Engineering Mathematics - II
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Lecture 18
Laurent's Series

So welcome back to lectures on Engineering Mathematics 2. So, this is lecture number 18 and we will be talking about Laurent Series, which is a generalized version of the Taylor series which we have discussed in the previous lecture.

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So, today we will cover what is the Laurent Series and then the practical methods for obtaining Laurent series quite similar to what we have done for the Taylor series.

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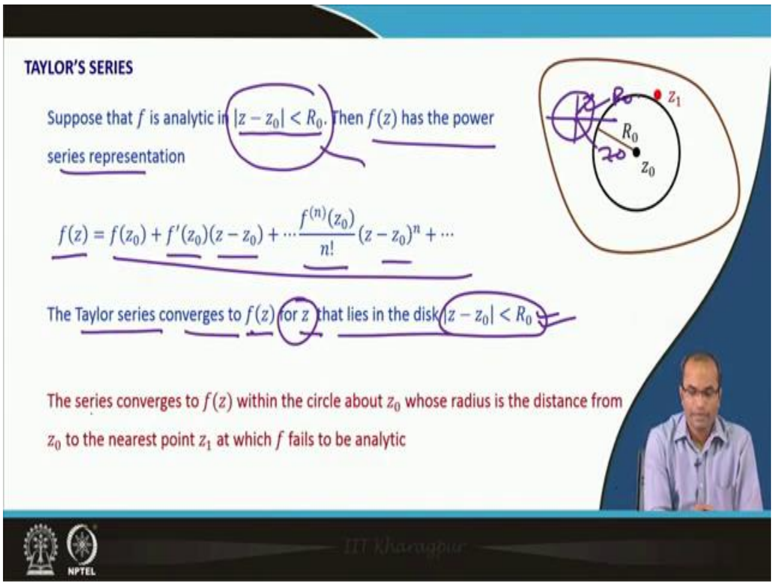

So, the Taylor Series just to recall suppose that f is analytic in this region z minus z 0 less than R naught so in this disc with radius R naught and this center z naught then f z has the power series representation this is what we have discussed. So, if f is analytic in this in this disc here with center z 0 and radius R naught in that case fz can be represented by such a series expansion fz naught the first derivatives z minus z naught and the nth derivative and so on.

And we have also discussed that the Taylor series converges to fz for z that lies in this disk. So, a z minus z 0 less than R naught, if this f z is analytic in this disk, then the series will converge to fz for any z that lies in this disk.

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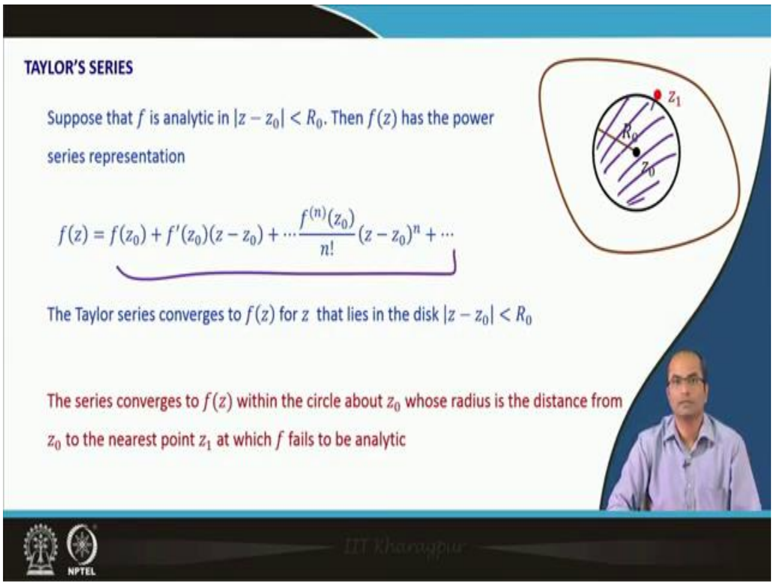

So, and we have also discussed that the Taylor series fz that the series which we calling Taylor series converges to fz and within the circle about z 0 and the maximum radius of this circle we can go up to the nearest point. So here its visualize there up to the nearest point where f fails to be analytic.

So, if this fz is analytic in this disk here, we can go on increasing the radius until we meet the point where fz is naught analytic or fz fails to be analytics. So, that is what we have discussed in the previous lecture. So, the most important here to note that, that this fz is analytic in the whole disc of radius R naught and then we can represent this fz by such a series representation which is called Taylor series representation and the maximum radius where this series will converge to fz can be obtained by extending this radius until we meet the point where fz fails to be analytic.

So, this was the summary of the last previous lecture and now, we will go for the further extension or the generalization because very often we will see that some functions they are not analytic at some point or in some region. So, what kind of representation we can have for those functions.

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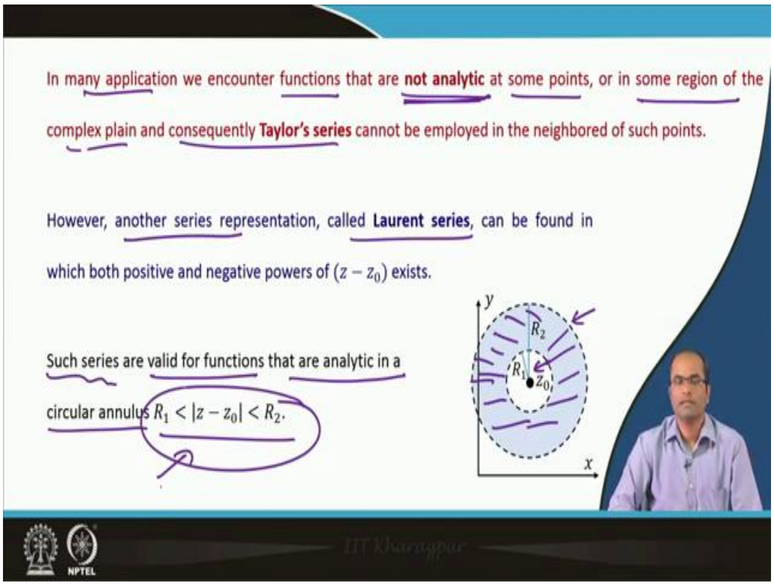

So, as written here, so in many applications we encounter functions that are not analytic at some points if they are analytic we have seen we can represent by this Taylor series representation, but if they are not analytic if the function is not analytic at some points or in some region of the complex plane and then this Taylor series cannot be written in the neighborhood of such points when we have the function which is not analytic at a certain point or a region.

In that case the another series which is called Laurent series that can be found and this Laurent series contains the positive power of z minus z0 and also the negative power of z minus z0. And this is what we will discuss today and we will also discuss that how to obtain such a series.

So, this Laurent series, they are valid for the functions that are analytic in the circular annulus. So, it is as discussed earlier, the Taylor series can be employed when you have the function which is analytic over the whole disk. But now, we are talking about that this more generalized power series which is called Laurent series can be written for the function which are analytic here on this annulus which contains these two circles that there is a cut here with this R one circle which is not a part of this domain now, and then we have the upper circle which is bounded whose radius is R 2.

So, in this annulus if our function is analytic we can have a power series representation but we too, but this power series representation will contain positive power as well as negative power. Whereas, in Taylor series we have seen that only the positive powers of z minus z 0 appears, but we will observe in Laurent series both positive and negative powers will appear.

