power whole square, one term is already there so that we will not touch when we do this partial fractions here, so 1 over this z plus 2i square we can expand in terms of z minus 2i powers. So, we can rewrite this z minus 2i and then to compensate this we have this 4i square and then we will take this 16 common from here the 4i this square terms.

So, 16i square that minus minus will become plus and we have here 1 plus z minus 2i over 4i whole square and this is valid now in this region z minus 2i less than 4 so that can be just done by setting this that z minus this 2i over 4i should be less than 1 from here we are getting this region of expansion.

So, exactly this is the region which we want to expand the whole function to see the, to see the, to classify the singularity and in this case, so we have 1 over z minus 2 square and then 1 by 16 and here when we expand so we have 1 minus z minus 2i over 2i and the higher order powers naturally will appear for z minus 2i.

So, this is the term which is coming in the negative power of so up to 2, the order 2 we are going here, so that means this is a pole of order, order 2. So, in this way also it is very easy to classify these singularities, so similarly 4z equals to minus 2i now we can expand in terms of z plus 2i and again we will observe the same situation that this is a pole of order 2.

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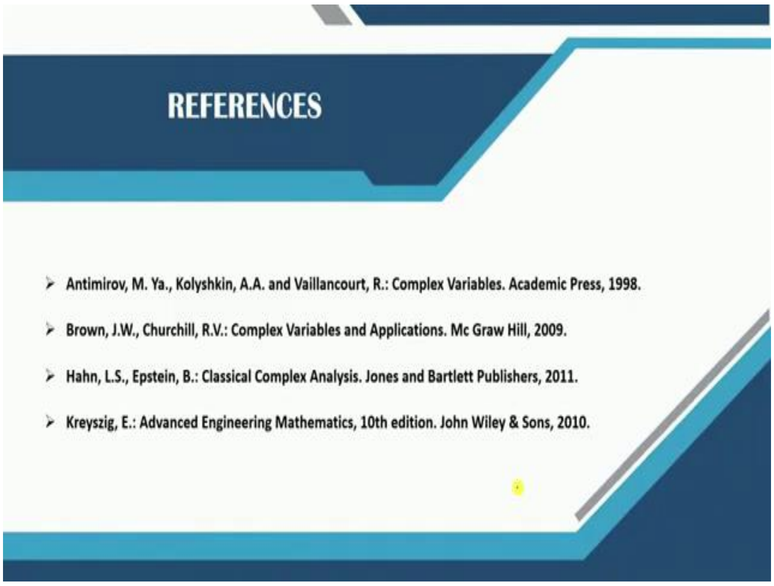

Ok well, so these are the references we have used for preparing this lecture.

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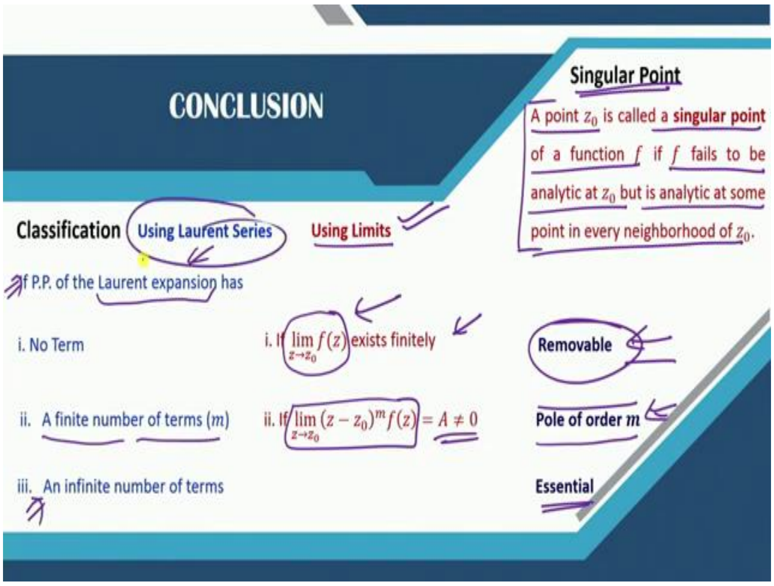

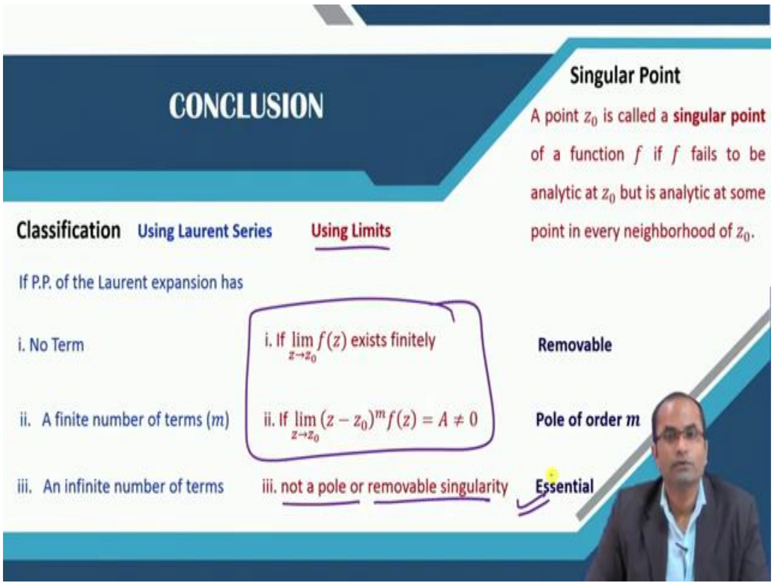

And just to conclude now, so we have discussed singular points today it is a very important topic in, in complex analysis and some of the applications we will observe in the next lecture. So, a point z0 is called a singular point of a function f if this f fails to be analytic and most important is the second part also that it is not only just failing to be analytic at z naught, but it should be analytic at some point in every neighbourhood of z naught and the example we have seen this mod z square which can be explain now with the this definition that this mod z square has now singular point.

So, then we have discussed the classification and the first was what is the removable singularity, so using the Laurent series we have observed that if the principal part of the Laurent series has no term then we call it as removable singularity, the second using the limit we have discussed and their we discussed that if this limit here exist finitely then also we call it as removable singularity.

So, depending on the convenience of whether the Laurent series expansion is easy to perform or using this limit we can check whether it is removable singularity or different kind of singularity. So, the pole of order m that was a second classification for the singularities isolated singularities, so a finite number of terms that is means these m, first m appear there, then we call this pole of order m or we can find such m where this z minus z0 power m fz is a finite number in that case also we can classify, so that is using limit.

The essential singularity we call when the this Laurent series has infinitely many terms and using the limit we observe that if it is not a pole or not a singularity by these methods we can classify this that this is essential singularity in that case. So, that is all for this lecture and I thank you for your attention.