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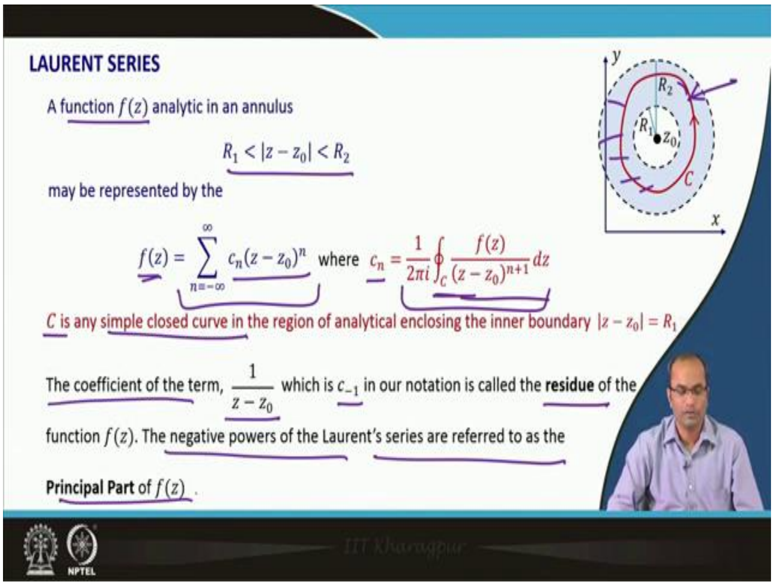

So, what is the Laurent series? So, suppose this function is analytic in this annulus which is bounded by these two circles here R 1 in whose radius R 1 and R 2. So, this can be represented then this function can be represented by this power series representation Cn z minus z 0 power n and then n is now from minus infinity to plus infinity. In the Taylor series it was from 0 to infinity.

So, now we have added from minus infinity to this minus 1 these terms and similar to the Taylor series this Cn, this coefficient can be written can be represented by this integral which comes to be the derivative of f basically. So, and the C here in this integral C is any simple close curve in this region of analytical and closing the inner boundary. So, we have this region where the function is analytics, so we can take any circle which is given here, so you can take any circle which enclosed the inner boundary and of course lies in this domain where the function is analytic.

So, we can do this integration on any curve, and then this fz can be written as with the help of this power series representation. So, again this proof of this series representation is quite involved and we are not talking about how to get exactly this power series representation. But now the coefficient term here of z minus z 0, which is C minus 1 in our notation, the coefficient of 1 over z minus z 0 is called the residue and that there will be one more lecture we will be talking about many applications or of the residue and what is the residue theorem which relates to the integral evaluation.

And the negative powers of this Laurent series are referred to as principal part of fz so this two more terminology here the residue is the coefficient of this 1 over z minus z 0 in this Laurent series expansion and all the terms which of the Laurent series which are having negative powers, we call the principal part of fz and we will be talking about that what we can do about this principle part or what is the importance of the residue in following lectures.

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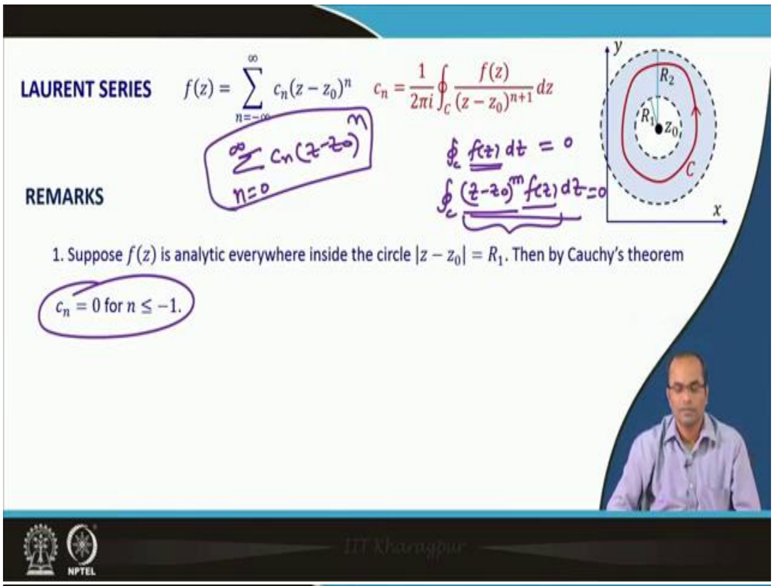

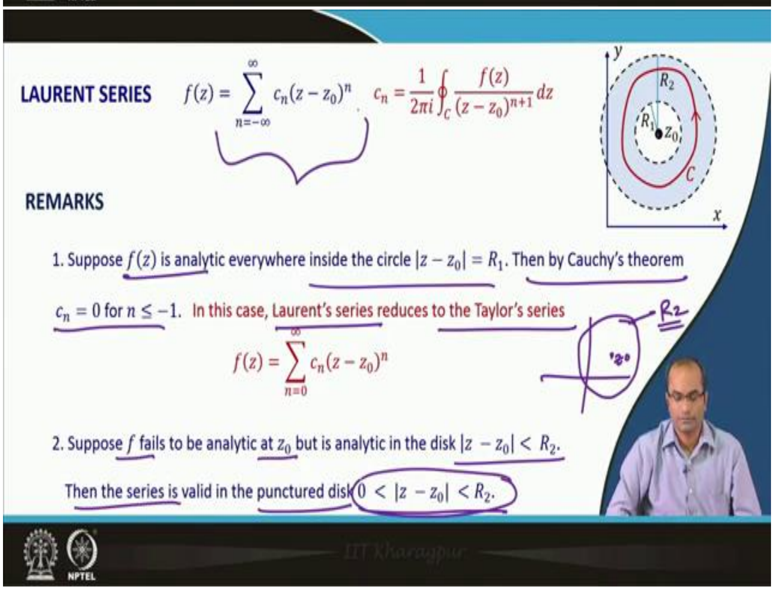

So, we have the Laurent series expansion where Cn can be evaluated by this integral and now few remarks related to these Laurent series expansion the first one suppose this fz is analytic everywhere inside this circle of radius this R 1. So, what we have observed while writing this Laurent series that our function is not analytic in this region or at some points in this region of this inner circle.

So, now we assume that our function is analytic everywhere. So, there is no such cut out of this region. So, the function is analytic. So, what will happen to this Laurent series In that case, we know from the previous lecture that when the function is analytic everywhere in this region, then we can write down the Taylor series expansion.

But the same expansion we should be able to deduce from the Laurent series because this is a bit more generalized version of the Taylor series. So, here we now we will see that indeed we can get the Taylor series also from this Laurent series. So, if the fz is analytic everywhere inside this circle also whose radius is R 1 and the center is z 1. So, then by the Cauchy theorem, what we know that the all the Cn these coefficients of z minus z 0 power n all the Cn for whenever n is less than or equal to minus 1.

So, what will happen? Why the Cn is 0. So, just consider this was our Cn. So, we have 1 over 2 pi and then the integral is over fz and then this z minus z 0 and power is n plus 1. So, if we consider this integral here, and this power the n is less than minus 1, then this power will be a negative number or 0. So either there will be a 0 power that means the integral will reduce. So in case n is minus 1, the integral will be fz dz and when the power is less than minus 1 that is minus 2 or minus 3.

So we will have something z minus z 0 power some positive power m and fz dz then we will have the situation in either case since this fz is analytic and C is a close curve, this is going to be 0. And again here the same argument fz is analytic and here z minus z 0 power positive. So, again this integrant is analytic and Cauchy theorem says that this value will be also 0.

So, all the Cn for n less than or equal to minus 1 will be 0. So, this summation will reduce to n equal to 0 to infinity and then we have Cn and z minus z 0 power n, so this is simply the Taylor series which we have already discussed.

So, that is the first remark that if fz is analytic everywhere inside the inner circle as well. And then by the Cauchy theorem we have seen that all these Cn that means the all the negative powers of this z minus z 0 all these terms will disappear and we will get the Taylor series.

So, in this case the Laurent series reduces to the Taylor series and therefore, we are calling that this is a more generalized version of the Taylor series. Well, the second one suppose f fails to be analytic only at z naught point, but is analytic in the in this disk of the outer radius this R 2 outer circle whose radius is R 2.

So, suppose the function is only failing to be analytic at a point z naught so, at the center here that z naught the function is not analytic otherwise it is analytic everywhere inside this outer radius this whose outer circle whose radius is R naught so, except this 1 point z naught the function is analytics.

So, again we cannot write of course, the Taylor series because the there is a point where function is not analytic, but what we now can reduce the inner the radius of the inner circle to tending to 0, because now the only point where the function is not analytic is z naught only. So, this R1 can be reduced to our wish now, we can go actually to 0.

So, what is the validity reason now, for this expansion is this puncture disk what we call only the 0 is excluded. So, 0 greater than the z minus z0 is greater than 0 and then less than R 2. So, this is the so called the puncture disk only this Z 2 is avoided. Otherwise at any point we can represent the function f z by that expansion the Laurent Series expansion well.

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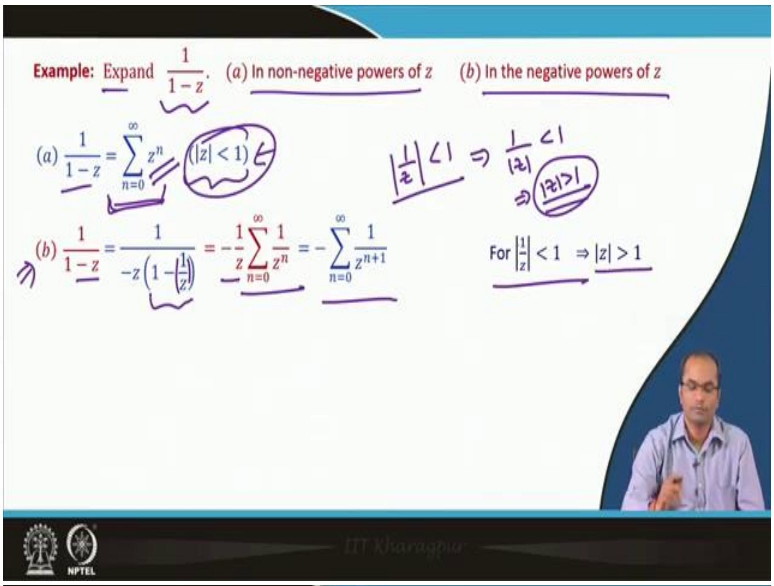

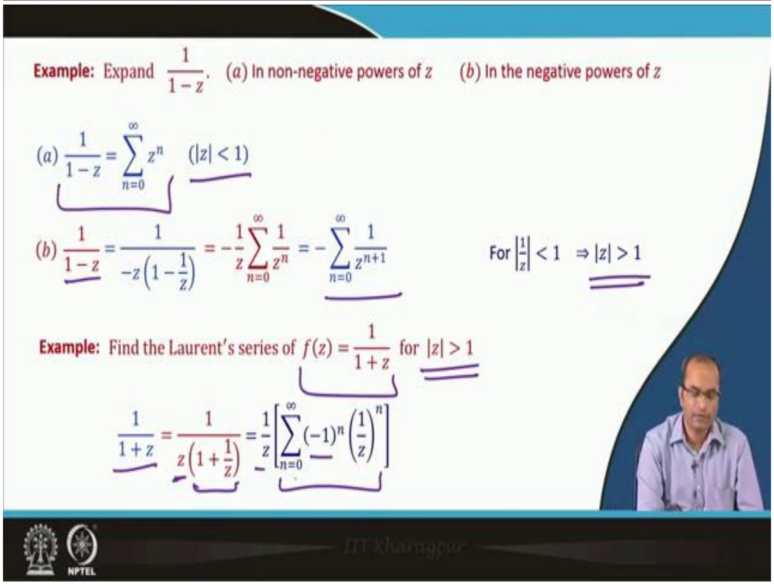

So, now we will come to the expansion that how to how to expand a function, which is not analytic at some point in the domain using this Laurent series. So, here also we will adopt the practical approach of writing this series because getting to this integral for computing coefficient Cn that could be very complicated. So, there are the ways which many functions can be treated using the expansion of this very fundamental function 1 over 1 minus z which we have already observed in the Taylor series.

So, firstly we will see that to expand in non negative powers of z and in the negative powers of z, so these two extreme expansions we will see for this function and then various combinations we can observe for any for many other functions with the help of this example. So, here are the non negative power of z naught and the negative power of z naught. So, in one we will get just the Taylor series expansion which we have seen before and in the other all the negative powers will come, so only the principal part there will be no positive power of z.

So, how to get these expansions the first one we have already observed before that this 1 over 1 minus z can be represented by this summation which goes from 0 to infinity and z power n and remember that the validity region for this was this, this disc of radius 1 and center 0.

Now, the second case with which you will observe and indeed we can just reformulate out of the previous one. So here if we take this minus z common then we have 1 minus 1 over z as similar expansion which we have written earlier for 1 over 1 minus z. So, instead of z we have this 1 over z term now, so we can do expand again using the earlier expansion, but the validate region will change because it was mod z less than 1 and now it will be in this new case 1 over in the modulus less than 1 that means, 1 over this modular z less than 1 which says that mod z greater than 1.

So, now this expansion the new expansion will be valid in this outer part of the disk, where mod z is greater than 1 whereas, the earlier expansion this from 0 to infinity z power n is valid when mod z is less than 1. So, if we expand now this we have 1 over z sitting outside before and then we have the summation from 0 to infinity 1 over z power n and the validity will be mod z greater than 1 which we can evaluate from this absolute value 1 over z less than 1.

So, we have two expansions for this 1 over 1 minus z, 1 which we have seen earlier, which is the so called the Taylor series expansion valid in this mod z less than 1 and we have another one which has only the negative power, negative powers of this z and this expansion is valid for mod z greater than 1. So, with these two extreme extension and one more example, we will see now we will be able to write down for many other functions we will see.

So, the another one we will observe here for 1 over 1 plus z but which is very similar to what we have done just now. And this we will just write for mod z greater than 1 because for mod z less than 1 again, we know from the previous lecture that was the Taylor series expansion. So, here 1 over 1 plus z we can again use the same trick, we take z common and then we have this 1 plus 1 over z.

So, again we can expand now, this 1 over 1 plus z using the Taylor series expansion, but the validity region will be computed from the absolute value of o1ne over z less than 1 and which will come out to be mod z greater than 1. So, here we can write down this expansion from the previous lecture. So this minus 1 power n will be extra term and 1 over z power n, n goes from 0 to infinity.

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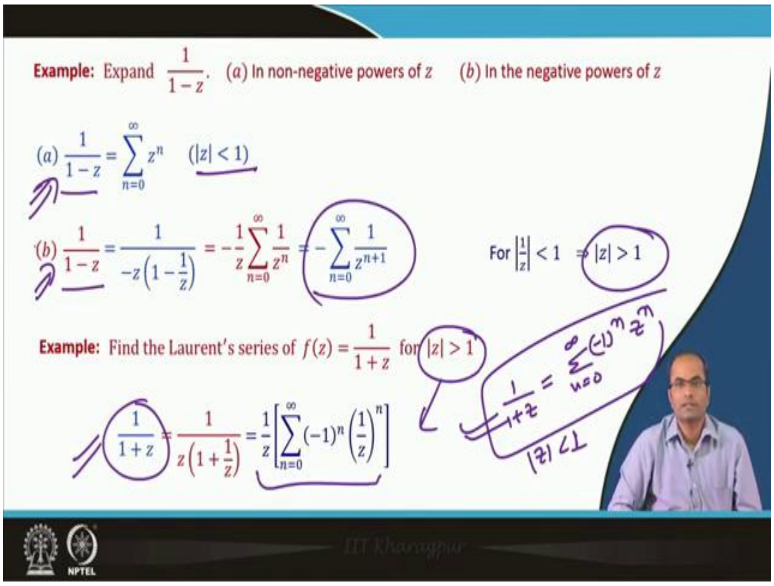

Well, so we have here this expansion and this is valid for mod z greater than 1. So what we have observe now we have basically the four expansions was one is for 1 have over 1 minus z valid for this mod z less than 1, we have the another expansion for 1 over 1 minus z which is given here by this series, but validities mod z greater than 1, the another one we have 1 over 1 plus z valid in this mod z greater than 1.

And we have also the Taylor series expansion 1 over 1 plus z which is simply n 0 to infinity and minus 1 power n and z power n which is valid here for mod z less than 1. So, these four these four expansions will be used now, for representing many functions in terms of the Laurent series expansion.

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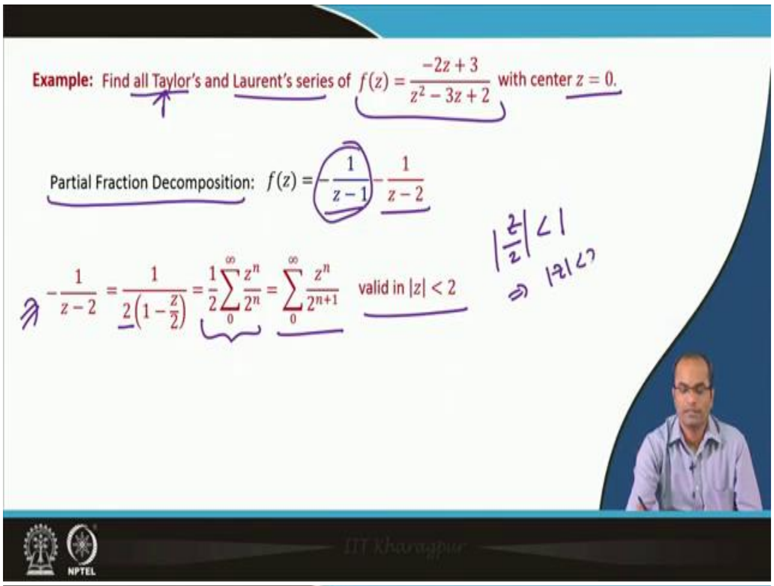

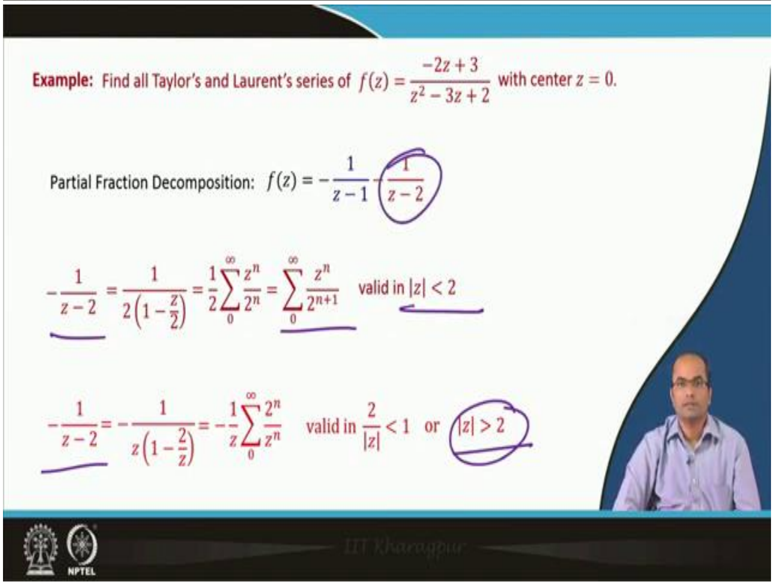

Coming back to this example, now we want to find all Taylor's and Laurent series expansion for this function, the function is given by minus 2 z plus 3 divided by z square minus 3 z plus 2 and the center here we want to take z equal to 0 that means, we want to have the power of the z positive and negative in case of the Laurent series and only positive power of z for the Taylor series, so all possible basically expansions we are going to look now for this function and we will provide according accordingly there validity region.

So, this is the trick which was used already in the previous lecture the partial fraction decomposition we have to decompose the given function in these simple functions whose expansion is known to us. So 1 over z minus 1 and minus 1 over z minus 2. So, this we know already the possible expansion for 1 over 1 minus z or z minus 1 but we want to see now what are the possible expansion for 1 over z minus 2.

So, the one way would be that we take minus 2 common that means this minus minus plus we have 2 1 minus z by 2, and that can be expanded in this z power n over 2 power n and the validity region will be z by 2 absolute value less than 1. So, here we have this expansion the validity mod z less than 2 which is coming by this relation, the modulus z by 2 less than 1 which says mod z less than 2.

So, this series here is valid for mod z less than 2. The another possibility we instead of taking this 2 common we can take the z common so, we have z there, and then we have again 1 minus, instead of z now we have 2 over z and we can again do the write this expansion, so we have minus 1 over z and then the expansion of 1 over 1 minus 2 over z which is 2 over z power n and the validity will be coming from this 2 over z less than 1.

So, the validity here for this expansion is going to be mod z greater than 2. So, we have seen these two expansions, one is valid here for mod z less than 2 another one is valid for mod z greater than 2 we have all possible expansions for 1 of these terms in the partial fractions.

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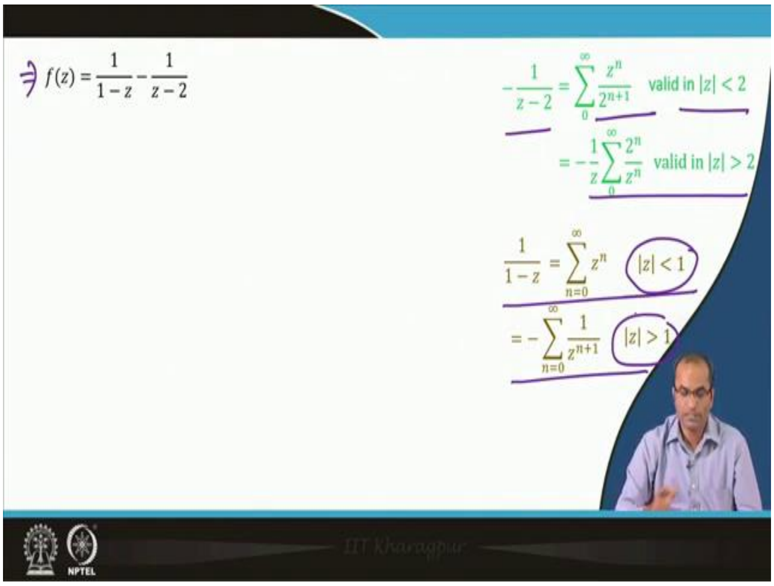

Well, so we have this function after this partial fractions and the possible expansions. So, for 1 over z minus 2 we have just seen before. So, we have these 2 possible expansions and for 1 over 1 minus z we have already seen in the first examples, so one is valid in mod z less than 1 another one is valid for a mod z greater than 1. So, all these possible expansions we have written here.

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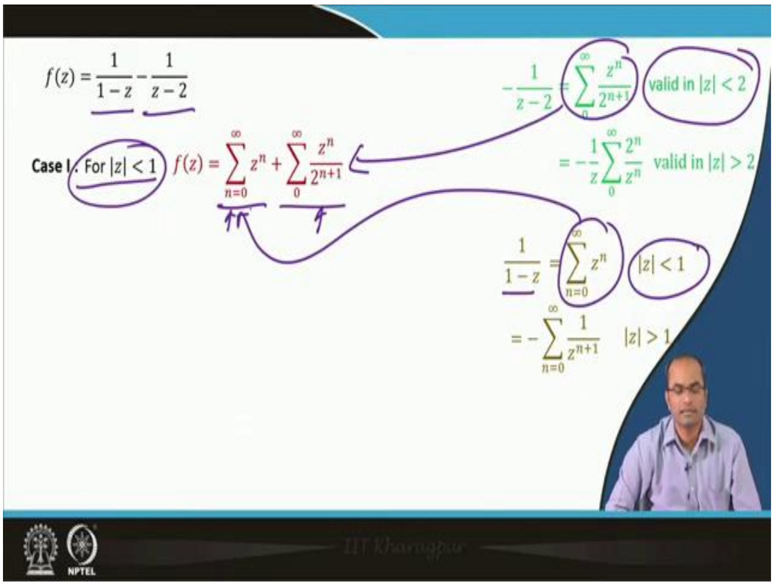

And now we can expand this given function using these expansions of the simple functions. So, the case 1, we will take when the mod z is less than when mod z is less than 1 in this region we will see that what will happen to the expansion. So, when mod z that is less than 1 the first function 1 over 1 minus z, so 1 over 1 minus z for mod z less than 1, we have this expansion, which is written already there. So, n 0 to infinity is z power n for the second one, for minus 1 over z minus 2 in this region mod z less than 1.

So, we have this valid series which is valid in this mod z less than 2 naturally that is also valid for mod z less than 1. So, we can use this expansion for the second term the z power n to power n plus 1. So this is valid for mod z less than 1 this is valid indeed for mod z less than 2, but as a whole this will be valid definitely for mod z less than 1 because both are valid in mod z less than 1. So, that is the trick we have used.

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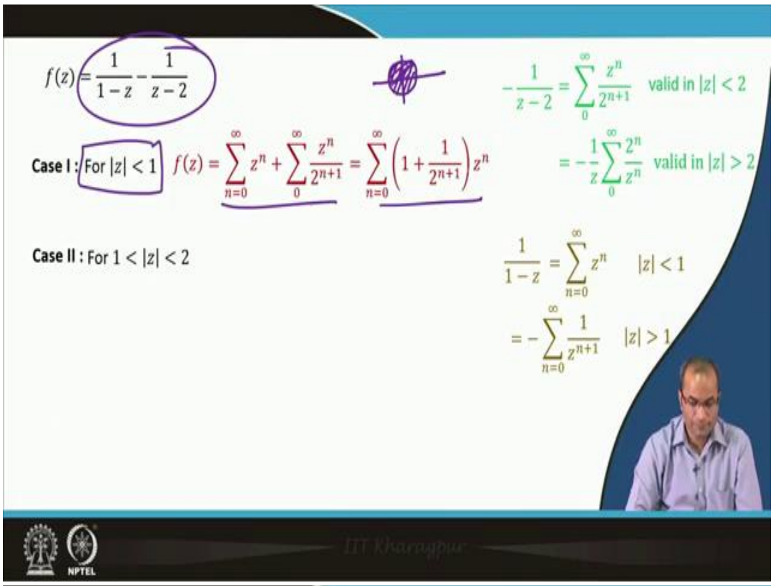

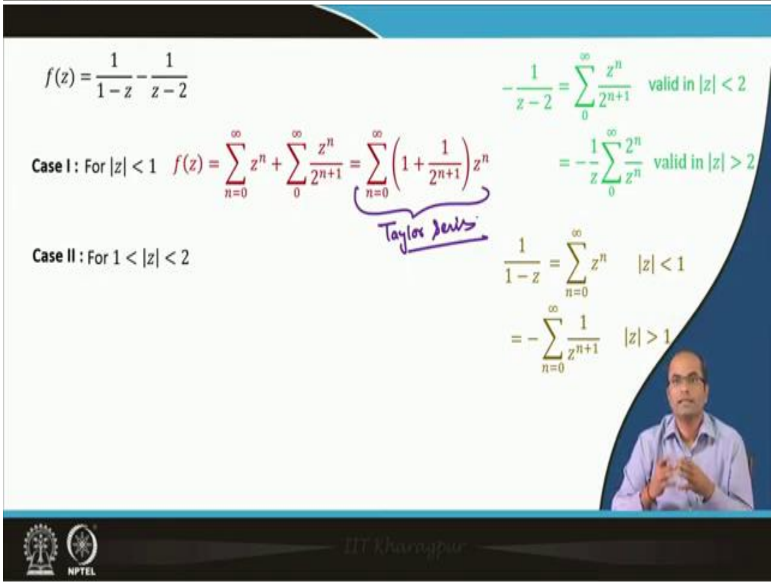

So, now we have one expansion, which is valid for mod z less than 1, this we can we can combine just to have this form. The second region we will consider. So, first we know that some mod z less than 1, the function the given function, which is given here 1 over 1 minus z and minus z minus 2. So, this is analytic in this domain with center 0 and this radius 1.

So, the function is analytic here the first point where the analyticity breakdown z equal to 1. So, in mod z less than 1 the function is analytic and hence we have the Taylor series expansion. This is the Taylor series expansion, Taylor series. So, we have the Taylor series expansion because the function was analytic in that disc.

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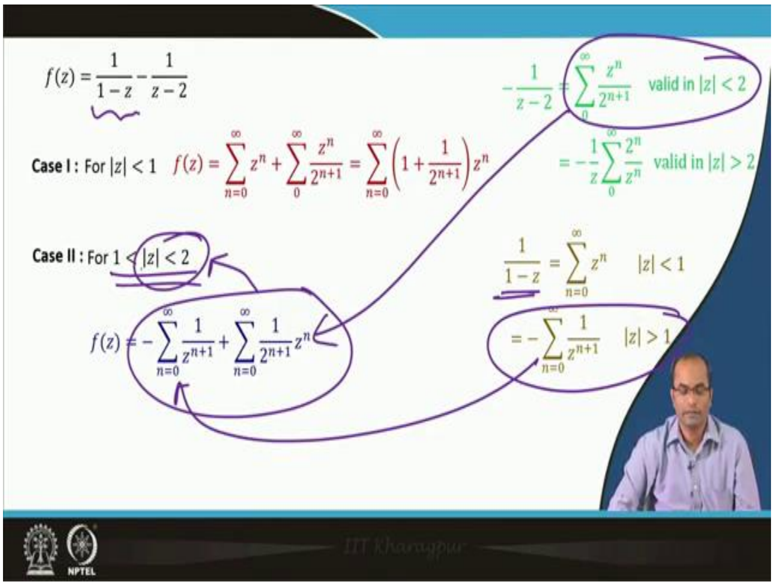

The second region now, this is the annulus we have taken mod z greater than 1 and less than 2 in this case, we will now from the first function 1 over 1 minus z 1 over 1 minus z we are now talking about mod z greater than 1. So, this is the valid expansion which we will take now with minus sign 1 over z n plus 1 and the second one we are going up to this mod z less than 2.

So, again this first one here valid mod z less than 2, we will use that expansion here and then we have this as a valid expansion in this region because the both the series are valid in the given domain.

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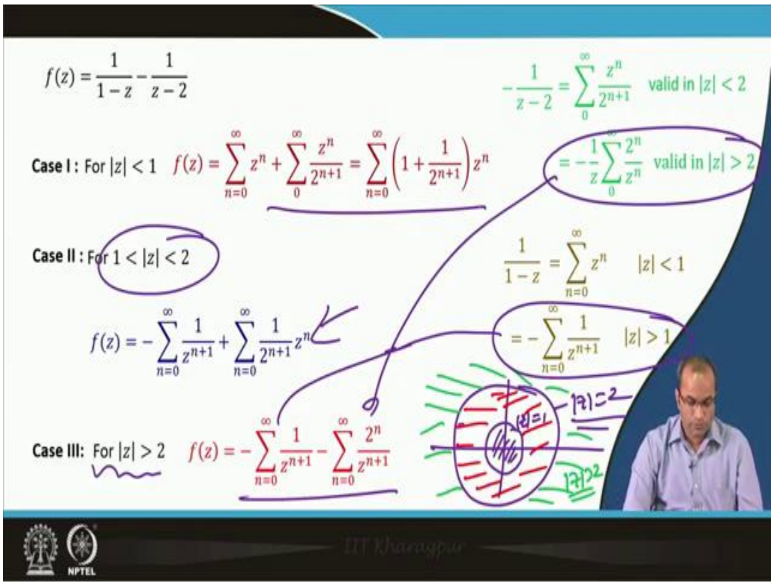

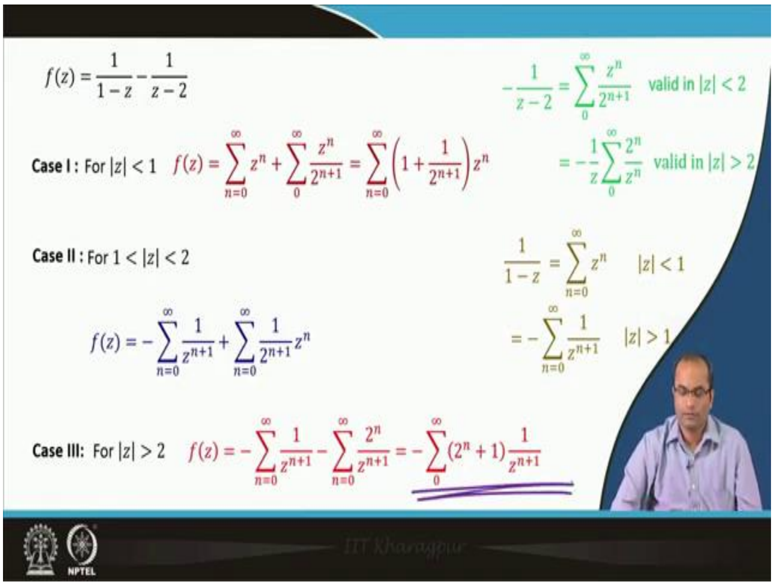

So, we have another expansion here which is valid in this annulus. And now what is interesting that we have positive and negative powers, we have both now in our expansion and this is the typical scenario where we actually have the Laurent series containing positive and negative powers of z.

The third case when mod z is greater than 2. So, basically here we have 3 region one was with radius one mod z 1, one was the mod z 2 we have seen that what kind of expansion we have inside this inner circle, we have seen that what kind of expansion we are going to have in this in this region. And now we will see that what kind of expansion we will expect. Now the outside of this disk here that is mod z greater than 2 mod z greater than 2 in this case.

So, having this situation mod z greater than 2, we have the expansions which is valid, this one in mod z greater than 2 and this is valid in mod z greater than 1. So, naturally it is also valid for mod z greater than 2. So, then we can use these two expansions here, the one here and the other one here. And then we have this series expansion and then we can club it. So here now in this expansion, we are going to have only the negative powers.

So in the outer region of this disc mounted greater than 2. That is the region where we are getting only the negative powers in the annulus, we were getting positive and negative power. And inside that disk where the function was analytic, we were getting only the positive powers of z, well so this weekend club and then we have this in compact form.

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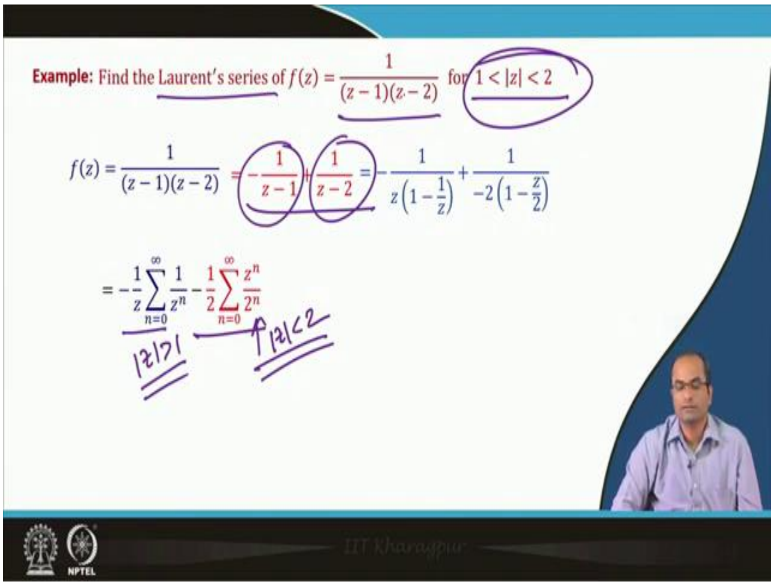

Now we want to write for instance, the Laurent series for this function in this annulus. And the idea is, it is not new now, we can easily frame this from the previous example itself. So we need to get this partial fractions first. And then we know that what is the expansion of this 1 over 1 minus z and what is the expansion of the other one 1 over z minus 2.

So, we can write down the one which is valid for more z greater than 1 and the other one is valid in mod z less than 2. So, this is valid for mod z greater than 1 and then this is mod z less than 2. So, as a whole we have this validity mod z between 1 and 2.

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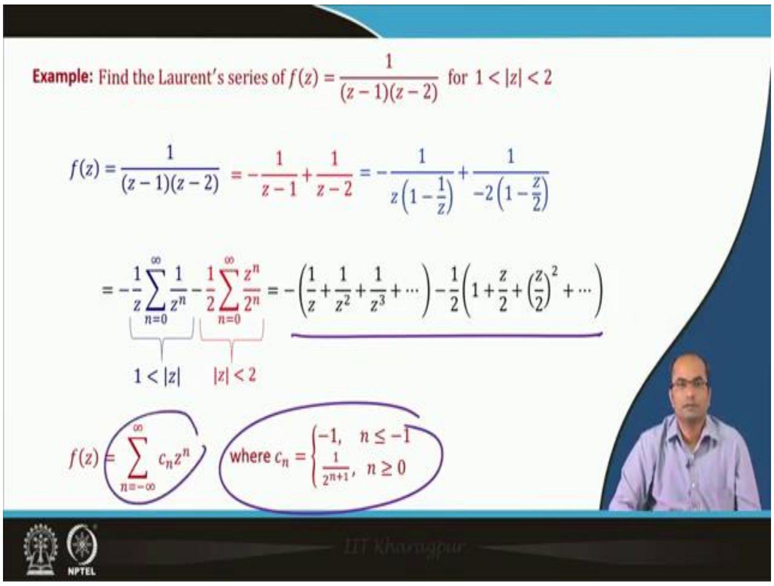

So, exactly the previous example, which we have, have seen, and then we have this expansion containing positive and the negative power because of the annulus. So, if we are talking about the inside they were the function is analytic we will get only positive power in the annulus where we have some points in the inner integral where the inner circle where the function is not analytic, we will get positive and negative powers and the complete outer region where again your function is analytic, you will get only the negative powers. So, we have just to combine this we have in the compact form this expansion.

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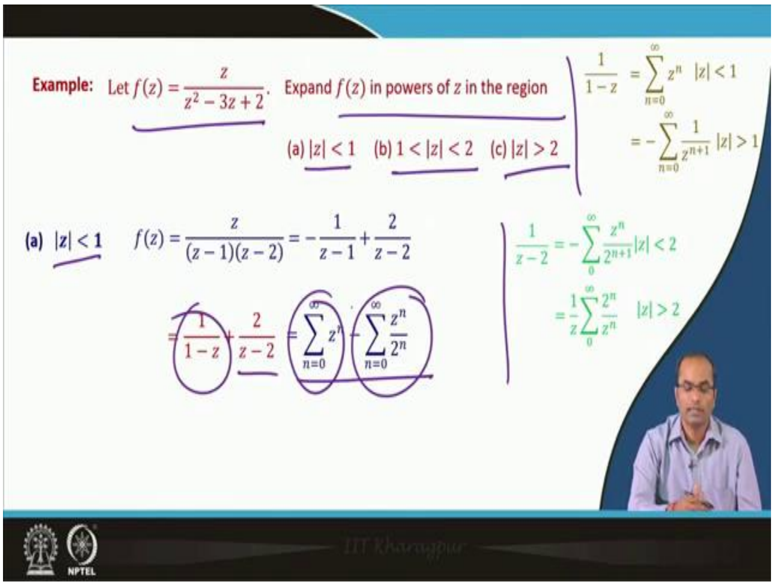

For instance, if we consider this expansion for this function again this region mod z less than 1 in the annulus and greater than 2. So, it is exactly similar to the previous examples we have done. So, for less than 1 we have to break it and then with these expansions these two expansions which we have already seen, we can write down the series expansion.

So, both we have to observe that this is valid now in mod z less than 1 and this is also valid in mod z less than 2 and therefore, as a whole we are in this range.

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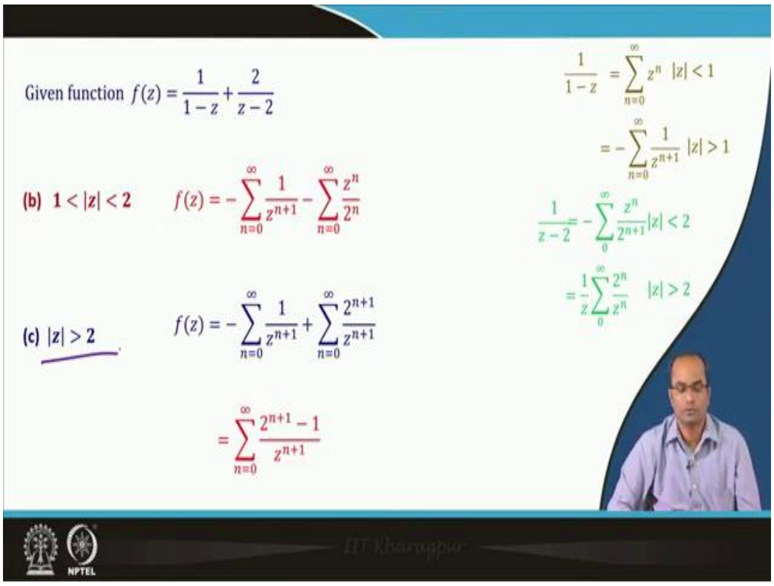

Here again, we can use the same trick. So, third with a mod z is greater than 2 and this can be also done similarly.

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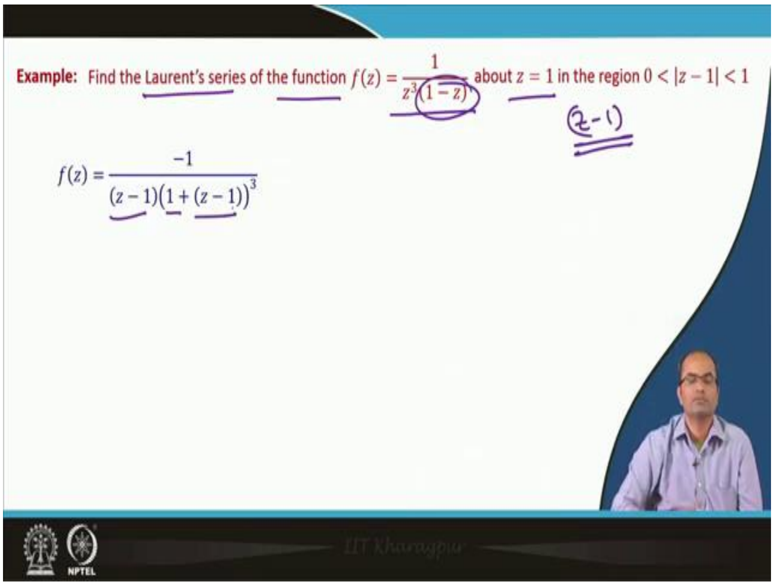

So, this is a little different example where the Laurent series of this function we will see about z equal to 1. So, instead of z equal to 0 in the previous examples, we have now z equal to 1 that means, we are going to have z minus 1 power. So, we have to frame now, this function so, that we have 1 plus z minus 1 or 1 minus z minus 1 kind of form and then we can again expand it.

So, like here we have done so, this z minus 1 which was already there 1 minus z so, we have with minus sign the z minus 1 because we are going to write everything in terms of z minus 1 and the z power 3 we have written 1 plus z minus 1 power 3 again.

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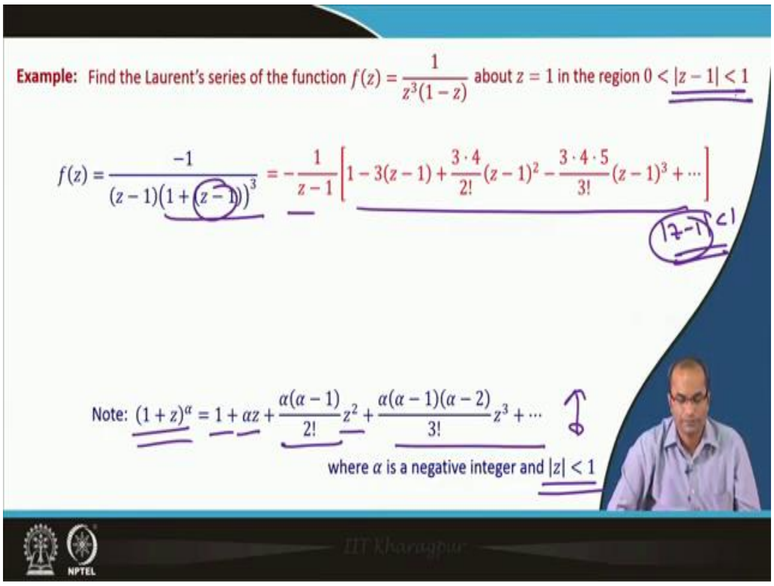

So, everything is written now in terms of z minus 1 and we know already this expansion. This is like binomial expansion, so 1 plus z power alpha is 1 plus alpha z and then we have alpha, alpha minus 1 factorial 2 z square and so on.

So, this expansion we can use here and then we will get all these powers of z minus 1. Well, so, we have expanded this in using this where the validity region was mod z less than 1. So, here also our validity region is mod z minus 1 less than 1 because we are expanding 1 plus z minus 1. So, mod z minus 1 less than 1 will be the validity region where we are expanding the series here, well.

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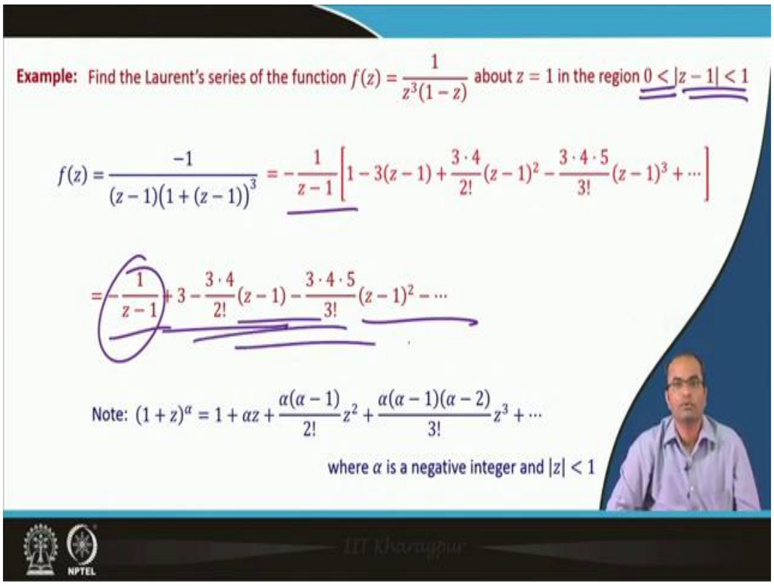

So, having this expansion and then we can just club it. So, we have these 1, 1 is the negative power and then all other we are getting the positive powers of z because this was also the situation where we have this deleted 0 point and otherwise we have we would have the annulus here, but only the 1 point the z equal to 1 is not in the domain otherwise, this valid in the whole domain except that z equal to 1 point.

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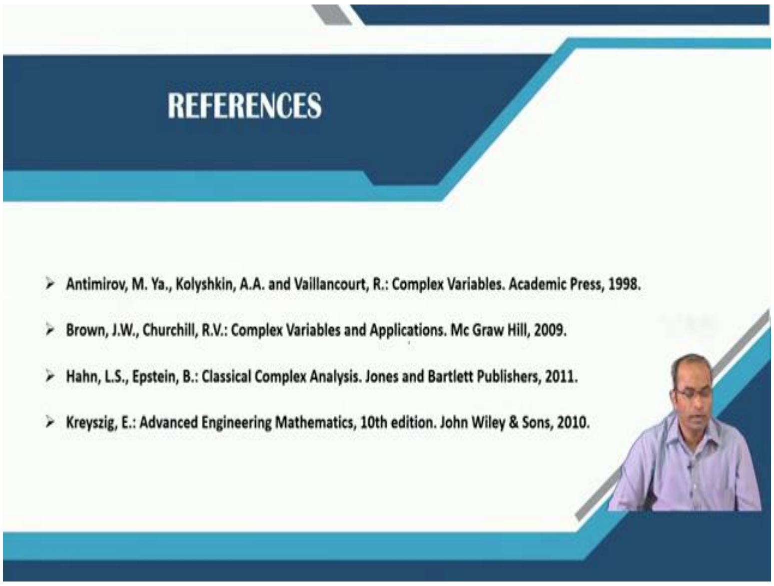

Well, so these are the references which we have used for preparing this lecture.

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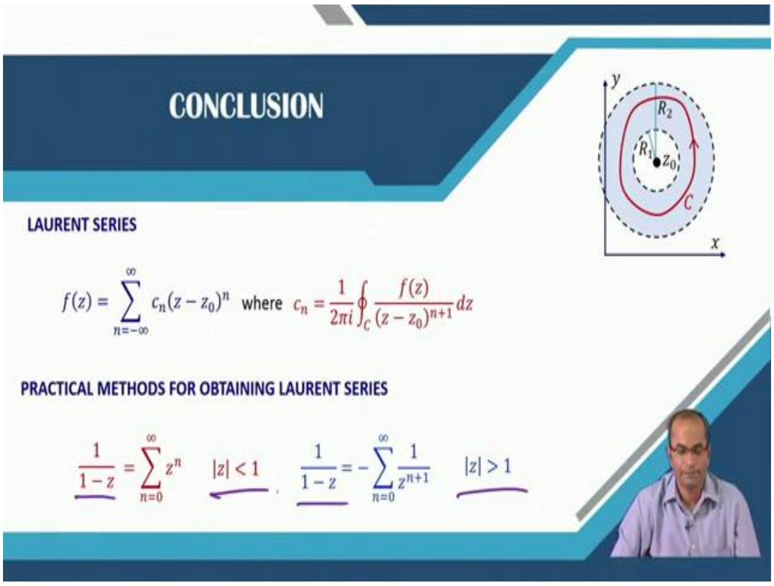

And just to conclude that this Laurent series will have now the positive and the negative power and this is usually valid in this annulus here and what we have seen that if your function is analytic over the whole disk then you will get the Taylor series, when you have the situation that the function is analytic only in the annulus then you will get positive negative powers.

And when we observe that what will be the expansion outside some disk, then we will get only the negative power. So, in all these examples we have observed that. So we have seen this practical method for obtaining Laurent series, which is very very important. And basically this function works very well.

So, if we have this expansion, the knowledge of this expansion which is valid for mod z less than 1 and the other 1, is mod z valid for mod z greater than 1 then many other functions can be written in this in terms of the simple functions and then we can have, again these expansions in their respective domains, so I thank you for your attention. That is all for this lecture.